Mathematical thinking in mathematical modelling activities

Bárbara Nivalda Palharini Alvim Sousa
Lourdes Maria Werle de Almeida

ABSTRACT

This paper refers to a research that aims to investigate the question “Do the cognitive processes that take place during modelling activities conduct students to advanced mathematical thinking?” We based our arguments on the theoretical frame related to advanced mathematical thinking and the characterization of the cognitive processes regarding mathematical thinking. We analyzed the progress of a student in a Degree in Mathematics during the development of mathematical modelling activities. The research question was investigated with a qualitative approach and an interpretative analysis. The development of the activities was recorded in audio and video. The students also wrote scripts (reports) in which they present detailed descriptions of each activity. These analyzes processes indicated that occurs elementary and advanced mathematical thinking processes as well as the interaction between them. The relationship between the thought processes is favored by the interaction between the real world situation and the mathematical content.

Keywords: Real World Situation. Mathematical Modelling. Mathematical Thinking.

Pensamento matemático em atividades de modelagem matemática

RESUMO

Este artigo descreve uma pesquisa que tem por objetivo investigar a questão “Os processos cognitivos que ocorrem durante as atividades de modelagem matemática conduzem os alunos ao pensamento matemático avançado?” Baseamos nossos argumentos no quadro teórico relacionado ao pensamento matemático avançado e à caracterização de processos cognitivos associados ao pensamento matemático. Analisamos o progresso de um aluno de um curso de Licenciatura em Matemática durante o desenvolvimento de atividades de modelagem matemática. A questão da pesquisa foi investigada por meio de uma análise qualitativa e interpretativa, com dados coletados em gravações de áudio, vídeo e registros escritos entregues pelos alunos. O processo analítico indicou que a caracterização do pensamento matemático ocorre na interação dos processos de pensamento matemático elementar e avançado, e essa

Bárbara Nivalda Palharini Alvim Sousa é Professora Doutora do Mestrado em Ensino da Universidade Estadual do Norte do Paraná. Endereço para correspondência: Av. Paraná, 340, Centro, 86300-000. Cornélio Procópio/PR. E-mail: barbara.palharini@uenp.edu.br

Lourdes Maria Werle de Almeida é Professora Doutora do Programa de Pós-Graduação em Ensino de Ciências e Educação Matemática da Universidade Estadual de Londrina. Endereço para correspondência: Rodovia Celso Garcia Cid, Pr 445 Km 380, Campus Universitário, Cx. Postal 6001, 8601-980. Londrina/PR.
E-mail: lourdes@uel.br

Recebido para publicação em 29 jun. 2017. Aceito, após revisão, em 11 out. 2017.

relação pode ser favorecida na interação ente a situação do mundo real e o conteúdo da matemática.

Palavras-chave: Situação do mundo real. Modelagem Matemática. Pensamento Matemático.

INTRODUCTION

People developed throughout life different cognitive processes, regarding the activity of acquiring awareness, and to the creative capacity of interpreting and representing the world. Cognitive processes generate mental structures or cognitive structures that lifelong and intellectual development, are adapted and changed.

In this paper, we deal with cognitive processes related to the development of mathematical thinking, as described by Dreyfus (1991) and Tall (1995, 2002). According to these authors, cognitive processes are mobilized when people are involved with mathematical activities. These processes are related to the development of students' capacity to think rationally, to analyze and to visualize. David Tall and Tommy Dreyfus have been dedicated part of their studies to the understanding of these processes in educational environments and considering mainly mathematical learning.

We are interested, particularly, in identifying cognitive processes used by students in mathematical modelling activities, and its relationship for the development of students' mathematical thinking in modelling activities.

The research has as its theoretical approach, on one hand, the arguments of Dreyfus (1991) and Tall (1995, 2002) regarding cognitive processes and mathematical thinking. On the other hand, it is based on the characterizations of mathematical modelling by Blum et al. (2002), English (2003), Crouch & Haines (2004), Lesh et al. (2008), Blum & Borromeo Ferri (2009), Schukajlow et al. (2011), Galbraith (2012) and Almeida & Silva (2012).

Mathematical thinking, according to Tall (1995), may be divided into elementary mathematical thinking and advanced mathematical thinking. Although researches consider that advanced mathematical thinking develops according to the complexity of the contents, associated, generally, to higher education contents, our research is based on the arguments of Rasmussen et al. (2000) that this "kind" of thinking may not be limited only to students on such school level, and that the thinking processes are not related to the complexity of contents, but to procedures and actions required for the execution of the activity.

We use a research's methodology that consists on the analysis of a mathematical modelling activity developed by groups of higher education students. Data were obtained during the development of mathematical modelling activities in a Degree in Mathematics on the discipline of Mathematical Modelling in the Perspective of Mathematics Education. The discipline was taught by the authors of this paper.

MATHEMATICAL MODELLING IN MATHEMATICS EDUCATION

According to Blum et al. (2002), in a mathematical modelling activity it is important to consider problems of reality as a starting point, by setting the activity as something on what:

> The starting point is normally a certain situation in the real world. Simplifying it, structuring it and making it more precise – according to the problem solver's knowledge and interests – leads to the formulation of a problem and to a real model of the situation. […]. If appropriate, real data are collected in order to provide more information about the situation at one's disposal. If possible and adequate, this real model – still a part of the real world in our sense – is mathematised, that is the objects, data, relations and conditions involved in it are translated into mathematics, resulting in a mathematical model of the original situation. Now mathematical methods come into play, and are used to derive mathematical results. These have to be re-translated into the real world, that is interpreted in relation to the original situation. At the same time the problem solver validates the model by checking whether the problem solution obtained by interpreting the mathematical results is appropriate and reasonable for his or her purposes. If need be (and more often than not this is the case in 'really real' problem solving processes), the whole process has to be repeated with a modified or a totally different model. At the end, the obtained solution of the original real world problem is stated and communicated. (BLUM et al., 2002, p.152-153)

In this context, mathematical modelling refers to the observation of a phenomenon, to the construction of a mathematical model that requires the development of mathematical constructs, mathematical concepts and mathematical tools. Mathematics involved in the model "must be made reasonable in two ways, not only in its mathematical "correctness" with regard to the domain in which it is resident, but also in the real-world situation which it represents" (CARREJO; MARSHALL, 2007).

According to Lesh et al. (2008), the modelling activities are placed as a "class of empirical thinking tasks we are designing that heavily utilize concepts and skills and that integrate numerical, algebraic, geometric and statistical/probabilistic thinking in realistic simulations of actual data modelling in the world outside of school" (p.115). Activities such as those require students to express the ways of their thinking in a manner so as that they are capable to record, execute tests and review their conjectures.

According to Schukajlow et al. (2011), it is possible to consider that the procedures executed by students constitute a modelling cycle, in which the steps would be:

> 1) understanding the problem and constructing an individual "situation model"; (2) simplifying and structuring the situation model and thus constructing a "real

model"; (3) mathematising, i.e. translating the real model into a mathematical model; (4) applying mathematical procedures in order to derive a result; (5) interpreting this mathematical result with regard to reality and thus attaining a real result; (6) validating this result with reference to the original situation; if the result is unsatisfactory, the process may start again with step 2; (7) exposing the whole solution process. (p.219)

According to these authors, even if the steps described above are not executed in a linear manner, the seven-step cycle described is both sufficiently detailed to capture the essential cognitive activities taking place in actual modelling processes and sufficiently simple to guide the necessary observations and analyses in a parsimonious way (SCHUKAJLOW et al. 2011, p.220).

Seeking to identify cognitive processes of the students in mathematical modelling activities, we consider what is characterized by Kaiser & Sriraman (2006) as the cognitive perspective for mathematical modelling. According to Kaiser & Sriraman (2006), the focus of this perspective is in the cognitive processes and in the understanding of these processes, as well as in the promotion of mathematical thinking processes using mathematical models as mental images, or emphasizing modelling as an activity that elicits processes such as abstraction and generalization, related to mathematical thinking.

ABOUT MATHEMATICAL THINKING

According to Japiassu & Marcondes (2001), thinking encompasses the effort that people do to reflect upon an object, turning the knowledge into something subjects to learn. Therefore, people activity when they are thinking it is intrinsic to mind, i.e., an intellectual activity, a cognitive activity.

Although it varies from one person to another, each one handles in a particular way with his/her thinking, reflections and understandings. In this sense, thinking can be a simple thing or a hard act that demands hard effort to understanding certain subject.

It is possible understanding thinking as something that generates the knowledge of people in relation with the objects. Thus, through thinking and reflecting the person appropriates something, or understanding something. In this paper, in particular, we refer to the knowledge of the person regarding mathematical concepts and mathematical tools, which can result in the development of mathematical thinking.

According to Sfard (1991), the apparent difficulty that many students reveal regarding learning of mathematics may be related to the features of mathematical thinking.

Some researchers have been devoted to studies on mathematical thinking. Dreyfus (1991; 1999), Rasmussen et al. (2000), Tall (2002), Magajna & Monaghan (2003), Sfard (1991; 2001), Gray et al. (1999), Dubinsky (2002), Domingos (2003), Costa (2002) and Yoon et al. (2011), are some of them. In this paper, we consider essentially, the research of Tall (1995-2002) and Dreyfus (1991) to approach the cognitive theory of mathematical thinking and the cognitive processes related to mathematical thinking.

Mathematical thinking may be characterized as elementary mathematical thinking and advanced mathematical thinking (DOMINGOS, 2003; TALL, 1995, 2002; COSTA, 2002; GRAY et al., 1999; DREYFUS, 1991).

According to Tall (1995), the elementary mathematical thinking starts with the perception of an action on the objects of the external world. It is possible to view such perception and action as a spacial-visual sense toward the verbal-deductive sense. Firstly, we have the perception of objects of the external world in a visual-spacial manner, and then, we apply actions on what we perceive of such object, using the verbal-deductive form.

Regarding to advanced mathematical thinking, Tall (1995) points out that it is characterized by a refinement in language and accuracy in the manner of dealing with mathematics, and it could lead individuals to the use of procedures that we may, in a certain manner, to consider advanced.

Tall (2002) argues that advanced mathematical thinking is developed as individuals come into contact with mathematical concepts and mathematical tools and when they start their mathematical development, as well as when they strengthen their contact/ties with these concepts. Such thinking development starts to require from them abilities that move beyond perceiving and observing, extending to the act of proving, generalizing, forcing subjects to develop more sophisticated manners of thinking in a mathematical manner and relate them to what the author characterizes as advanced mathematical thinking.

To investigate the transit between elementary mathematical thinking and advanced mathematical thinking we must consider the cognitive processes related to mathematical thinking.

COGNITIVE PROCESSES RELATED TO MATHEMATICAL THINKING

Dreyfus (1991) argues that the transition between elementary mathematical thinking and advanced mathematical thinking is influenced by the manner in which students deal with processes of representation and abstraction. In this context, Dreyfus (1991, p.30) believes that advanced mathematical thinking consists in a series of processes that interact among themselves, such as the processes being of representing, visualizing, generalizing, classifying, conjecturing, assuming, analyzing, abstracting and formalizing.

The cognitive processes of students when dealing with mathematical objects, according to Tall (2002), enable the construction of mathematical knowledge and can provide the learning of mathematics.

Both, elementary and advanced mathematical thinking, depend on processes developed by subjects in the development of their activities. Dreyfus (1991) characterizes these as cognitive processes related to the learning of mathematical concepts, and characterizes them as representation and abstraction, and the sub-processes related to those.

The representation process is connected to the act of representing something, and the representation may be symbolic or mental:

> […] A symbolic representation is externally written or spoken, usually with the aim of making communication about the concept easier. A mental representation, on the other hand, refers to internal scheme or frames of reference which a person uses to interact with the external world. It is what occurs in the mind when thinking of that particular part of the external world and may differ from person to person. (DREYFUS, 1991, p.31)

According to Dreyfus (1991), the representation processes may be related to three other processes: visualization, changes of representations and modelling. The process of representing-visualizing is related to concrete artifacts, which we can visualize. The process of change of representations and translation is used when a representation of a certain concept is used to facilitate the understanding of the concept. In such event, the transformation of a representation of a concept turns into another representation of the same concept. The process of modelling, according to the author, consists in seeking a mathematical representation for an object or a non-mathematical situation. This process involves the construction of a mathematical structure, or a theory that incorporates essential features of the object, of the system or of the situation to be described, and it generates a model that we may use to study the behavior of the object, in the situation under modelling.

The process of abstraction relates to, according to Dreyfus (1991), the processes of generalization and synthesis. The author characterizes the process of generalization as deriving or assuming, based on particularities, to identify common expects and expand validity domains. Generalization becomes more important at the rate in which it establishes a result for a large group of events, or at the rate in which it establishes the formulation of a concept. The process of synthesis, on the other hand, implies in synthesizing, combining or composing several manners of forming a whole, an entity.

Dreyfus (1991, p.34) considers that advanced mathematical thinking becomes evident when the person develop the capacity of abstracting. According to the author, if a student develops the ability of making abstractions, consciously, of the mathematical situations, he/she is developing a level of advanced mathematical thinking. To develop this capacity of abstracting may be an important objective in mathematical education. In addition to the capacity of abstracting, the author approaches the articulation of the processes of abstraction and representation, as well as the use of different representations of mathematical objects as denoting the development of mathematical thinking.

In mathematical modelling, under the cognitive perspective it is possible to focus on the cognitive processes of abstraction and generalization. For this purpose it is necessary to focus on the processes of representation, visualization, change of representation or translation during the development of mathematical modelling activities.

With the purpose of presenting a few ideas regarding the mathematical thinking of students when they develop mathematical modelling activities, and in the expectation to identifying and characterizing the cognitive processes of the students in this context, we analyze data collect with students engaged in modelling activities.

The information, data and activity that support our arguments were acquired with students during a higher education class, during a subject of Mathematical Modelling in the Perspective of Mathematics Education. Students developed modelling activities and it is possible to state that the seven steps idealized by Schukajlow et al. (2011) were executed by them.

Considering the complexity of studying the development of mathematical thinking, the results presented in this paper refer to one student, named as student A. He participated in all classes in which we developed modelling activities, during a school year, and he also answered to questionnaires and conceded an interview carried out by one of the authors of the text. Although we developed some activities in the classes, we chose the description of an activity and of one student that participated in its development.

THE MATHEMATICAL MODELLING ACTIVITY: ANALYSIS OF THE CONCENTRATION OF NICOTINE IN THE BODY OF A SMOKER

The activity we analyze was developed by two groups of students, coming from a problem-situation chosen by them. The theme they investigated is Concentration of nicotine in smokers. Taking into account the specificities of each student in relation thinking processes, we analyzed each student individually and present here the data on student A. The activity we refer aims to analyze the concentration of nicotine in the body of a smoker, considering the consumption of different quantities of cigarettes.

The data that the students used in the activity were obtained from a specialized website of the World Health Organization (WHO), Masters Dissertations, articles from the Internet and on the book Nicotine, Universal Drug (Nicotina Droga Universal, in Portuguese) by José Rosemberg, doctor and president of the committee coordinator of tobacco control in Brazil.

Essential information that the students obtained in their researches regarding the problem is that the half-life of nicotine in the body is of two hours, and that a single cigarette contains approximately eight milligrams of the nicotine. Considering that there are smokers who consume a different daily number of cigarettes, students decided to study two situations.

Situation 1: How much time does it take for the organism to eliminate nicotine when a person smokes a single cigarette?

Situation 2: For a person who smokes approximately one pack of cigarettes in a day how is the concentration of nicotine in their organisms throughout time?

Figure 1 presents a summary of the resolution of students to the first situation.

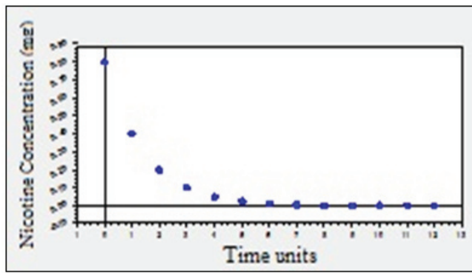
FIGURE 1 – Mathematical approach to situation 1.

1st Half-life of a cigarette according to the data

t (hours)	n	C_n (mg)
0	0	0,8
2	1	0,4
4	2	0,2
6	3	0,1
8	4	0,05
10	5	0,025
12	6	0,0125
14	7	0,00625
16	8	0,003125
18	9	0,0015625
20	10	0,00078125
22	11	0,000390625
24	12	0,000195312

Table 1

2nd Graphic representation of the data of Table 1



"through the graphic, we observed that the points we obtained behave similarly to the graphic of a decreasing exponential function" (Record of the students during the presentation of the activity).

3rd Table used to obtain the model

t (hours)	n	C_n (mg)	$C_{n+1} - C_n$	$\frac{C_{n+1} - C_n}{C_n}$
0	0	0,8	- 0,4	- 0,5
2	1	0,4	- 0,2	- 0,5
4	2	0,2	- 0,1	- 0,5
6	3	0,1	- 0,05	- 0,5
8	4	0,05	- 0,025	- 0,5
10	5	0,025	- 0,0125	- 0,5
12	6	0,0125	- 0,00625	- 0,5
14	7	0,00625	- 0,003125	- 0,5
16	8	0,003125	-0,0015625	- 0,5
18	9	0,0015625	-0,00078125	- 0,5
20	10	0,00078125	-0,000390625	- 0,5
22	11	0,000390625	-0,000195312	- 0,5
24	12	0,000195312		

Table 2

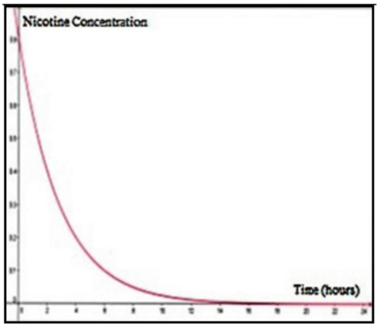
4th Acquisition of a mathematical model

$$C(t) = 0,8e^{-0,3465736 32t}$$

5th Validation of the model

t (hours)	C_n (mg)	$C(t) = 0,8e^{-0,3465736 32t}$
0	0,8	0,8
2	0,4	0,399999966
4	0,2	0,199999966
6	0,1	0,099999974
8	0,05	0,049999983
10	0,025	0,024999989
12	0,0125	0,012499993
14	0,00625	0,006249996
16	0,003125	0,003124997
18	0,0015625	0,001562498
20	0,00078125	0,000781249
22	0,000390625	0,000390624
24	0,000195312	0,000195312

Table 4 Model Validation

6th Graphic representation of the model $C(t) = 0,8e^{-0,346573632t}$ 	**7th Resuming the problem-situation and conclusion of students** *Then, when we obtained the model, we raised a new question: In how long will the nicotine absorbed be eliminated by the organism? Using our model, we found that, for the concentration in only one cigarette, we could not equal to zero, then we equaled it to a very low amount, to see how long would the nicotine remain in the organism. By solving these calculations, we found that the time, in days, will be of 2 days, 17 hours, 47 minutes and 41 seconds for the elimination of the nicotine from the organism from the time one smokes it.* (Records of students during the presentation)

Source: Students written data.

For the second situation, students had to consider different hypotheses and other mathematical models, as the summarized description in Figure 2.

FIGURE 2 – Problem Resolution of the 2th situation.

First: Definition of variables

t – time (hours); / C(t) – Concentration of nicotine.

Second: Elaboration of a figure on the nicotine concentration in the body

Nicotine concentration in the body

$$C(0) = 0{,}8e^{0} = 0{,}8$$

$$C(0) + C(1) = 0{,}8 + 0{,}8e^{-0{,}346573632.1} = 0{,}8.\left(1 + e^{-0{,}346573632}\right)$$

$$C(0) + C(1) + C(2) = 0{,}8.\left(1 + e^{-0{,}346573632.1}\right) + 0{,}8e^{-0{,}346573632.2} = \\ = 0{,}8.\left(1 + e^{-0{,}346573632.1} + e^{-0{,}346573632.2}\right)$$

$$C(0) + C(1) + C(2) + C(3) = \\ = 0{,}8.\left(1 + e^{-0{,}346573632.1} + e^{-0{,}346573632.2} + e^{-0{,}346573632.3}\right)$$

$$\vdots$$

$$C(0) + C(1) + \cdots + C(m) = \\ = 0{,}8.\underbrace{\left(1 + e^{-0{,}346573632.1} + e^{-0{,}346573632.2} + \cdots + e^{-0{,}346573632.(m-1)} + e^{-0{,}346573632.m}\right)}_{\text{Sum of a finite geometric progression}}$$

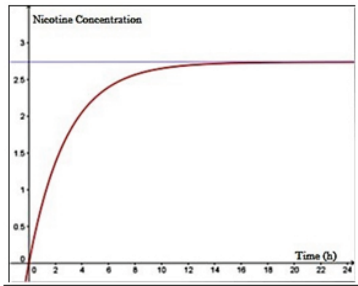
Third- Construction of a mathematical model using mathematical concepts

$$C(t) = 0{,}8 \cdot \frac{1 - e^{-0{,}346573632t}}{1 - e^{-0{,}346573632}}$$

Fourth-Conclusion

"even if a person smoked infinite cigarettes a day, the tendency of the content of nicotine in the organism is to stabilize due to its exponential decay."

(Report of student during the activity presentation)

Fifth-Calculation of the concentration of nicotine limit in the organism $\lim_{t \to \infty} C(t) = 2{,}731370575$	**Graphic Representation** 	*"We may realize that the concentration of nicotine throughout time in the organism of a smoker who smokes, regularly, one cigarette every hour, tends to stabilize and not to surpass the mark of 2,731370575 mg.* *(Report of student)*

Source: Students written data.

To finalize the activity, student A emphasizes that:

> According to the elaborated models, it is clear the nicotine leaves an organism in a time that we can consider short, and it is also noticeable that this is the reason for it to cause a dependency. Dye to this dependency, smokers consume several cigarettes a day, only to maintain its addiction to nicotine, and they end up absorbing the other substances dangerous to health, with several of them being highly carcinogens, as is the case with cadmium, of which concentration is much lower than nicotine's per cigarette, but, on the other hand, it has a longer half-life (Report produced by student A).

MATHEMATICAL THINKING OF STUDENT A IN THE ACTIVITY

The student A participated of all sessions during the activity development, in the classroom and outside it. In the interview after the development of the activity, he evaluated as positive the experience of the group, stating that:

> Our work was excellent, modesty aside, and we had an excellent group experience. Everyone contributed with the development of the work. When someone did not remember a concept used, another one reminded and explained it to the rest of the group. It was an excellent experience [...] it made us excited to put in practice the concepts we learned throughout the course. This is where concept application must be seen. Sometimes we think that it is difficult to apply them, since we need very sophisticated concepts, but, in reality, it is possible to execute a very good job with the little we know. (Student A)

Mathematical concepts used during the modelling activity (exponential function, mathematical recurrence, sum of the terms of a finite geometrical progression, limit of a function) were already known by the student. In this sense, the difficulty of student A, probably, is not in the concept itself, but it is in the relationship between the concepts to the real world situation, the study of nicotine concentration in the body. Made the mathematical concept and tools reasonable in two ways – in its mathematical "correctness" with regard to the domain in which it is resident, and also in the real world situation which it represents – it was the biggest challenge for the student. It seems that we can thereby evidence an idea of mathematical thinking in which the translation between languages and modelling, as Dreyfus (1991) characterizes, are necessary cognitive processes.

The student described the development of the activity through certain stages, signaling his comprehension regarding the necessary procedures for the development of a modelling activity. In this sense, mathematization, interpretation and validation, are features of the activity that Blum et al. (2002) identifies, which the student also indicates in his interview after the activity development:

> 1st comes the choice of the subject. This is the hardest and one of the most important stages. In this stage, we do not know the path that the work can take, and it may come to nothing, or require much more advanced knowledge than those available in our course. The 2nd stage is to research everything about the subject, every possible information, from reliable sources, of course. In this stage, we have the risk of not finding much information about the subject. In this stage, it is necessary to exhaust all possible information. The 3rd stage is the development of the problem. This stage depends a lot on the information and mathematics available to us. The 4th is the resolution of the problem. At least we hope to resolve it. In this stage, we take the information we researched with the mathematical concepts available in our arsenal. The 5th stage is the validation of the results we obtained. In this stage, we verify whether or not we managed to reach our objective, or at least come closer to it. The 6th stage is conclusion. (Student A)

This stages identification, and the procedures and results obtained in each of them, signals an exercise of thinking that now seems to be anchored in isolated cognitive processes, such as representation and transition between representations, and, at other times, however, reflects more sophisticated processes, such as abstraction, synthesis and generalization, as well as the interaction among them, in tune with what Dreyfus (1991) identifies as advanced mathematical thinking.

We observe that the student, to study the concentration of nicotine in the organism of a smoker, goes through a path that consider different possibilities for the consumption of cigarettes, and is capable of realizing that, also from a mathematical point of view, these possibilities present distinct properties and that, for such reason, it is necessary to model them in separate.

Hence, even though the mathematical objects activated by the student may seem elementary for a student of a Degree in Mathematics, the articulation between these objects in the context of the modelling is an indication of cognitive processes related to the advanced mathematical thinking.

An example of this can be identified on the first situation when the student makes an analysis on the data (as indicated in item 3, figure 1) to conclude that, from $\frac{C_{n+1} - C_n}{C_n} = 0{,}5$ it may establish the hypothesis for the problem that leads to $C_{n+1} - C_n = 0{,}5C_n$ and that, considering the continuous variable "t" for the time, it leads the model to $C(t) = 0{,}8\, e^{-0{,}3465736 32t}$ to indicate the quantity of nicotine in an instant "t". The Student A, in his interview, made attempts to explain this formulation:

> We made that equation linear, the one we had to be able to use the method of the minimum squares, to be able to find the line that adjusts to those points, and ... after making it linear, we used the method of minimum squares to find the amount of a and b. Then, we used the method of minimum squares here, found the straight line, made the calculations, and then went back to where we made it linear, and built that table with the concentration of nicotine. Then, we placed the data in a table and made the comparison with those amounts that we already had, and the results were approximate, amounts that we found with that function. Here we have the graphic of this decrease, of the function that we found through the method of minimum squares. (Student A)

In item 7 of figure 1, the student validates the model by checking whether the problem solution obtained by interpreting the mathematical results is appropriate and reasonable for his purposes:

> […] when we obtained the model, we raised a new question: In how long will the nicotine absorbed be eliminated by the organism? Using our model, we found that, for the concentration in only one cigarette, we could not equal to zero, then we equaled it to a very low amount, to see how long would the nicotine remain in the organism. By solving these calculations, we found that the time, in days, will be of 2 days, 17 hours, 47 minutes and 41 seconds for the elimination of the nicotine from the organism from the time one smokes it. (Records of students during the presentation)

In this event, the cognitive processes of analysis and synthesis seem to support the statement of the student regarding the concentration of nicotine in an organism when people smoke only a cigarette. According to Dreyfus (1991) and Tall (2002), cognitive processes as synthesis when interacting with other mathematical thinking processes gives us indications of advanced mathematical thinking.

The problem investigated in the situation varies exactly to consider a more common problem, or, in other words, it would deal with a situation in which several people find themselves in current times: smoking a pack of cigarettes per day.

This situation is probably closer to reality than the previous. Besides that, the mathematical approach of this situation is more sophisticated than the previous one.

According to Bassanezi (2002), how much closer to reality the student wants to arrive, the more he/she needs to invest in the mathematization of the situation. This seems to be revealed in the statement of student A, when he questioned regarding the mathematical construction of item 2 of figure 2 states that:

> We needed to build another model to analyze the concentration of people who consume several cigarettes, in a period interval that we can call hours, or even less. In our case, we considered the interval from hour to hour, which will also come into the next model, in this second situation. (Student A)

In this context, we can assume that the cognitive processes related to abstraction and generalization mediated the mathematical thinking of the student in the mathematical formulation of the model through a mathematical recurrence related to the concept of the sum of a geometrical progression terms.

The conclusion of the stabilization of the quantity of nicotine in the organism of the smoker indicated in item 5 of figure 2 came with basis on several mathematical procedures by the student, such as the limit calculation, and construction of the model graph obtained. However, the student trust in the answer started to become clearer based on the interaction between results. Hence, representation, translation between representations, and synthesis, seem to be the cognitive processes articulated for the presentation of the answer by the student for this situation, which, based on arguments by Dreyfus (1991) and Tall (2002), are signs of advanced mathematical thinking.

In this manner, we may consider student A actions during the activity as involving advanced mathematical thinking, according to the theory of mathematical thinking that David Tall established, and the cognitive processes of mathematical thinking that Dreyfus (1991) approaches.

FINAL REMARKS

We present in this paper a mathematical modelling activity developed by a group of students, evincing the development of a particular student – student A. The analysis we presented aims at identifying cognitive processes used by one of them and its relationship to the development of the mathematical thinking.

Based on this analysis we may consider that, in this specific mathematical modelling activity, there are interactions between the cognitive processes that seem to reveal nuances of elementary mathematical thinking, on one hand, and on another, nuances of advanced mathematical thinking. Neither one nor another prevails, since these types of thoughts take place when required, or through the cognitive structure of the student, or by the developed activity.

According to the analysis, the student starts with procedures that denote the occurrence of elementary mathematical thinking. However, along the modelling process, he used mathematical tools, procedures and concepts that allow them to infer on the cognitive processes and their articulations, mobilized during the modelling steps that indicate advanced mathematical thinking.

We observed the cognitive processes that Dreyfus (1991) describes in the development of the activity, with the observation of more use of the processes of abstraction in the activity. Besides that, it is possible to assume the steps from one mathematical modelling activity identifying the cognitive processes that the student mobilized when going through these steps, in tune with the arguments of Schukajlow et al. (2011) and Blum et al. (2002), regarding mathematical modelling activities.

In general terms, we can argument that mathematical modelling activity has the potential to unleash cognitive processes related to elementary as well as advanced mathematical thinking.

The analysis that we present for student A reveals processes related to elementary thinking as well as more sophisticated cognitive processes, and in their interaction, providing signs of an advanced mathematical thinking. The mathematizing of the nicotine concentration in the body of a smoker, takes place, basically, mathematical objects as recurrence and geometrical progressions, which were already known by the student A. However, the sophistication of the cognitive processes and of thought was necessary for the articulation between the concepts, and in its use for the approach of the problem-situation.

Hence, when dealing with mathematical modelling, the characterization of mathematical thinking occurs in the interaction of elementary mathematical thinking with advanced mathematical thinking processes. And this relationship is associated with the relationship between the situation of the real world and mathematical content, and specifically with the actions with which the student engages in the activity.

REFERENCES

ALMEIDA, L. M. W.; SILVA, K. A. P. da. Semiotic and cognitive actions of students in mathematical modeling activities: A look on the inference manners. *Ciência e Educação* (UNESP), v.18, n.3, 2012, p.623-642.

BASSANEZI, R. *Education-learning with mathematical modeling*: A new strategy. Ed. Contexto, 2002.

BLUM, W, et al. ICMI Study 14: applications and modelling in mathematical education – discussion document. *Educational Studies in Mathematics*, v.51, n.1-2, p.149-171, 2002.

BLUM, W., BORROMEO FERRI, R. Mathematical Modeling: Can it be taught and learned? *Journal of Mathematical Modeling and Application* v.1, n.1, p.45-58, 2009.

CARREJO, D. J.; MARSHALL, J. What is Mathematical Modelling? Exploring Prospective Teachers' Use of Experiments to Connect Mathematics to the Study of Motion. *Mathematics Education Research Journal*, v.19, n.1, p.45-76, 2007.
COSTA, C. Mental processes related to advanced mathematical thought Visualization. In: XI MEETING OF INVESTIGATION IN MATHEMATICAL EDUCATION SPCE – Work group 4 – Development of a mathematical advanced rational thought. *Proceedings…* Coimbra, 2002.
CROUCH, R.; HAINES, C. Mathematical modeling: Transitions between the real world and the mathematical model. *International Journal of Mathematical Education in Science and Technology*. Taylor and Francis, v.35, n.2, p.197-206, 2004.
DOMINGOS, A. *Comprehension of advanced mathematical concepts* – mathematics in higher education. (PhD thesis) – College of Sciences and Technologies of Universidade Nova Lisboa). Lisboa, 2003.
DREYFUS, T. Advanced mathematical thinking processes. In: TALL, David (Org.). *Advanced mathematical thinking*. p.25-41. Dordrecht: Kluwer, 1991.
DREYFUS, T. Why Johnny can't prove. *Educational Studies in Mathematics*, v.38, p.85-109, 1999.
DUBINSKY, E. Reflective abstraction in Advanced Mathematical Thinking. In: TALL, David (Org.). *Advanced mathematical thinking*. p.95-123. Dordrecht: Kluwer, 2002.
ENGLISH, L, D. Reconciling theory, research, and practice: a models and modeling perspective. *Educational Studies in Mathematics*, v.54, p.225-248, 2003.
GALBRAITH, P. Models of Modeling: Genres, purposes or perspectives. *Journal of Mathematical Modeling and Application*, v.1, n.5, p.3-16, 2012.
GRAY, E.; PINTO, M.; PITTA, D.; TALL, D. Knowledge construction and diverging thinking in elementary & advanced mathematics. *Educational Studies in Mathematics*, v.38, n.1-3, p.111-133, 1999.
JAPIASSÚ, H.; MARCONDES, D. *Basic Philosophy Dictionary*. Rio de Janeiro: Zahar, 2001.
KAISER, G.; SRIRAMAN, B. A global survey of international perspectives on modeling in mathematics education. *The International Journal on Mathematics Education*, v.38, n.3, p.302-310, 2006.
LESH. R.; MIDDLETON, J. A.; CAYLOS, E.; GUPTA, S. A science need: Designing tasks to engage students in modeling complex data. *Educational Studies in Mathematics*. v.68, p.113-130, 2008.
MAGAJNA, Z.; MONAGHAN, J. Advanced Mathematical Thinking in a technological workplace. *Educational Studies in Mathematics*, v.52, p.101-122, 2003.
RASMUSSEN, C. L.; ZANDIEH, M.; KING, K. D.; TEPPO, A. Advanced mathematical thinking: aspects of students' mathematical activity. In: 22 ANNUAL MEETING OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION – North American Chapter. *Proceedings…* Tucson: AZ, 2000.
SCHUKAJLOW, S.; LEISS, D.; PEKRUN, R.; BLUM, W.; MÜLLER, M.; MESSNER, R. Teaching methods for modeling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. *Educational Studies in Mathematics*, v.79, p.215-237, 2011.

SFARD, A. On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, v.22, p.1-36, 1991.

SFARD, A. There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, v.46, p.13-57, 2001.

TALL, D. *Advanced Mathematical Thinking*. Mathematics Education Library. Kluwer Academic Publishers. A. J. Bishop, Cambridge, U. K, 2002.

TALL, D. Cognitive growth in elementary and advanced mathematical thinking. In: 19TH INTERNATIONAL CONFERENCE FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION. *Proceedings*… v.1, p.61-75. Recife, Brazil, 1995.

YOON, C.; THOMAS, M. O. J.; DREYFUS, T. Grounded blends and mathematical gesture spaces: Developing mathematical understandings via gestures. *Educational Studies in Mathematics*, v.78, p.371-393, 2011.