Teacher's Actions to Promote Students' Justifications

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ABSTRACT

Justification is a mathematical reasoning process that relies on concepts, properties or mathematical ideas and, in certain situations, particular cases, being a fundamental part of the proof. The teacher needs to promote justification in the classroom, as it is essential to the development of students' mathematical knowledge. This study aims to understand how a set of design principles regarding tasks and teacher's actions contributes to enhance students' justifications in whole-class mathematical discussions and to understand what kinds of justifications emerge in those discussions. The intervention, part of a design-based research, occurs in a grade 7 class of an experienced teacher, in nine classes about linear equations. The data collection includes classroom observations (video and audio recorded) and a logbook. Data analysis considers a set of design principles, a conceptual framework for teacher actions, and a conceptual framework for student justifications. The results show that certain sequences of teacher actions based on the design principles allow students to present quite complete justifications based on logical coherence and mathematical aspects of the situation.

Keywords: Mathematical reasoning. Justification. Teacher's actions. Design-based research.

Ações do Professor para Promover Justificações dos Alunos

RESUMO

A justificação é um processo de raciocínio que se baseia em conceitos, propriedades ou ideias matemáticas e, em algumas situações, em casos particulares, sendo um elemento fundamental da demonstração. O professor deve promover a justificação na sala de aula pois esta é essencial para o desenvolvimento do conhecimento matemático dos alunos. Este estudo tem por objetivo compreender de que modo um conjunto de princípios de design referentes a tarefas e ações do professor contribui para promover as justificações dos alunos em momentos de discussão coletiva e compreender que tipos de justificação surgem dessas discussões. A intervenção, parte de uma investigação baseada em design, ocorre numa turma de 7.º ano de uma professora experiente, em nove aulas sobre equações lineares. A recolha de dados decorre de observações de sala de aula (vídeo e áudio gravados) e de um diário de bordo. A análise de dados considera um conjunto de princípios de design, um quadro conceptual referente às ações do professor e um quadro conceptual referente às justificações dos alunos. Os resultados mostram que determinadas sequências de ações do professor que se baseiam nos princípios de design

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permitem aos alunos apresentar justificações bastante completas baseadas na coerência lógica e em aspetos matemáticos da situação.
Palavras-chave: Raciocínio matemático. Justificação. Ações do professor. Investigação baseada em design.

Mathematical reasoning and mathematical proof are fundamental aspects of mathematics. Justification is structural to proof and proving and, hence, is essential to the development of students' mathematical knowledge. Among other mathematical reasoning processes, justifying allows students to envision mathematics as a logical, inter-related and coherent subject. For instance, justifying procedures can help students overcome using them with none or little understanding. With Brousseau and Gibel (2005), we consider mathematical reasoning as making justified inferences, that is, using known mathematical information to obtain new information. This may be done inductively, deductively or abductively (Lannin, Ellis, & Elliot, 2011; Pólya, 1954; Rivera & Becker, 2009). Justifying is, thus, part of the mathematical reasoning that also involves processes as formulating questions and solving strategies, and formulating and testing generalizations and other conjectures (Lannin et al., 2011).

To promote students' mathematical reasoning in the classroom demands the establishment of challenging learning environments that go beyond proposing exercises to solve with well-known procedures. Thus, understanding how teachers can organize those learning environments is essential to promote students' mathematical reasoning. Aiming to understand how teachers can help students to engage in mathematical reasoning, we carried a design-based research (Cobb, Jackson, & Dunlap, 2016). This research addresses mathematical whole-class discussions, unleashed by exploratory tasks, as privileged moments to promote students' mathematical reasoning. Focusing on justification, this article aims to understand how a set of design principles regarding tasks and teacher's actions promote students' justifications in whole-class mathematical discussions and to understand what kinds of justifications emerge in those discussions.

STUDENTS' JUSTIFICATIONS

In the classroom, justifying is a process that rarely emerges spontaneously. Frequently, students accept as valid conjectures and generalizations, without feeling any need to test or justify them (Harel & Sowder, 2007). In addition, in several scenarios, students focus mainly on what is familiar or on ideas that they recall, focusing little or no attention in the mathematical properties or concepts involved (Lithner, 2008). However, justification is a mathematical process fundamental for students' mathematical learning. Justification leads students to make connections among mathematical concepts, representations and procedures, to present arguments to support claims and conjectures, to solve problems, and to develop new mathematical ideas (Brodie, 2010). In order to justify mathematically, students produce statements to convince themselves and others that a claim is true or false (Harel & Rabin, 2010), but the assumptions of such statements need to be of mathematical nature. Justification is a means to sustain claims based on mathematical properties, procedures and ideas (Lannin et al., 2011). As Lannin et al. (2011) indicate, in some situations, justifications based on particular cases can be an important stage when students do not have the tools to justify in a deductive way. As involving students in formal justifications too early in their schooling might not be suitable to their developmental level (Conner, Singletary, Smith, Wagner, & Francisco, 2014), teachers can consider students' empirical justifications as acceptable according to their knowledge. Moreover, justifications based on particular cases that aim to represent a broader class of mathematical objects (Sowder & Harel, 1998), may be regarded as proper justifications according to the knowledge of the class. However, teachers should discuss with students the mathematical validity of such justifications, as students' empirical explorations are not equivalent or a substitute for mathematical proofs (Stylianides & Stylianides, 2009).

Despite the goal of having students' justifying based on mathematical assumptions, not all justifications that emerge in the classroom are of a purely mathematical nature. In fact, justifications can occur at different levels of formality and complexity. In addition, a student's justification, at any level of formality or complexity, may be correct, partially correct or incorrect. It is essential that students understand what validates a justification and that they reject justifications based on authority, perception or common sense (Lannin et al., 2011).

By being a broad concept, justification comprises classifications of various kinds and different grain sizes. A lens to provide an understanding of students' justifications regards its formality and complexity levels (Figure 1). Brousseau and Gibel (2005) propose three different levels regarding the formality of a justification: Level A – a justification that is not formally presented, but that can be associated with student's actions as a model of his/her action; Level B – a formal justification, however incomplete or with inferences based only implicitly in elements of the situation or in what is considered as shared knowledge; Level C – a formal justification based in a sequence of related inferences, with explicit reference to the situation or what is considered as shared knowledge. The concept of formal justification referred to in these three levels concerns what is considered to be formal in a concrete situation, namely according to students' grade level and their knowledge which is not necessarily the usual mathematical idea of formal justification regarding mathematical proof (Stylianides, 2007). As students advance in their schooling, formal justifications should progressively be more formal from a mathematical standpoint, often becoming equivalent to proof or significant parts of proofs.

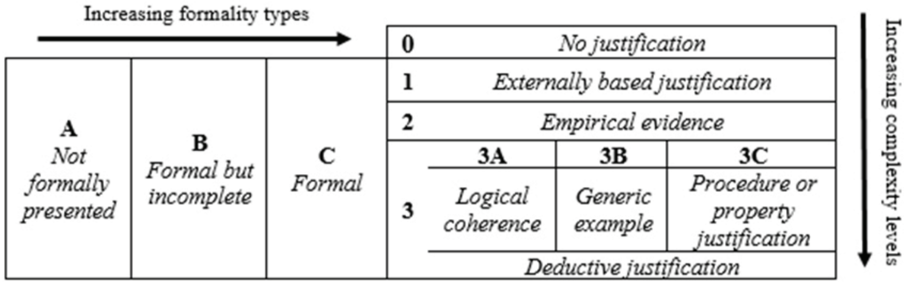

Figure 1. Justification levels of formality and complexity

The work of Balacheff (1988), Sowder and Harel (1998), Lannin (2005) and Carraher, Martinez and Schliemann (2008) on the classification of justifications may be summarized by considering four different increasing levels of complexity: Level 0 – *no justification*, if students' answers do not include a justification; Level 1 – *externally based justification,* if students' justifications rely on someone else or in reference materials; Level 2 – *empirical evidence*, if a justification is based on particular examples; Level 3 – *deductive justification,* if a justification has a deductive nature. Within Level 3 justifications, it is possible to distinguish among Level 3A – *logical coherence*, if a justification is based on logical principles; Level 3B – *generic example*, if a justification is deductive, but stated regarding a particular example; and Level 3C– *procedure or property justification*, if a justification is based on deductive arguments that are independent of particular cases or examples. Levels 3A, 3B and 3C are varieties of deductive justifications, all the same level of complexity.

Level 1 justifications, based on external sources, can be based on authority or rituals or can be symbolic (Sowder & Harel, 1998). In a justification based on authority, a student relies on a textbook, in a teacher statement or even in a more knowledgeable student. A justification based on rituals occurs when students regard only the structure of the argument and not its content. An example is a justification based on a well-known procedure, such as the division algorithm. A justification based on symbolic processes is the one that might lead students to consider mathematical symbols as independent from any meaning or relation with a specific situation. In this justification, students can write, for instance, $4x + 2 = 6$ as equivalent to $4x = 6 - 2 = 4 : 4 = 1$.

Regarding justifications based on empirical evidence (Level 2), it is possible to consider perceptual justifications, justifications based on examples (Sowder & Harel, 1998) and justifications based on crucial examples (Balacheff, 1988). Perceptual justifications are based on a perception of the situation, often based on diagrams or drawings. Justifications based on examples rely on particular cases of a situation. Balacheff (1988) designates these justifications as naïve empiricism and considers them as an obstacle to generalization. With these justifications, students often validate a mathematical generalization on the basis of a single naïve experiment (Stylianides & Stylianides, 2009) or state a generalization by justifying that it works for all the cases they tested (Knuth, Chopin, & Bieda, 2009). In justifications based on crucial examples, examples are selected attending to the hypothesis that is settled, the strategy of choosing examples aims to obtain a clearly distinct result that leads to exclude all the hypothesis but one (Balacheff, 1988). In these justifications, a crucial experiment may validate a generalization (Stylianides & Stylianides, 2009) that is settled by the characteristics or the known facts of the situation.

Deductive justifications (Level 3) are either non-empirical or consider an example not as a particular case, but as a representative of a class of objects. Justifications of this level of complexity are often part of a proof and proving and can be subdivided into three distinct levels mentioned above as levels 3A, 3B and 3C. Presenting justifications at level 3 can be a signal of recognizing empirical arguments as insecure methods for validating a mathematical statement (Stylianides & Stylianides, 2009).

Justifications based on logical coherence (Level 3A), also denominated analytical justifications of axiomatic nature (Sowder & Harel, 1998), consider mathematics as a body of knowledge that may be organized in a way that new results are logical consequences of previous results. These justifications are based on logical principles rather than mathematical computations (Schliemann et al., 2003). Generic example justifications (Level 3B), also referred as analytical justifications of transformational nature (Sowder & Harel, 1998), focus on general aspects of a particular situation and may involve other reasoning processes as a generalization. Procedure or property justifications (Level 3C), aim to make explicit why a statement is valid, either by operations or transformations of an object that is considered as a representative of a class of objects (Balacheff, 1988), or by relying on mathematical properties, definitions, assumptions and theorems (Bergqvist, 2005).

TASKS AND TEACHER'S ACTIONS THAT ENHANCE JUSTIFICATION

Students learn mathematical reasoning by reasoning and by analyzing their and others' mathematical reasoning (Ponte & Sousa, 2010). Therefore, it is necessary to promote situations that require students to justify their answers. When the students are explicitly engaged in presenting justifications, they develop a broader understanding of the mathematical aspects of the situations (Kosko, Rougee, & Herbst, 2014). Moreover, probing the students for justification stimulates them to re-examine their solving processes and to offer more adequate justifications (Martino & Maher, 1999).

A central aspect to enhance students' justifications is to propose suitable mathematical tasks. Therefore, it is important to understand the nature of tasks, how the students get involved in those tasks and the interactions that may emerge in the classroom (Brodie, 2010). Different research studies (e.g., Francisco & Maher, 2011) highlight problems and exploratory tasks as particularly indicated to enhance students' mathematical reasoning. However, it is not mandatory or desirable that all tasks include highly challenging questions (Brodie, 2010). Too many challenging questions might be unsuitable due to time limitations and lead students to lose their interest. The structure and level of challenge of a task must consider the students who will solve it. However, by themselves, tasks are not enough to develop students' mathematical reasoning. Teacher's actions are equally central to engage students in situations that enhance their mathematical reasoning processes, such as justification.

An important moment to support the development of students' justifications are whole-class discussions in which the teacher prompts students to share their thoughts. In these discussions, the teacher questions the students to describe or explain their mathematical reasoning, leading them to understand the mathematics involved (Kosko, Rougee, & Herbst, 2014). A model to analyze teacher's actions in whole-class discussions is proposed by Ponte, Mata-Pereira, and Quaresma (2013) (Figure 2). This model considers actions related to mathematical processes such as justifications, namely, inviting actions, informing/suggesting actions, supporting/guiding actions and challenging actions as well as actions related to classroom management.

Inviting actions initiate a whole class discussion or a segment of discussion, with the teacher prompting the students to participate and share their solving processes. During the discussion, the teacher calls upon the other three kinds of actions that are central to support students' learning. With informing/suggesting actions, the teacher makes information available to the students or validates their statements, while with supporting/guiding actions the teacher leads the students to explain their thinking or to move forward in their thinking. With challenging actions, the students are encouraged to go beyond the knowledge previously presented. In these three kinds of actions central in whole-class discussions, Ponte et al. (2013) also consider different mathematical processes that are involved, not necessarily disjoint: (i) representing, that includes providing, using or changing a representation, revoicing, and making procedures, (ii) interpreting, that includes giving meaning to the wording of a question or of an idea and making connections, (iii) reasoning, that includes raising questions about a statement or a justification, generalize a procedure, a concept or a property, justify and present arguments, and (iv) evaluating, a method or solving process and comparing different methods. This model relates teacher's actions while conducting whole-class discussions with the mathematical processes involved. Despite being part of the reasoning, justifying is often associated with all the other mathematical processes considered in this model.

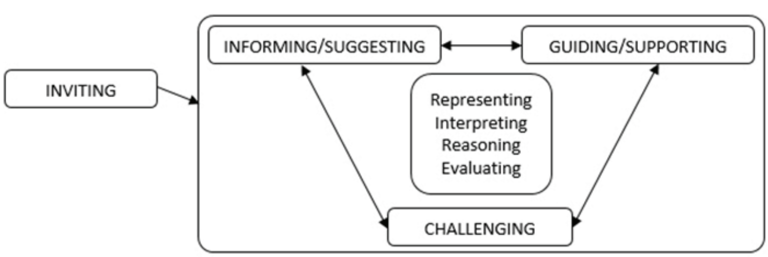

Figure 2. Teacher's actions in the whole-class discussion (adapted from Ponte et al., 2013)

Regarding teacher's specific actions that enhance justification, Bell (2011) suggests that the teacher should help the students to give meaning to justifications, ask for alternative justifications, emphasize what validates a justification and lead the students to explain "why". In addition, it is relevant that the teacher encourages the students to share ideas and reasoning, and look forward to considering students' invalid or partial contributions and broaden valid contributions (Brodie, 2010). Harel and Rabin (2010) highlight also actions that aim to enhance justifications that are not supported by authority leading students to discuss and solve disagreements, namely, evaluating students' mathematical reasoning or asking them to evaluate their colleagues' reasoning, and presenting deductive justifications to students.

RESEARCH METHODOLOGY

Research design and design principles

The research reported in this article is part of a broader design-based research (Cobb et al., 2016) that aims to develop a local theory about how to enhance students' mathematical reasoning in the classroom. In this research, a set of design principles, i.e., heuristics that structure an intervention, are defined with a particular focus on tasks and teachers' actions to enhance students' mathematical reasoning. Such design principles were refined considering the continued literature review and the previous cycles of intervention. Defining design principles, besides structuring the intervention, brings together the often-contrasting understandings of what is considered by teachers and by researchers as being students' mathematical reasoning and the ways to enhance it (Kosko, Rougee, & Herbst, 2014).

This article concerns the third cycle of intervention on linear equations, after a first cycle with lessons about sequences and a second cycle addressing linear equations. We focus on the design principles that directly relate to students' justifications. Regarding task design, a principle states that tasks should include questions that ask for justifications of answers or of solving processes. Regarding teacher's actions, the design principles state that the teacher should propose situations that lead students to (a) justify and present alternative justifications; (b) identify valid and invalid justifications, indicating why; and (c) share ideas, namely considering and valuing invalid or partially valid contributions, deconstructing, complementing or clarifying them.

Intervention and participants

The third cycle of intervention included nine lessons in a grade 7 class with 27 students. A detailed plan of each lesson was prepared attending to the design principles, namely with tasks designed to enhance students' mathematical reasoning and considering possible teacher actions. Each lesson plan was proposed by the first author and discussed in detail with the teacher, who made all the changes and adjustments she deemed necessary, taking into account the characteristics of the class and the available means. We invited the participant teacher given her experience and her availability to consider changes in her practice. All participants in the study (teacher and students) are volunteers, were informed about the characteristics of the research, agreed to participate and their anonymity is assured by the use of fictitious names.

Data collection and analysis

We present illustrative episodes of justifications of different kinds that emerge in whole-class discussions. The lessons of such episodes were directly observed, video and audio recorded, and led to written field notes. Data analysis focuses on the justification-related design principles about tasks and teacher's actions in whole-class discussions based on the presented model (Figure 2) and also on students' justifications regarding levels of formality and complexity (Figure 1). In the next section, we present several situations by describing the tasks that prompted the whole-class discussions and the context in which such tasks were proposed to students, and then illustrate and analyze whole-class discussion episodes.

USING MATHEMATICAL PROPERTIES TO JUSTIFY

Task and context

The task that prompt the whole-class discussion reported in this segment aimed to introduce the process of solving equations based on the addition principle of equality. This task was proposed for the class to be solved collectively and its first question required to convert into symbolic language an equation represented in a twin-pan balance with sugar packages and weights (Figure 3). To this particular question, the lesson plan did not foresee a particular justification but indicated that the whole-class discussion should consider the design principles formulated in the intervention design.

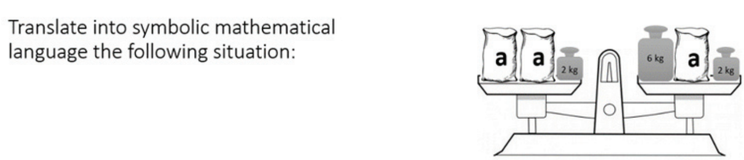

Figure 3. The first question of twin-pan balance task.

This task was proposed in the second lesson of the intervention. The first lesson addressed the concepts of equation, members and terms of an equation, and solution of an equation. It also considered the process to verify if a certain value is or is not the solution of an equation.

Justification based on mathematical properties

The teacher begins the whole-class discussion by briefly presenting the task and the aim of the stated question. After this presentation, the students begin to answer and the second student that participates in the discussion presents the sought answer:

Gustavo: 2 *a* plus 2 equals... 6 plus *a* plus 2.

The teacher registers the equation $2a + 2 = 6 + a + 2$ in the whiteboard and encourages the students to present alternative answers (principle (c)):

Daniel: I wrote 6 plus 2 plus *a*.

Teacher: 6 plus 2 plus *a*.

Gustavo: Or 8 plus *a*.

Teacher: Or 8 plus *a*. Isn't it? It could be. Would someone translate differently?

Several students: No.

Gustavo: No, only changing the order.

As the students share these ideas, the teacher focuses on ordering terms, *challenging* the students to justify the possibility of changing the order of terms (principle (a)):

Teacher: Right, but does it make any difference?

Several students: No.

Teacher: Why is it irrelevant to change the order?

Gustavo: Because *a* is always the only [unknown] value...

The justification provided by Gustavo to the teacher's challenge is invalid, as it is based on characteristics of that particular situation (level 2 justification). In addition, this justification is not formally presented (type A justification) as the student does not use the proper terms to refer to the unknown value. Still, the teacher values Gustavo's participation (principle (c)), implicitly *informs* the students about the validity of the justification (principle (b)) and *challenges* the students to enrich the justification (principle (a)):

Teacher: More than that.

Leonardo: Because it is an addition and it is a property of addition.

Addressing the teacher's challenge, Leonardo justifies the possibility of changing the order with a statement regarding the properties of addition (level 3C justification). This justification is presented in a rather formal way, despite being incomplete (type B justification). However, it can be improved; the teacher *guides* the students aiming to complete the justification (principle (c)):

Teacher: That is...? What is its name?

Several students: Commutative property.

Teacher: Commutative. Commutative property, so we can change the order of the terms, it will be the same. That's it, it's true.

In view of students' answer, the justification is recognized as valid, being based on mathematical properties independent of the particular situation under discussion (level 3C justification) and properly stated (type C justification). As such, the teacher decided to finish this whole class discussion moment by *informing* the students of the obtained conclusion (principle (c)) and validating it (principle (b)).

In this episode, a challenging action enhances an invalid and not formally presented justification. Based on the teacher's informing and challenging actions, the students enrich it by providing a valid but incomplete justification. The teacher's guiding actions allow the students to complete the justification and to present it formally, by referring to mathematical properties. As such, a path of challenging, informing, challenging and guiding actions allows students to justify properly. Also, this process of completing and formalizing a justification arises when the teacher's actions focus on enhancing such justification and on valuing invalid and incomplete justifications.

JUSTIFICATIONS THAT DO NOT JUSTIFY

Task and context

By the end of the second lesson, the teacher recommended the students as homework to solve several equations using the addition principle of equality. The teacher expected the students to justify the solving process based on the addition principle of equality. At the beginning of the third lesson, there was a whole class discussion moment about solving some of the equations. The following episode concerns the solving process of the equation presented in Figure 4.

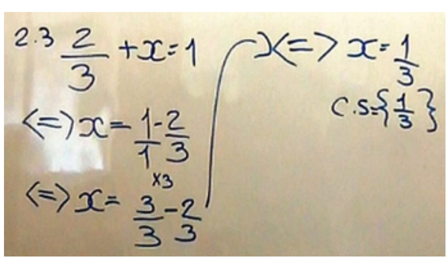

Figure 4. Solving process of Daniel.

No justification and justification based on authority

Daniel solved the equation on the whiteboard, and afterwards, the teacher *invites* the students to share questions or suggestions. Tomás intervenes presenting an alternative solving process:

Tomás: What I have done... Instead of writing three thirds minus two thirds, I wrote two thirds minus three thirds.

Following this contribution, the teacher *challenges* the students to validate this solving process by asking them to identify which processes are wrong, Daniel's or Tomás' (principle (c)):

Teacher: And which one is right? Where's the error? Here [on the whiteboard] or there [in Tomás' solving process]?

Tomás: Here [on my solving process].

Teacher: Why?

Tomás: Because I swapped it.

Tomás himself answers the teacher's challenge, and the teacher moves forward with another *challenging* action, asking for a justification (principle (a)). However, Tomás does not present a justification (level 0 justification), stating solely the procedure that he used to solve the equation. As such, the teacher *guides* the students in order to achieve the sought justification (principles (a) and (c)):

Teacher: But why... Why is it wrong to exchange?..

Ricardo: You [teacher] said that over there [on the whiteboard] it was right.

Tomás is once again unable to justify, and Ricardo justifies based on authority (level 1 justification, type A justification), albeit the teacher had not explicitly stated that Daniel's solving process on the whiteboard was correct. Attending to students' difficulties to justify, the teacher provides *information* regarding the use of the inverse operation that allows the students to move forward:

Teacher: What exchanges... To the inverse operation?

Daniel: It was just two thirds or *x*...

Teacher: The question is, how do I know if a term is going to change its operation or not?

Leonardo: If we change members.

Adding to the statement of the teacher, Leonardo's answer completes the sought justification. This justification, based on addition principle of equality, and thus, on mathematical ideas, is based on arguments independent of the particular case under discussion (level 3C justification), despite being a somewhat informal statement (type B justification). However, this justification is complemented by the teacher with *information* regarding its application in this particular case (level 3B justification) and subsequent *information* to reinforce and formally present the justification that had been completed by Leonardo (level 3C justification, type C justification):

Teacher: If we change members, the operation changes to the inverse operation ... The ones that don't change members, stay exactly as they were. Let's check ... So, 1 has to continue 1... What changed member? The two thirds, it was adding and changed to subtracting, because it changed member. Therefore, the terms that stay in the same member where they were, those stay the same, don't change a thing... The terms that change member are the ones that have to change to the inverse operation.

In this segment of the discussion, the first challenging action from the teacher leads to no justification and a subsequent guiding action leads to an invalid justification, based on authority. Due to the students' difficulties in achieving the sought justification, the teacher's actions rely on informing students of relevant ideas to the justification. Such informing action provides the students with the tools to justify based on mathematical ideas, albeit without the expected formality. As the students have difficulties in achieving the justification, the teacher's actions also consider informing actions to clarify the justification. Due to students' difficulties, in this segment, the sequence of actions has an emphasis on guiding and informing, with the overall path of actions including challenging, guiding, and two sets of informing actions. To formally present the justification and clarify it, the teacher began by switching between justification levels, and formalize the justification by revoicing students' justification.

USING PREVIOUS KNOWLEDGE TO JUSTIFY

Task and context

Both episodes presented in this section focus on the part of a task that was proposed in the eighth lesson of the intervention. This segment of the task (Figure 5) aims to lead students to establish a procedure to figure out the intersection point of two functions. This class had previously studied linear functions, considering both algebraic and geometric representations

> Francisca received a plant as a gift, and she kept a record of its growth. Santiago thought it was a really nice idea and, in the very same day, bought a plant and also kept a record of its growth. The functions that follow represent the height of both plants in their first days with the students:
>
> Francisca's plant: $f(x) = 0.4x$ Santiago's plant: $s(x) = 0.2x + 2.2$
>
> 1. Graphically represent functions f and s.
>
> 2. Based on the analysis of the previous graphic representations, identify in which day do the plants have the same height.
>
> 3. Consider the comment: "Graphs are not necessary for us to know in which day the plants have the same height. Knowing the functions that represent the growth of each plant is enough to verify when they are equal". What would be another way to figure the day in which the plants have the same height? Justify your answer.

Figure 5. Proposed task regarding functions and equations.

In the first two questions, the students are expected to use GeoGebra app, as the school is equipped with iPads and they have used it already in the past. Regarding design principles about the task, question 3 explicitly asks for a justification.

Justification based on knowledge about functions

At the beginning of the lesson, the teacher asks the students to read the questions and clarifies the aims of the task and the tools to use. Afterwards, the students work autonomously on the task, in pairs, for a couple of minutes. After inserting the algebraic expression in GeoGebra, some students state that the plants have the same height in the eleventh day. The teacher begins the discussion by asking students to justify such statement:

Teacher: How did you realize that it was on the eleventh day? Isa.

Isa: Because, if we look closely, both lines intersect on eleven.

The teacher's *challenge* to justify (principle (a)) leads Isa to justify her answer to question 2 on the basis of her prior knowledge about functions. This justification is incomplete with regard to the statement "on eleven", however, it refers to elements of the situation, namely the graphic representation of both functions and the point of intersection. Thus, Isa presents a generic justification given the available data (level 3B justification), despite lacking the proper formality (type B justification).

In order to complete Isa's answer (principle (c)), the teacher *guides* students, revoicing Isa's answer and leading to a more precise justification:

Teacher: On eleven...

Isa: On point eleven.

Teacher: On point eleven?

Gabriel: Abscissa.

Teacher: On the point with abscissa eleven.

By referring parts of students' answers, the teacher is implicitly informing students about what is missing in the justification (principle (b)), and, based on student responses, she highlights what completes it (principles (b) and (c)).

After validating Isa's response, the teacher goes further in the justification, *challenging* the students to come up with another justification (principle (a)):

Teacher: And why is it, Isa and not only, why am I going to read the intersection on the x-axis? Why am I going to look for the value on the x-axis?

Isa: Because x-axis the axis of objects...

Teacher: Right... And how do I know if I am looking for an object or an image?

Isa's justification is based on mathematical concepts (type B justification); however, her statement is not sufficient as a justification in this particular situation since it is not related to the context of the problem (invalid level 3C justification), being an invalid justification. Once again, the teacher validates the student's partial contribution and encourages the students to complete this contribution (principles (b) and (c)), *guiding* them. Another student tries to justify but does not add information to what Isa said earlier. At that moment, Gabriel participates in the discussion:

Gabriel: I think it's because the height is in ... In ... I forgot the name.

Teacher: In the axis...

Gabriel: Of the ordinate, in the axis of the ordinates and the days are in the abscissa.

At this point in the discussion, Gabriel adds information relevant to the justification by relating objects and images of these functions to the context of the situation (level 3B justification), supported by the presentation of a small *suggestion* by the teacher. Despite the relevant relationship that has been added, the justification remains incomplete, and teacher continues to *guide* students to justify (principle (c)):

Teacher: What do functions *s* and *f* represent?

Several students: The height.

Teacher: The height of the plant, right? In function of what?

Several students: The time.

Teacher: The time that elapses. The time that elapses in days. OK, very well, 11.

This *information* given by the teacher leads the students to easily identify the dependent and independent variables, thus completing the intended justification (level 3B justification, type C justification).

As in previous episodes, the teacher's first action is a challenging action to justify. In this episode, this first action leads to an incomplete justification that is completed based on guiding actions. In order to achieve a more accurate justification based on students' previous knowledge, the teacher also uses informing actions and a challenging action to go further in justifying. This last action leads to another sequence of moving between guiding and suggesting actions aiming to complete and formalize the sought justification. As such, the path of actions considers challenging, guiding, informing, challenging, guiding and suggesting actions.

Justification based on knowledge about equations

After the discussion of question 2, the teacher introduces question 3. At this point in the discussion, a student immediately proposes a strategy to solve the question. This leads the students to engage immediately in a new segment of whole-class discussion:

Santiago: So, teacher, we have that thing that was G.C.D...

Santiago brings to the discussion a strategy based on a mathematical concept that would not be expected in this situation. Although it seems an idea with little meaning, the teacher *informs* the class of the idea of Santiago and lets him continue his explanation (principle (c)):

Teacher: Greatest common divisor?

Santiago: Yes, something like that. Can't we use it to answer to when they intersect? ... I can't recall it, but wasn't there something in common? Doing each number and then...

By allowing Santiago to justify his statement, it is possible to understand that, although incorrect and not formally presented (type A justification), this justification is based on an idea with some logical coherence (level 3A justification). Effectively, both in the greatest common divisor and in the intersection of functions what is sought is "something in common", as the student refers. At this point, the teacher poses more questions in order to deconstruct the student's perception of G.C.D., which leads other students to identify Santiago's strategy as inappropriate for the situation.

After this clarification, Clara presents her strategy:

Clara: We can use an equation (referring to $0.4x = 0.2x + 2.2$), and the number that we get is the day they have [the same height]...

Teacher: What are you expecting as a solution of this equation?

Several students: 11.

Teacher: 11. So, confirm that.

In retaking the information obtained in the previous questions, the teacher *supports* Clara's strategy of solving this equation and, by *challenging* the students to confirm the result, she leads them to justify (principle (a)) that 11 is the solution of the equation. The students solve this equation in an autonomous work, and Daniel intervenes:

Daniel: Teacher, it isn't.

Teacher: It isn't? So, solve the equation over there (on the whiteboard).

By *inviting* the student to solve the equation in the whiteboard, the teacher realizes that the student only forgot x in one of the steps and, in *guiding* his solving process (principle (b) and (c)), the sought justification is adequately achieved (level 3B justification, type C justification).

In this segment of the discussion, previous knowledge about G.C.D. is brought to bear, and a teacher's informing action leads to a justification. As the mobilized knowledge is not accurate with the situation, the justification is invalid albeit posed in at a logical level. By supporting another strategy and challenging the students to confirm a statement, previous knowledge about solving equations is used to justify using procedures properly. However, this proper justification is consolidated only after teacher's guiding actions. The first path in this segment considers only informing actions, while the second began by guiding actions and continues with challenging and further guiding actions.

DISCUSSION

In the analyzed episodes, the task proposed triggered all situations. Two of these tasks gave the students the opportunity to develop procedures, one to discover the process of solving equations and another to discover where two functions intersect. In the moments of whole-class discussion presented, there were opportunities for justification that were not directly related to the situation presented by the task, but rather to other ideas, concepts and mathematical properties that emerged. This reinforces the idea that collective activity in mathematical discussions allows students to share, discuss and clarify their thinking and mathematical knowledge (Galbrait, 1995).

As referred by Kosko, Rougee, and Herbst (2014), in order to understand the kind of students' justifications, it is not enough to focus on a specific question made by the teacher, but it is necessary to take into account his/her sequences of actions. This research shows that particular sequences of teacher actions based on the design principles make it likely that justifications will arise in whole-class discussions. In the episodes presented, when the principle of requesting a justification was followed, often by means of a challenging action, the students presented justifications. As could be expected, these justifications stand essentially on prior knowledge of mathematical concepts or ideas or known mathematical procedures and are often incomplete and sometimes incorrect. Thus, as in previous research (Galbrait, 1995), the use of available information about a particular concept or mathematical idea was not always adequate to what was defined or assumed in the task. When the justifications were incomplete, the teacher tended to guide the students to complete it, validating or invalidating their statements mostly implicitly. Depending on her perception of the support the students needed to mobilize their knowledge, the teacher provided them with more or less information. When an invalid justification arose, and in accordance with the formulated principles, the teacher valued the contribution of the students and insisted on encouraging them to present ideas. In these situations, when the student's justification was incorrect, the teacher's actions focused on abandoning such justification and challenging the students to present a new justification or guiding them to reformulate the justification so that it would become valid. As such, based on the design principles to enhance justifications and by means of a sequence of teacher's actions, complete justifications often arose in whole-class discussions.

In the episodes presented, the students' justifications, although sometimes incomplete or invalid, tended to be reasonably formal given that they were based on mathematical aspects of the situation. However, some of the justifications presented, either were not justifications at all or were based on authority. These cases emerged when students did not have the required mathematical tools to answer the justification challenge posed by the teacher, and the appropriately formal justification emerged when the teacher's informing actions introduced the required ideas for justification. In the context of whole-class discussions based on the formulated design principles, justifications of all levels of complexity emerged. In addition, in a whole-class discussion episode, as justifications emerged and teacher's actions supported it, justifications tended to increase its level of complexity. As these episodes illustrate, in order to promote opportunities for the students to move forward between levels of justification, it is not enough to ask them to justify and validate their justifications – it is also necessary to accept and appreciate partial and incorrect justifications.

CONCLUSION

The sequences of challenging, guiding and informing actions from the teacher, constitute promising support for enhancing students' justifications and, thereby, students' mathematical reasoning. In order to achieve such sequences of actions, a close match between the design principles formulated and what happens in the classroom is necessary. In this study, as in other design-based research studies (e.g., Stylianides & Stylianides, 2009), this emerges from the systematic approach followed and the close interaction between researchers and teachers. However, it will be important to understand how this match can occur in professional development processes.

Another relevant finding of this study is that whole-class discussions oriented by the design principles resulted in opportunities for students to justify at different and increasing levels of formality and complexity. Moreover, looking at justifications led to clarify how they contribute to students' understanding of mathematical knowledge (Miyazaki, Fujita, & Jones, 2017). Thus, regarding higher levels of complexity, to alternate between levels of deductive justification may promote opportunities to understand better the mathematical situation at stake.

By detailing task characteristics and sequences of teacher's actions supported by design principles regarding justifications, this study contributes to understanding how to promote students' justifications of different kinds. It will be important to know if this set of design principles can improve teachers' practice to enhance students' mathematical reasoning in other school grades, such as middle and high school levels. As stated above, another aspect to be further researched is how these design principles may inform professional development processes.

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