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An Investigation of the Bivariate Complex Fibonacci Polynomials Supported in Didactic Engineering: An Application of Theory of Didactics Situations (TSD)

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ABSTRACT

A research cut will be presented in the Academic Master of the Programa de Pós-Graduação em Ensino de Ciências e Matemática (PGECM) of the Instituto Federal de Educação, Ciência e Tecnologia do Ceará (IFCE). This research used Didactic Engineering with a focus on the Theory of Didactic Situations, evidencing epistemological, cognitive and didactic elements articulated among themselves. This made it possible to mobilize the student's intuitive thinking towards inferential reasoning during the study of the Bivariate Complex Fibonacci Polynomials. Moreover, it had the purpose of inserting an epistemological conception in the teaching of History of Mathematics, considering that the research was applied in the course of Degree in Mathematics in the discipline of History of Mathematics.

Keywords: Didactic Engineering. Theory of Didactic Situations. Bivariate Complex Fibonacci Polynomials. Didactics of Mathematics.

Uma Investigação dos Polinômios Bivariados e Complexos de Fibonacci Amparada na Engenharia Didática: uma Aplicação da Teoria das Situações Didáticas (TSD)

RESUMO

Será apresentado um recorte da pesquisa realizada no Mestrado Acadêmico do Programa de Pós-Graduação em Ensino de Ciências e Matemática (PGECM) do Instituto Federal de Educação, Ciência e Tecnologia do Ceará (IFCE). Essa investigação usou a Engenharia Didática com enfoque na Teoria das Situações Didáticas, evidenciando elementos epistemológicos, cognitivos e didáticos articulados entre si. O que possibilitou mobilizar o pensamento intuitivo do aluno em direção ao raciocínio inferencial durante o estudo dos Polinômios Bivariados e Complexos de Fibonacci. Além do mais, teve a finalidade de inserir uma concepção epistemológica no ensino de História da

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Matemática, tendo em vista que a pesquisa foi aplicada no curso de Licenciatura em Matemática na disciplina de História da Matemática.

Palavras-chave: Engenharia Didática. Teoria das Situações Didáticas. Polinômios Bivariados e Complexos de Fibonacci. Didática da Matemática.

INTRODUCTION

The research developed in France, which investigate the mathematical concepts and how they are arranged in the pedagogical plan, contributed to the creation and international recognition of a scientific field that became known as Didactics of Mathematics. This fact, according to Alves (2016, p.132), made it possible to observe the existence of several tendencies, as well as certain questions concerning the mathematics teaching, as well as "perceptions about the canonical paradigms followed in mathematics subjects, which have been revised".

On the other hand, Alves (2016b) explains that in the construction of mathematical concepts about the "historical gaps" have emerged, representing epistemological inertia in the development of scientific knowledge. That is, in a historical-evolutionary approach to mathematical relationships, there was a time when concepts did not develops. These kind of historical gaps are verified when a bibliographical revision is made in the History of Mathematics books, which have an inexpressive approach to the mathematical concepts inherent to the Fibonacci model, especially when we focus on the most current evolutionary historical aspects. (Alves & Borges Neto, 2011; Alves, 2018).

Considering the scenario indicated, the research presented in this article is justified and was developed guided by the following question: how to make didactic situations that allow to explore the properties and mathematical definitions from the generalized Fibonacci model in order to understand, in the context Epistemological, the generalized representation of Bivariate and Fibonacci Complex Polynomials?

Thus, this research had the purpose of providing an epistemological conception in the teaching of the History of Mathematics (HM). This conception is associated to the exploration of the formal theorems and properties of the Fibonacci model in Mathematical History classes, explaining the evolution and the generalization of mathematical relations. For this, the phase of empirical experimentation of this research was carried out in the Mathematics History course in the Mathematics Degree course of the Federal Institute of Education, Science and Technology of the State of Ceará (IFCE).

In addition, this article presents a review of the Fibonacci Model, which was carried out in the Master's Degree in Science and Mathematics Teaching (PGECM/IFCE), and which assumed the epistemological, cognitive and didactical aspects as fundamental presuppositions. In this sense, this research, as part of the dissertation of Master of Oliveira (2018), follows a methodological structure based on the design of Didactic Engineering (DE), adopting the approach in the Theory of Didactic Situations (TSD), seeking to understand the dialectical movement of mathematical knowledge. Thus, it is relevant to mention that the research project was submitted to the Research Ethics Committee (CEP), obtaining the approval opinion about it´s development.

Regarding the initial questioning or guiding question, we searched for a teaching scenario, where didactic situations were carried out and structured by the didactic contract between teachers and students, which instigated the development of the student's inferential reasoning during the understanding of mathematical concepts that are inserted in pedagogy through didactic transposition. In addition, it is intended with this work, to publicize an investigation that until then is only done in Pure Mathematics, Bivariate and Fibonacci Complex Polynomials supported by a teaching theory in the context of Didactics of Mathematics (DM).

Still on this research, it is valid to comment that in the epistemological dimension, which concerns the mathematical structure of what is intended to teach to the students-teachers in initial formation at (IFCE), we have the elements related to the historical and evolutionary scope of the generalization process of the model of Fibonacci, with emphasis on the respective ones and not discussed by the (HM) books, nominated by polynomial and matrix representations in the context of the Bivariate and Fibonacci Complex Polynomials. At the cognitive and didactic level, the situation is considered where students (students – teachers) learn, that is, how they assimilate and accommodate the contents taught in the classroom. Next, we will discuss, in a preliminary way, some elements related to the Didactics of Mathematics.

DIDACTICS OF MATHEMATICS IN A FRENCH RESEARCH

In France in the 1960s and 1970s, a series of educational changes took place under the influence of the Modern Mathematics movement and, after some time, gained prominence from the creation of the IREMs (*Institut de Recherche sur l'Enseignement des Mathématiques*), Institute of Research on Teaching Mathematics, and the recognition of the works of J. Piaget on the psychological theories related to the development of intelligence. This fact favored the emergence of Mathematics Didactics (DM) as a scientific area of research involving specialists of several scientific contexts, whose main purpose is to investigate "the problems of teaching mathematical concepts due to the specific requirements of mathematical knowledge" (Almouloud, 2007, p.25-26). For this, these researches have approaches in the epistemological, cognitive and didactic dimensions (Artigue, 1995, p.98).

In this sense, Pais (2002, p.9) understands that the (DM) covers aspects related to the teaching of Mathematics, emphasizing the difficulties arisen in the learning process of the mathematical concepts. "As a consequence, we observe the evolution of theories that have tried to make controllable, reproducible and predictable, certain didactic transpositions and / or structured teaching approaches" (Alves, 2016c, p.2).

Researches emphasize the obstacles identified during the epistemological construction of scientific concepts. In this context, Alves (2016, p.9) emphasizes that "through a dialectical movement, characteristic of its evolution and systematization, we see a theoretical corpus that starts from mathematics, acquires a scientific robustness and has the capacity to rejoin, again, to Mathematics."

It should be noted that the French part of the Didactics of Mathematics involves the multi-theoretical use of several frames of reference. However, we cannot only consider a restrictive scenario of the process, since, as a matter of priority, we observe a whole discussion from a strongly determined and conditioned by mathematical knowledge. Moreover, these concepts have their genesis in the scientific field of Pure Mathematics and are inserted by educators in pedagogical environments. For this reason, it is valid to understand that this didactic transposition involves epistemological, cognitive and didactic elements, what will be highlighted in the next section.

Epistemological, Cognitive and Didactic Aspects of (DM)

A (DM) covers epistemological, cognitive and didactic aspects. Thus, in the conception of Almouloud (2007, p.149), epistemology incorporates structure, where scientific concepts are organized, so an analysis is made from its historical genesis to its evolution and how these concepts are manifested and assimilated by during the construction phase of the concepts. Thus, the author himself explains:

> The Epistemological analysis is based on the historical development of the concept. Thus, it allows to identify the different conceptions about a given object, as well as to group them into pertinent classes so that a didactic analysis can be done. This type of analysis can help the researcher in mathematics didactics to better understand the relationships between mathematical objects and to control the didactic variables related to the teaching and learning process of such objects. (Almouloud, 2007, p.156)

Moreover, Almouloud (2007, p.152) explains that the epistemological analysis is an investigation of the mathematical concepts and their foundation. However, the problems related to the genesis of concepts only became object of study after centuries of application of the concepts as support in the resolution of questions. In turn, Alves (2016a, p.137) stresses that the "epistemic ground" explored by the researcher demands attention, "because in view of their nature, obstacles and obstacles, often insurmountable, can arise. And in this context we speak of an epistemological obstacle."

The obstacles can be categorized into two classes: one composed of the difficulties manifested by the students in the teaching and learning process and the other that covers the obstacles that appear in the construction of the mathematical concepts in the scientific field. However, the epistemological obstacles do not represent the absence of knowledge, but rather a stagnation as to its reformulation and resistance of the acceptance of the development of other forms of representation. In this sense, Pais (2002, p.39) states that there are people who understand these changes as a threat to the "[...] intellectual stability of those who hold knowledge."

On the other hand, according to Bachelard (1996, p.18-19), knowledge arising from scientific construction may wear off, causing the formation of "an epistemological obstacle embedded in unquestioned knowledge". Although Bachelard explicitly and in detail argues that mathematics is the only area that has no epistemological obstacles, so that this notion only applies to natural (experimental) sciences, the Bachelardian notion influenced the thinking of Brousseau (1976) in the elaboration of his theses about obstacles and obstacles present in teaching and learning situations, so Brousseau (1976) considers, for example, the didactic aspects that are ignored in the studies of Bachelard.

Obstacles are relevant factors when researching the historical and evolutionary process of certain mathematical concepts. Thus, the development of mathematical concepts is evidenced in two stages: the initial stage, where the primitive axiomatic notions of Mathematics are discovered and the final stage, where a produced text is presented, which registers the concepts constructed and supported by a rigorous paradigm (Alves, 2018). However, the obstacles that arose at this point in production are difficult to identify historically, that is, "the advances, setbacks, doubts and errors committed at the stage where the conjectures are made by the mathematician, practically, disappear in the final result presented by the scientific text" (Pais, 2002, p.41).

The validity of mathematical concepts, responsible for theorems and properties, is verified, most of the times, by induction methods. It is worth commenting that although generalized thinking can behave as an obstacle in the development of Mathematics, it must be understood that "[...] the technique of mathematical induction is not based on an inductive logic. The observation of particular cases does not serve as a basis for a demonstration, at most it may suggest a conjecture" (Pais, 2002, p.48). Alves (2016) develops some considerations about the reasoning models employed in the proof and reasoning mobilized by the student.

> In this context, it is worth warning the epistemological character that lies in impressing the student's reasoning, the monosemic and inferential character, characteristic of formal theories. [...] Just as the theorems and founding theories, which confer their character of certainty, are intertwined with an "epistemic grid" of conceptions and knowledges that are not neglected by the Didactics of Mathematics. (Alves, 2016a, p.140-141)

In this sense, Pais (2002, p.45) stresses that the epistemological development of scientific knowledge only happens when one can overcome the barriers that limited the knowledge previously established as absolute truth. Thus, according to Pais (2002, pp.44-45), epistemological obstacles arise in the historical, social and cultural context. Thus, in a cognitive and historical approach, Artigue (1995, p.113), in turn, explains that epistemological obstacles exist in the classroom and are categorized into operational and structural, the first related to learning the mathematical structure and another associated with the understanding of the context and process that allowed the construction of mathematical concepts.

Thus, a relation can be established between epistemology and cognition, because epistemological obstacles arise, first, in the cognitive structure of the subject and, consequently, influence the development of scientific knowledge, providing the appearance of epistemological obstacles. In view of this, in the cognitive dimension, didactic obstacles are subclassified as ontogenic and psychological. In the conception of Brousseau (1976, p.108), ontogenic obstacles originate from the neuropsychological deficiencies of individuals when they are faced with a learning situation. Thus, Brousseau clarifies that:

> The Genetic epistemology brings to light the stages, accommodations, and assimilations, which sometimes resemble the stages of development of concepts by the law of regulation that make them appear, and which differ from the exact nature of the limitations that determine this regulation. (Brousseau, 1976, p.108, our translation)

Moreover, some psychological obstacles can be pointed out in teaching situations that contradict the common sense conceptions of the subjects, thus inducing them to "an unacceptable destabilization, as, for example: mathematical logic is not the logic of life from day to day "(Almouloud, 2007, p.144). Therefore, constant vigilance on the part of the teacher is urged, aiming at the adequate evolution of such student conceptions (Alves, 2018).

Considering that the cognitive obstacles are evidenced in the didactic context, it can be understood that the (DM) includes epistemological, cognitive and didactic elements in order to investigate methodological procedures that allow didactic transposition (Chevallard, 1998) of the mathematical concepts of the epistemological plane to the pedagogical. Almouloud (2007, p.141-142) describes that didactic obstacles are generated at the time of didactic transposition, that is, they are identified in teaching situations, in which the student expresses difficulty in learning or when he questions the validity concepts. Thus, we can verify that:

> For example, the current presentation of decimals at the elementary level is the result of a long evolution in the context of an educational choice (didactique) [...]. Given their usefulness, decimal numbers would be taught as quickly as possible, associated with a system of moderation and in reference to the techniques of application as a whole. Thus, today, decimals are, for students, "natural integers with a change of unity," so "natural" (with a comma) and measure. And this project, supported by a mechanization of the student, will create obstacles until the D.E.U.G. It is characteristic that the main factor of discrimination of the students in a recent

> questionnaire (IREM of Rouen) is the calculation that involves both decimal and the products of a power of ten. Thus, it is the same "understanding" of the decimal definition that explains student behavior. But today, this obstacle becomes, at times, didactic and sociocultural. (Brousseau, 1976, p.108, our translatioin)

In this way, Alves (2016a, p.141) explains that in the context of mathematics teaching, the didactics addressed by the teacher must consider the particular experiences and idiosyncrasies that "gives the apprentice the origin of a wide repertoire of problem situations that allow him / not to explore and gradually elaborate and rework effective constructions and mental models of action "in order to internalize the mathematical concepts through an efficient trajectory of learning.

In a dialectic between didactics and cognition, Alves (2016a, p.143) describes that the student elaborates two types of mental maps to explore a problem situation. The first map has a specific "theoretical corpus" whereby the student is instructed to formulate and express his or her conjectures. While the other presents an intrinsic aspect of the student's cognitive structure, that is, tacit and intuitive thinking, which is expressed when the student strives to solve some problem situation, especially in his moments or stages of preliminary heuristics. Finally, it is understood that research in (DM) emphasizes the obstacles that, according to Pais (2002, p.44), arise in the intersection of the scientific field with the didactic plane, thus evidencing epistemological, cognitive and didactic aspects. Next, it will be discussed how these aspects are articulated in (ED) in complementarity with the Theory of Didactic Situations (TSD).

A Complementary Perspective Involving (TSD) and the (DE)

In the early 1980s, the context of research in the field of (DM) contributed to Artigue (1995) fourmala a research methodology or research design that became known as Didactic Engineering (DE). This nomenclature, according to Pais (2002, p.100), refers to the stages of conception, elaboration and execution of a project developed by an engineer, who conceives and elaborates a precise project, however, aiming at teaching Mathematics.

Artigue (1995, p.36-37) explains that (DE) refers to an experimental methodological course based on teaching situations, with emphasis in the classroom, where structured and controlled didactic sequences can be realized, which are conceived, observed and analyzed considering two typologies: the inherent microengineering to the phenomena of the classroom and the macroengineering, that covers the methodological aspects and institutional variables for the execution of didactic situations.

In this sense, a (DE) is structured in four consecutive classic stages: preliminary analysis, a priori design and analysis, experimentation and a posteriori analysis and validation. Almouloud and Silva (2012, p.26) describe that in the preliminary analyzes an "epistemological analysis of current teaching and its effects, student conceptions, difficulties and obstacles, and analysis of the field of constraints and requirements in which it is situated the effective didactic accomplishment “. In addition, a bibliographical review is done on the mathematical concepts that one intends to explore in teaching situations. According to Artigue (1995, p.42-43), the researcher-teacher determines a set of variables related to the object under study. These command variables are classified as macrodidactic variables, these are associated to the general structure of (DE) and microdidactic, which refer to a local subset of (DE), that is, to the organization of a specific situation.

In this way, Pommer (2013, p.22) recommends that the concepts to be investigated in the classroom should be presented to the students in the form of problem situations, which allow the student to participate actively in their learning process, are stimulated to develop their autonomy in the elaboration of strategies of solutions and, thus, considering their previous and intuitive knowledge.

From this perspective, it is understood to apply a methodology based on a theory of student-centered teaching. Thus, Pais (2002, p.69-70) explains that problem situations can allow students to develop their knowledge from their previous and intuitive conceptions. This characterizes learning by adaptation. This scenario served as an inspiration for Brousseau (1976) to conceive of (TSD), with an emphasis on the approximation and repercussion of “assimilation and accommodation schemes, which were initially described by Piaget.”

Thus, the (TSD) is inserted in the (DE), mainly, in the phases of conception, a priori analysis, experimentation and a posteriori analysis. Thus (TSD) is organized as a system of didactic situations elaborated to be applied in educational environments in order to analyze student behavior. That is, “the central object of study in this theory is not the cognitive subject, but the didactic situation in which the established interactions between teacher, student and knowledge are identified” (Almouloud, 2007, p.31-32).

In this sense, the (TSD) presents four stages: action, formulation, validation and institutionalization. In the action stage, the student has the freedom to think of a solution strategy based on his previous knowledge and intuitive, thus, being able to define and redefine his supposed path to solve the problem proposed (Almouloud, 2007, p.37). In the formulation, the student presents a solution strategy, in which he will construct conjectures to be validated or refuted in the later stage. In this way, the student mobilizes a reasoning of a theoretical nature, presenting in his productions a more elaborate language (Pais, 2002, 72). In stage of the validation, according to Pais (2002, p.73), the validity of the formulated arguments happens, the student must already have internalized the mathematical concepts, thus presenting an inferential reasoning and using methods of mathematical demonstration. In addition, we see that:

> A validation problem is one more problem of comparison, evaluation and rejection of evidence and investigation of the demonstration. [...] for a validation approach, thinking must be based on previous formulations. The developed language, in the dialectic of the formulation, is less specific than that of validation. Communication

> plays an important part independently of validity issues. (Brousseau, 1976, p.110, our translation).

Returning to the phases of (DE), we have the experimentation, where the teaching situations happen, through the proposal of problem situations. This application is followed by a posteriori analysis, which addresses the discussion of the data collected during the experimentation, this collection is made with the use of some external resources such as: questionnaires, interviews, among others (Artigue, 1995, p.48).

In addition, Artigue (1995, p.48) explains that in the validation phase of a (DE), the comparison between a priori and a posteriori analyzes is performed in order to validate the hypotheses of the research. In this way, the validation is internal, with respect to the experience in the classroom, being linked to the command variables and to the "epistemological state of didactics". The effectiveness of didactic situations with a focus on (TSD) occurs due to the effectiveness of didactic transposition and didactic contract, what will be discussed next.

The Didactic Transposition and the Didactic Contract

The classic notion of didactic transposition is responsible for articulating elements of the epistemological and cognitive dimensions. In this sense, it is worth to understand that the concept of didactic transposition was conceived by Chevallard (1998) in order to distinguish the only scientific knowledge of the students. Moreover, didactic transposition starts from an epistemological analysis of knowledge, which is categorized into: mathematical and paramathematics that are related to the investigation of mathematical concepts and the protomathematics, which have properties to solve mathematical problems (Almouloud, 2007, p.113). Thus, we can observe that:

> The transformation of the content of knowledge into a didactic version of this object of knowing, more appropriately, is called didactic transposition stricto sensu. However, the scientific study of the process of didactic transposition (which is a fundamental dimension of mathematics didactics) implies taking into account the didactic transposition sensu lato, represented by the scheme: object of knowing $\Rightarrow$ object to teach $\Rightarrow$ object of teaching. The first link that marks the passage from the implicit to the explicit, from practice to theory, from the pre-built to the built (Chevallard, 1998, p.45, our translation).

After the mathematical concepts are transposed into the didactic plan, it is relevant to make a didactic contract between the main agents: teachers and students. In this context, Almouloud (2007, p.89) describes that the didactic contract covers teaching situations, in which one intends to construct the understanding of certain mathematical concepts.

Thus, the didactic situation only happens if there is an engagement of the teachers, in the planning and the application of the situations, and the students necessary engagement in the resolution of the proposed problem situations.

Therefore, in the didactic contract, the roles of both the teacher and the student are defined. In the case of (TSD), the three initial stages of action, formulation and validation are carried out by the student, and the teacher can intervene, while the institutionalization is carried out by the teacher. From now, we will present the preliminary analysis phase of the research on the investigation of Bivariate and Complex Fibonacci Polynomials.

PRELIMINARY ANALYZES CORRESPONDING TO FIBONACCI BIVARIATE AND FIBONACCI COMPLEX

The preliminary analyzes cover a set of epistemological, cognitive and didactic elements (on the current teaching). In this sense, there is an epistemological analysis of the mathematical concepts that one intends to teach, of students' conceptions, of traditional teaching and its repercussion and of the obstacles that arise in the historical development of mathematical concepts (Artigue, 1995, p.38).

In view of this, the research described in this article has the scope to investigate the Fibonacci Model (FM) modeling process, with emphasis on the Bivariate and Complex Fibonacci (PBCF) polynomials. Thus, in the mathematical-epistemological context of Mathematics, a bibliographic review on the polynomial representations of the MF is made. In this way, a bibliographical survey is begun by Brother (1963), Hoggatt and Long (1974), Witford (1977), Asci and Gurel (2012) and Alves (2017, 2018).

Considering the epistemology of the (FM), King (1963) describes that the Fibonacciano model had its genesis from the problematization of the reproduction of rabbits, such problem situation, proposed by Leonardo Pisano in 1202, has its resolution represented by the sequence {1, 1, 2, 3, 5, 8, ...). Currently, this mathematical model is described by the "modern notational apparatus" $f_{n+2} = f_{n+1} + f_n, \forall n \in \mathbb{N}$, with initial conditions $f_0 = 0 \text{ e } f_1 = 1$ described by Alves and Catarino (2017).

Thus, in the process of historical development of the (FM), we can understand an evolution of the Fibonacci sequence (FS), with respect to the emergence of other generalized forms of (FM), evidenced in the polynomial and matrix representations. In this sense, the introduction of the imaginary unit i and of variables in the one – dimensional Fibonacci recursive model characterized the extension of the model to the context of the polynomials, registered from the seventies.

Moreover, the first works on the Fibonaccian polynomials were proposed by Ernest Erich Jacobsthal (1881-1965) and Eugene Charles Catalan (1814-1894) in 1983. Consequently, the sequences of the Fibonacci polynomials and (PBCF) are presented by Asci and Gurel (2012). Let's look at the following definitions:

Definition 1. The polynomial Fibonacci sequence is given by the recursive relation:

$$f_n(x) = x \cdot f_{n-1}(x) + f_{n-2}(x) \text{ with } f_1(x) = 1, f_2(x) = x \text{ and } n \geq 1.$$

Definiton 2. The sequence of (PBCF) $\{F_n(x, y)\}_{n=0}^{\infty}$ is described by the recurrence relation:

$$F_{n+1}(x, y) = ix \cdot F_n(x, y) + F_{n-1}(x, y) \text{ with } F_0(x, y) = 0,\ F_1(x, y) = 1 \text{ and } n \geq 1.$$

In addition, to explore the behavior of the tridiagonal matrix of Figure 2, Asci and Gurel (2012) highlight the following property indicated in the theorem below:

Theorem 1. $\det D_n(x, y)_{nxn} = f_n(x, y),\ n \geq 0 \text{ and } \det D_0(x, y) = 0.$

Still in the context (PBCF), an analogous formula of Binet, explored by Witford (1977), is presented:

Binet´s like formula. $f_n(x, y) = \dfrac{\alpha^n(x, y) - \beta^n(x, y)}{\alpha(x, y) - \beta(x, y)}, \text{for } n \geq 0.$

Starting from the investigation of these mathematical concepts inherent to the (FM), some formal definitions and properties were selected, in order to be explored in practical teaching situations, with a view to the transposition of Pure Mathematics to the didactic plan in order to give the students the understanding of the construction of these relations. For this, it is necessary to instigate the mobilization of the student's intuitive thinking towards inferential reasoning. Thus, the didactic situations proposed in this research and discussed below have a focus on (TSD).

THE (TSD) IN THE CONCEPTION AND THE A PRIORI ANALYSIS OF DIDACTIC SITUATIONS

The didactic situations were conceived with a basis in (TSD), since this teaching theory has the purpose of stimulating the cognitive component of the student in the sense of developing a theoretical mathematical knowledge, through the accomplishment of the phases of action, formulation and validation. In order to elaborate the problem situations, the class of (PBCF) was selected, thus, the proposed planned situations have an implicit purpose of working the validity of properties derived from the (FM).

Hereafter, we will describe the problem situations elaborated, followed by their a priori analysis with a focus on (TSD), evidencing the prediction and anticipation of possible behaviors that students can manifest during the phases of action, formulation and validation, as well as the positioning of the teacher in the final stage of the institutionalization of mathematical knowledge. The following is the first problem situation.

Table 1
Problem-situation (1).

(1) According to this table, it is requested to verify if there is any relation of the same with the sequence (0,1,1,2,3,5,8,13, 21, ...), if it understands that yes, it explains in detail, then determine other terms in the (SPBCF).

The table mentioned in the statement in Table 1 refers to the initial terms of the sequence of (PBCF) arranged in Figure 1. Thus, in the action phase, starting from trials, the student is expected to determine a relationship between SF and $f_n = (x, y)$ (Figure 1), in order to identify each element of the second sequence as a polynomial representation for the Fibonacci sequence and thus obtain a recursive relation to the (PBCF).

In the formulation situation provided by (TSD), students should use the definition $f_{n+1} = f_n + f_{n-1}, \text{ para } n \geq 1$ to write $f_{n+1}(x, y) = ixf_n(x, y) + yf_{n-1}(x, y)$, with emphasis on the insertion of the imaginary unit i and the variables x and y. In validation, using the bivariate expression for the Fibonacci sequence, explored in the previous step, one must determine some particular terms of (PBCF).

n	$F_n(x,y)$
0	0
1	1
2	ix
3	$-x^2 + y$
4	$-x^3 i + 2xyi$
5	$x^4 - 3x^2 y + y^2$
6	$x^5 i - 4x^3 yi + 3y^2 xi$
7	$-x^6 + 5x^4 y - 6x^2 y^2 + y^3$
⋮	⋮

Figure 1. Some initial elements of the sequence (PBCF). (Asci & Gurel, 2012).

In addition, in the institutionalization stage provided by the (TSD), the teacher must resume the didactic situation and confer the students' productions, in order to resign an extension of SF through PBCF. The problem situation (2) (Table 2) presents a matrix approach for the (PBCF) with the purpose of ascertaining the tridiagonal matrix (see Figure 2) proposed by Asci and Gurel (2012).

Table 2
Problem-situation (2).

(2) [...] we have a matrix proposed by Asci & Gurel (2012). Explain, with your words, the function and properties of this matrix, including its behavior for the 2x2, 3x3 orders. 4x4, 5x5, 6x6, etc. Also, consider the theorem teorema det $D_n(x, y)_{nxn} = f_n(x, y), n \geq 0$.

The second problem situation, at the moment of action, must be solved starting from the reflection on the aspects of the proposed matrix, with this the students must suggest the description of the square matrices seeking a generalized understanding of the matrix, in addition, it is worth discussing as supposedly this matrix can be classified, since Asci and Gurel (2012) define it as tridiagonal. In the formulation phase, students must calculate the determinants of the matrices constructed in the previous phase, besides understanding that the terms of the matrix are the same as the sequence of the (PBCF). Thus, students can present a more elaborate and formal argument with the purpose of proposing a mathematical model that approaches the theorem indicated by the condition $\det D_n(x,y)_{nxn} = f_n(x,y), n \geq 0$ for $D_0(x,y) = 0$.

$$D_n(x,y) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & ix & 1 & \ddots & \vdots \\ 0 & -y & ix & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -y & ix \end{bmatrix}, \quad n \geq 1$$

Figure 2. Tridiagonal matrix. (Asci & Gurel, 2012).

Thus, in the validation stage predicted by the (TSD), students are expected to use inferential reasoning, such as mathematical induction, to evaluate the validity of the mathematical model formulated, being able to use the definition $f_{n+1}(x,y) = ix \cdot f_n(x,y) + y \cdot f_{n-1}(x,y)$ with $n \geq 1$ and starting from the initial values $f_0(x,y) = 0$ and $f_1(x,y) = 1$. Finally, the dialectical phase of institutionalization allows the teacher to systematize, prove and recognize, through generalization, the validity of $D_0(x,y) = 0 \Rightarrow \det D_n(x,y)_{nxn} = f_n(x,y)$, for $n \geq 0$.

Table 3

Problem-situation (3).

(3) The Binet Formula known by $f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ was formulated by Jacques-Phillipe-Marie Binet (1786-1856). Is there a similar formula for the (PBCF) class? If it exists, deduct the same. Consider the equation $t^2 - ixt - y = 0$.

It is worth commenting that, during the resolutions of the questions, the student can use the knowledge built in the resolution of the previous problems. Therefore, during the resolution of the problem situation (3), at the moment of action, the student should try to rewrite the Binet Formula by inserting the variables x and y. Thus, in the formulation dialectic stage, students should suggest the rewriting of

$f_{n+1}(x,y) = ixf_n(x,y) + yf_{n-1}(x,y)$ for $n \geq 1$ according to the variant formula of Binet $f_n(x,y) = \dfrac{\alpha^n(x,y) - \beta^n(x,y)}{\alpha(x,y) - \beta(x,y)}$, for $n \geq 0$.

Moreover, in the context of validation, the Binet variant formula for the index (n + 1) must be assumed assuming a reasoning of a theoretical nature, such as the mathematical induction method and thus obtaining $f_{n+1}(x,y) = \dfrac{\alpha^{n+1}(x,y) - \beta^{n+1}(x,y)}{\alpha(x,y) - \beta(x,y)}$ for $n \geq 0$. Finally, the teacher must institutionalize, that is, formalize and demonstrate that there is in fact a representation analogous to Binet's formula for (PBCF).

Table 4

Problem-situation (4).

(4) [...] when comparing the left column with the right column, is it possible to interpret and signify a historical and evolutionary perspective of the sequence conceived by Leonardo Pisano in 1202?

The column mentioned in the problem situation statement (4) refers to the table in Figure 9. Thus, in the action phase, the student is expected to compare the two columns (Figure 9) and understand that the properties addressed in the context (PBCF) are an extension of the Fibonacci model. At the time of formulation, students should suggest that there was a "historical gap" in the process of building generalized Fibonacci models. However, it can be observed that there was an evolution of the Fibonacci model from its extension to negative indexes and the insertion of matrix representations, generating functions and the Binet variant formula, among other relations.

The validation of this situation presents a historical perspective, by which the student is expected to understand that this issue emphasizes the extension of (FM) through the introduction of the imaginary unit ie of the variables x and y into (FM), thus characterizing a process of model. This conception is formalized in the institutionalization by the teacher when reporting that, in the epistemological dimension, the sequence modeled by Leonardo Pisano goes through an evolutionary-historical process, which is explored through the class of (PBCF). Let's look at the fifth problem situation.

Table 5

Problem-situation (5).

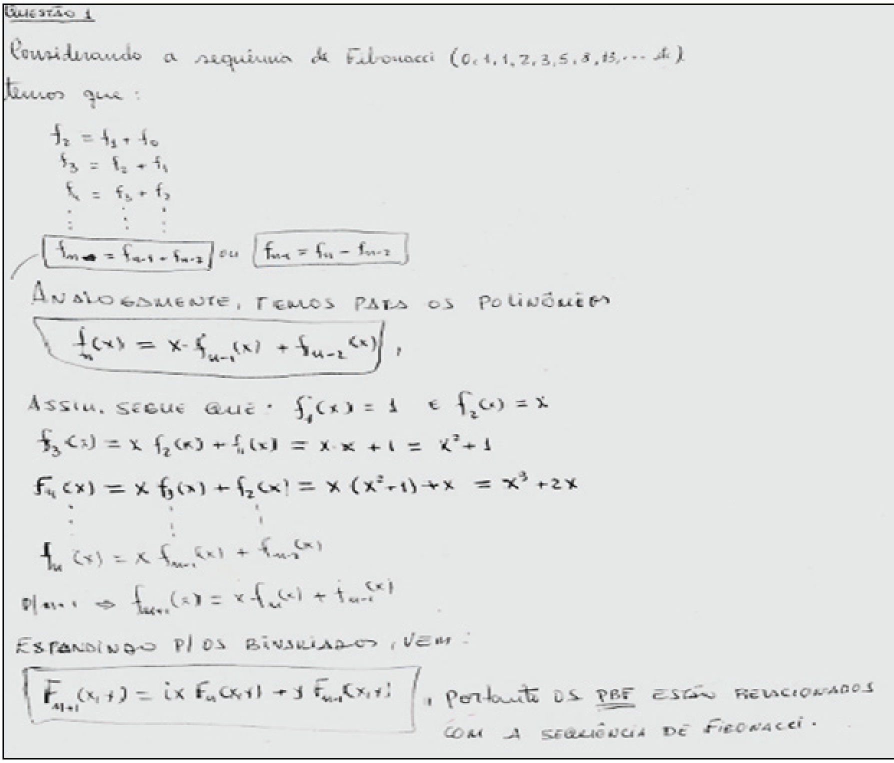
(5) argues and deduces any property of the Fibonacci sequence and that can be related to the model of the Bivariate and Fibonacci Complex Polynomials. Justify this relationship. Consider the equation $x^2 - x - 1 = 0$

The problem situation (5) has the purpose of giving the student an understanding of the mathematical construction of the properties resulting from the complexification of the MF. Thus, at the time of action, students should select a property of the (PBCF) class. Assuming that the properties chosen were $f_{-n} = (-1)^{n+1}.f_n$ and $f_{-n}(x,y) = \frac{-f_n(x,y)}{(-y)^n}$. Thus, to deduce them, the student must use the definition $f_{n+1}(x,y) = ixf_n(x,y) + yf_{n-1}(x,y)$, for $n \geq 1$, arguing that this recursion validates some relations derived from (FM), as for example, we see that $f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \Rightarrow f_n(x,y) = \frac{\alpha^n(x,y) - \beta^n(x,y)}{\alpha(x,y) - \beta(x,y)}$, for $n \geq 0$.

Na fase formulação, uma das sugestões dos alunos deve ser a reescrita das propriedades selecionadas em função, respectivamente, da Fórmula de Binet e de sua fórmula variante. Desse modo, espera-se que essas duas fórmulas sejam verificadas para índices negativos a fim de determinar as propriedades $f_{-n} = (-1)^{n+1}.f_n$ e $f_{-n}(x,y) = \frac{-f_n(x,y)}{(-y)^n}$. Com isso, na institucionalização, deve-se formalizar que a primeira propriedade é intrínseca da SF, enquanto a outra é uma extensão dos (PBCF). A seguir, serão descritos os momentos de aplicação e analisados os dados coletados durante a experimentação.

THE (TSD) IN EXPERIMENTATION AND ANALYSIS A POSTERIORI

The work covered in this article is a research cut that was applied in the Mathematics History course of the Mathematics Degree course of the IFCE. It is worth noting that teaching and learning moments were performed using (TSD) as a teaching methodology. Thus, in this section, the data obtained in the didactic situations were analyzed with a focus on the (TSD) evidencing its stages of action, formulation, validation and institutionalization. In this way, when the problem-situation 1, in the stage of action and formulation, the student A1 (see Figure 3), tried stablish the recurrence relation in the Fibonacci´s sequence for the case $(0, 1, ix, -x^2 + y, -x^3i + 2xyi, x^4 - 3x^2y + y^2, \ldots)$, obtaining the recurrence relation $f_{n+1}(x,y) = ixf_n(x,y) + yf_{n-1}(x,y)$ for every integer $n \geq 1$. Moreover, en virtue to validate the strategie, the student A_2 realized some arguments (see Figure 4) applying the $f_{n+1}(x,y) = ixf_n(x,y) + yf_{n-1}(x,y)$, for every integer $n \geq 1$ from this, the student find some particular elements of the (PBCF).



QUESTÃO 1

Considerando a sequência de Fibonacci (0, 1, 1, 2, 3, 5, 8, 13, ... etc)
temos que:

$f_2 = f_1 + f_0$
$f_3 = f_2 + f_1$
$f_4 = f_3 + f_2$
$\vdots$
$f_n = f_{n-1} + f_{n-2}$ ou $f_{n-1} = f_n - f_{n-2}$

ANALOGAMENTE, TEMOS PARA OS POLINÔMIOS

$f_n(x) = x \cdot f_{n-1}(x) + f_{n-2}(x)$,

ASSIM, SEGUE QUE: $f_1(x) = 1$ e $f_2(x) = x$

$f_3(x) = x\, f_2(x) + f_1(x) = x \cdot x + 1 = x^2 + 1$

$f_4(x) = x\, f_3(x) + f_2(x) = x\,(x^2+1) + x = x^3 + 2x$

$\vdots$

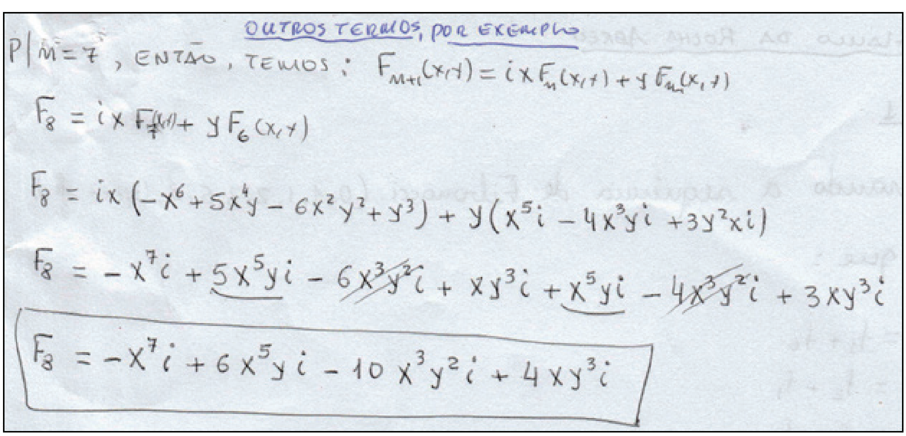
$f_n(x) = x\, f_{n-1}(x) + f_{n-2}(x)$

p/ $n+1 \Rightarrow f_{n+1}(x) = x\, f_n(x) + f_{n-1}(x)$

ESTENDENDO P/ OS BIVARIADOS, VEM:

$F_{n+1}(x,y) = ix\, F_n(x,y) + y\, F_{n-1}(x,y)$, PORTANTO OS PBF ESTÃO RELACIONADOS COM A SEQUÊNCIA DE FIBONACCI.

Figure 3. Stage of action and formulation: relationship between the (SF) and (PBCF).



OUTROS TERMOS, POR EXEMPLO

P/ $n = 7$, ENTÃO, TEMOS: $F_{n+1}(x,y) = ix\, F_n(x,y) + y\, F_{n-1}(x,y)$

$F_8 = ix\, F_7(x,y) + y\, F_6(x,y)$

$F_8 = ix\,(-x^6 + 5x^4y - 6x^2y^2 + y^3) + y\,(x^5 i - 4x^3 y i + 3y^2 x i)$

$F_8 = -x^7 i + 5x^5 y i - 6x^3y^2 i + xy^3 i + x^5 y i - 4x^3y^2 i + 3xy^3 i$

$F_8 = -x^7 i + 6x^5 y i - 10x^3y^2 i + 4xy^3 i$

Figure 4. Stage of validation: determination of particular elments of the (PBCF)..

Thus, in the resolution of the problem situation (2), students encountered a matrix (Figure 2), thus, as an action behavior and in the sense of determining a nomenclature for the matrix proposed by Asci and Gurel (2012), we have the comment:

> *Aluno A_3: [...] Essa matriz proposta por esses estudiosos vão originar os Polinômios Bivariados. A matriz diagonal é aquela que todos os elementos que não estão na diagonal principal são zero. Comparando aqui, essa tem três "camadas" como sendo três diagonais [...].*

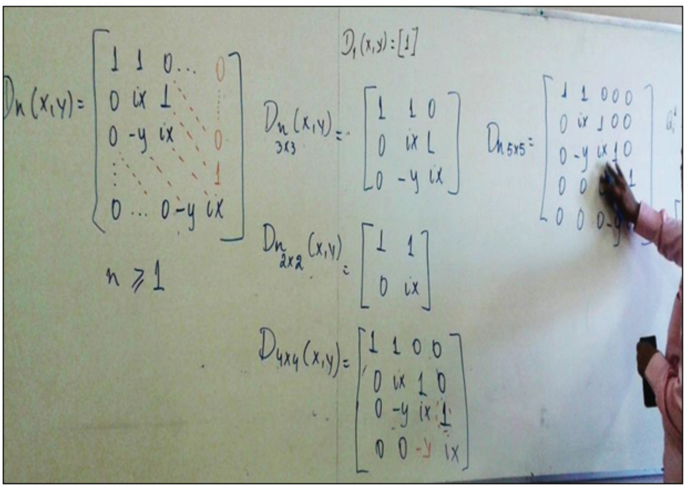


Figure 5. Stage of action: student´s action analysing the tridiagonal matrix.

Moreover, in Figure 5, it can be seen that A3 sought to explore the behavior of the matrix. While, in the formulation, student A1 (see Figure 6) suggests that the determinant of each square matrix is a term of the (PBCF) sequence, thus addressing the theorem det $D_n(x, y)_{nxn} = f_n(x, y), n \geq 0$ with initial condition $D_0(x, y) = 0$.

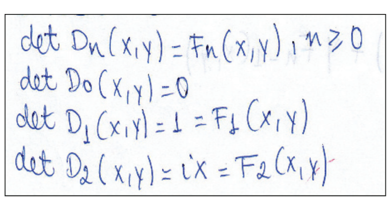


Figure 6. Fase de formulação: relação do determinante da matriz com os (PBCF).

In the validation situation, it was observed that many students were following the construction of mathematical relationships, so some students evaluated the validity of

Theorem 1 by mathematical induction. One aspect of this process is evidenced in the introduction of the inductive step (Figure 7) in the proof of Theorem 1.

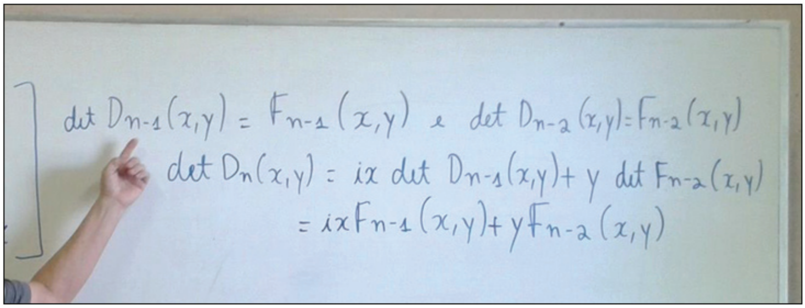


Figure 7. Inductive student´s step for the theorem 1.

The problem situation (3) allows students to explore the Binet formula in the context of two-variable polynomials. Thus, in the action stage, it was proposed to rewrite the Binet formula in the form $f_n(x,y)=\dfrac{\alpha^n(x,y)-\beta^n(x,y)}{\alpha(x,y)-\beta(x,y)}$, for $n \geq 0$. Thus, in the formulation stage, the characteristic equation $t^2 - ixt - y = 0$ was explored in order to obtain relations originating from its roots: $\alpha(x,y)=\dfrac{ix+\sqrt{4y-x^2}}{2}$ e $\beta(x,y)=\dfrac{ix-\sqrt{4y-x^2}}{2}$.

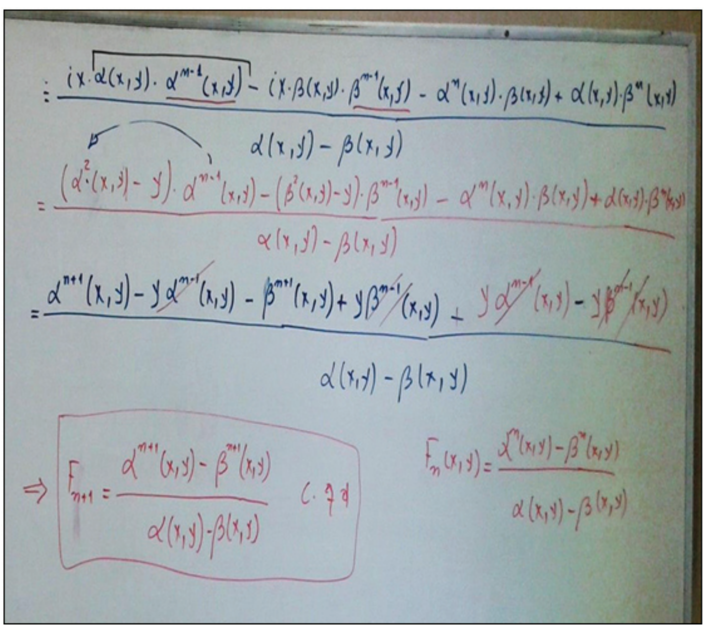


Figure 8. Demonstration of the Binet´s like formula.

In the validation process, the students verified the Binet variant formula for the index $(n+1)$ obtaining $f_{n+1}(x,y) = \frac{\alpha^{n+1}(x,y) - \beta^{n+1}(x,y)}{\alpha(x,y) - \beta(x,y)}$ fot $n \geq 0$. In Figure 8, a mathematical demonstration made by student A1 is explicit.

The comparative scheme between the (SF) and the sequence of (PBCF) was explored in the problem situation (4). In this sense, in the action (Figure 10), student A3 explains that when comparing the columns of the table (see Figure 9) it is possible to consider that the scheme deals with a bibliographic survey of the properties inherent to (FM), representing its extension. Continuing, in the formulation (see Figure 11), student A2 identified the existence of a "historical gap" in the research on the generalization of (FM), observed in the dates of registration of properties.

Descrição histórica	Propriedades dos polinomios bivariados de Fibonacci
$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ Fórmula fornecida por Jacques-Phillipe-Marie Binet (1786 – 1856).	$F_n(x,y) = \frac{(\alpha^n(x,y) - \beta^n(x,y))}{\alpha(x,y) - \beta(x,y)}$ Fórmula variante de Binnet
$g(t) = \frac{t}{1-t-t^2}$ Abraham De Moivre (1667-1754) empregou a noção de função geradora ao modelo de Fibonacci.	$g(t) = \frac{t}{1 - ix \cdot t - y \cdot t^2}$ Função geradora do modelo PBF
$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ Representação matricial estudada em 1960, por Charles King (GOULD, 1981)	$Q(x,y) = \begin{pmatrix} ix & y \\ 1 & 0 \end{pmatrix}$ Representação matricial introduzida por Asci & Gurel (2013).
$f_{-n} = (-1)^{n+1} f_n$ Processo de extensão da SF ao campo dos índices inteiros discutida por Brousseau (1963)	$F_{-n}(x,y) = \frac{-F_n(x,y)}{(-y)^n}$ Processo de extensão da SPF ao campo dos índices inteiros
$f_{n-1} f_{n+1} - f_n^2 = (-1)^n$ Identidade formulada por Domênico Cassini (1625 – 1712)	$F_{n+1}(x,y)F_{n-1}(x,y) - F_n(x,y)^2 = (-1)^n y^{n-1}$ Fórmula variante de Cassini fornecida por Asci & Gurel (2013).
$f_m \backslash f_n \Leftrightarrow m \mid n$ Caso de divisibilidade relacionadas com a SF	$F_m(x,y) \backslash F_n(x,y) \Leftrightarrow m \mid n$ Divisibilidade dos PBF introduzidas por Jacob, Reutenauer & Sakarovitch (2006)
$mdc(F_n(x,y), F_{n+1}(x,y)) = 1$ Caso de divisibilidade relacionadas com a SF	$mdc(F_n(x,y), F_{n+1}(x,y)) = 1$ Divisibilidade dos PBF introduzidas por Jacob, Reutenauer & Sakarovitch (2006)
$\begin{cases} H_0(x,y) = a_0, H_1(x,y) = a_1, \\ H_{n+1}(x,y) = x \cdot H_n(x,y) + y \cdot H_{n-1}(x,y) \end{cases}$ Polinômios bivariados introduzidos por Catalani em 2004.	$\begin{cases} F_0(x,y) = 0, F_1(x,y) = 1, \\ F_{n+1}(x,y) = ix \cdot F_n(x,y) + y \cdot F_{n-1}(x,y) \end{cases}$ Forma complexa dos PBF introduzidos por Asci & Gurel (2012; 2013).

Figure 9. Comparative scheme between (FS) and the sequence of (PBCF). (Alves & Catarino, 2017).

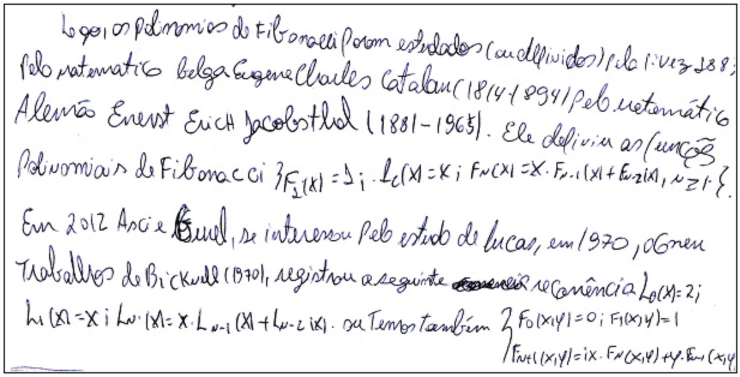


Figure 10. Stage of action: historical-evolutionary interpretation of (FM).

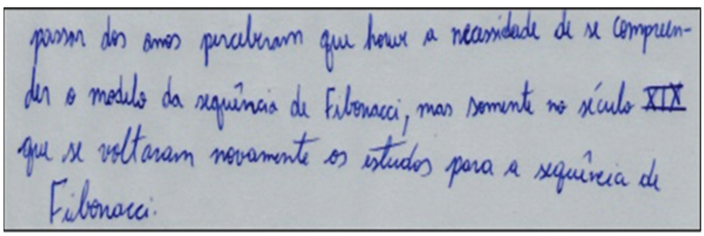

passar dos anos perceberam que houve a necessidade de se compreender o modelo da sequência de Fibonacci, mas somente no século XIX que se voltaram novamente os estudos para a sequência de Fibonacci.

Figure 11. Stage of formulation: the record of a "historical hiatus" of (FM).

Thus, in order to validate the elaborated arguments, the student A5 explained that in the history of the concepts of (SF) there is an evolution evidenced with the insertion of the imaginary unit ie of variables in (FM) and the (PBCF) represent a generalized typology of the model in complex form. This conception is emphasized as follows:

> *Aluno A_5: [...] a fórmula da primeira é muito básica, elementar. Na segunda ele aperfeiçoou o que já existia, acrescentou as variáveis, agora dá para trabalhar com polinômios a partir dessas fórmulas. Na última linha, ele já trabalhava com polinômios só que aí no Fibonacci, ele vai trabalhar nos complexos. Sempre tendo uma extensão [...].*

Concluding the sequence of activities, in the problem situation (5), in their action phase, the students resorted to Figure 9 to select the properties. Thus, some students decided to explore the identities $f_{-n} = (-1)^{n+1}.f_n$ and $f_{-n}(x, y) = \dfrac{-f_n(x, y)}{(-y)^n}$.

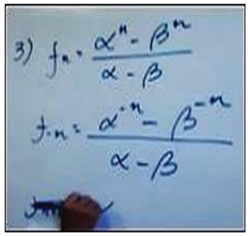


Figure 12. Phase of formulation for the Binet Formula.

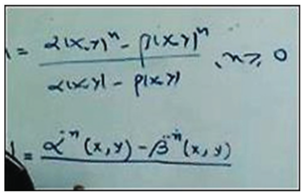


Figure 13. Formulation phase for the Binet variant formula.

In this context, in the formulation (Figures 12 and 13), the students explored the Binet formula and its variant in order to obtain the selected identities. Finally, in validation (Figure 14), students were able to demonstrate the two formulas for negative indexes proposed by the teacher-researcher.

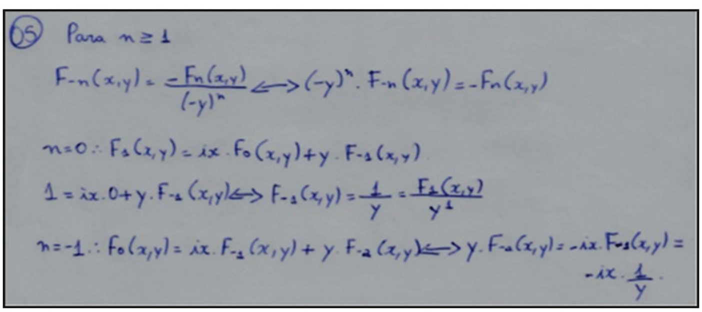


Figure 14. Fase de validação: propriedade inerente aos (PBCF).

Finally, it is worth commenting on the institutionalization phase made by the teacher-researcher at the end of the resolution of each problem situation. At the moment, the students' productions are analyzed in order to organize, through generalizing reasoning, the knowledge constructed. Thus, in the problem situation (1), it is possible to understand the (PBCF) class as a generalized form of (FM), in situation (2), we can validate Theorem 1 and in situation (3), if one Binet variant formula for the (PBCF). In the situation (4), the (FM) complexification process was formalized with emphasis on the (PBCF) class from the insertion of the imaginary unit i and variables. In the latter situation, the properties

(Figure 9) of the (PBCF) are generalized as an extension of (FM). From now on, the validation of this research will be discussed.

INTERNAL VALIDATION

The validation of the research is done internally, according to (DE), through the results obtained from the analysis of the application of didactic situations. And, it is worth mentioning that a comparison of the data collected in this research with productions external to this application was not made. What characterizes the internal validation of this research. Thus, as the work presented here is a part of a research, it can be concluded that there was validation of the didactic hypotheses specific and inherent to the study of the (PBCF) class, which can be evidenced through the construction and learning of a bivariate polynomial generalized model (SF), with emphasis on the mathematical validity of properties and theorems investigated in the classroom.

In view of this, the (TSD) was a relevant factor for the validation of this research. In an epistemological perspective, it is possible to emphasize that the institutionalization with focus in the (TSD) made by the teacher, evidenced the success in the construction of the relations originating from the (FM) through mathematical demonstrations. However, in the internal validation, besides the investigation of definitions and properties of the (PBCF), were considered cognitive and didactic aspects.

At the didactic level, the didactic situations provided an opportunity to understand the process of construction of properties and theorems, with the mobilization of the generalizing reasoning, and the understanding of the SF historical evolutionary process. Thus, in terms of cognition, it was observed that during the phases of the action, formulation and validation of (TSD), some students were able to evolve by developing an inferential reasoning. Finally, it is understood that, in fact, the Mathematical History classes, where the research was applied, were approached in an epistemological perspective with respect to the (FM) complexity process with emphasis on the (PBCF) class. This was possible due to the effectiveness of the didactic contract between the teacher and the students in face of the didactic situations.

FINAL CONSIDERATIONS

This paper presents a research study carried out in the Master's Program in Teaching Science and Mathematics (PGECM-IFCE) and investigated the mathematical, epistemological and historical process of Fibonacci model complexation. However, it was restricted to the class of (PBCF). It is understood that the mathematical definitions and theorems explored in this research have their origin in Pure Mathematics, so it was necessary to perform a didactic transposition (Chevallard, 1998) of these mathematical concepts for the pedagogical plan, in order to be taught to the students (teachers in initial formation) in the course of Degree in Mathematics.

In view of this, it can be observed that the didactic situations applied with a focus on (TSD), incorporated into (DE), made it possible to carry out a learning process of mathematical relations by implicitly following the structure of a mathematical demonstration, and mobilizing a inferential reasoning (Alves; Alves Dias, 2017). Thus, it was understood that the construction process of the Bivariate Polynomial and Fibonacci Complex model represents a generalization of the Fibonacci model, evidencing its historical and evolutionary process. This also allowed for the development of an epistemological conception in the teaching of the History of Mathematics. This conception is associated to the epistemic-mathematical field, that is, to the study of the algebraic structure of a non-trivial mathematical model. This is little detailed in Mathematical History classes, which are mostly limited to discussing the biographies of mathematicians and the historical context in which generalized relations have emerged.

Finally, we advocate a robust perspective of the mathematics teacher, in that he acquires an in-depth mathematical culture, which is not restricted to the style of rhetorical information, which conveys conceptions about the teaching of mathematics, without explicitly distancing itself from the interests and classroom activities. In this case, an epistemological, historical and evolutionary understanding of the Fibonacci sequence or model, since it is nowadays a branch of interest and contribution of several specialists in other countries, can provide an expanded and in-depth view on a sequence that, standard, we see a dossier adopted by the authors of Mathematical History books prioritize their anecdotal, picturesque character and conveys little more than the curious significance of the production of immortal rabbits. With origin in Figure 15, we invite an appreciation on the mathematical, epistemological process evolutive of the Generalized Sequence of Fibonacci (Oliveira, 2018).

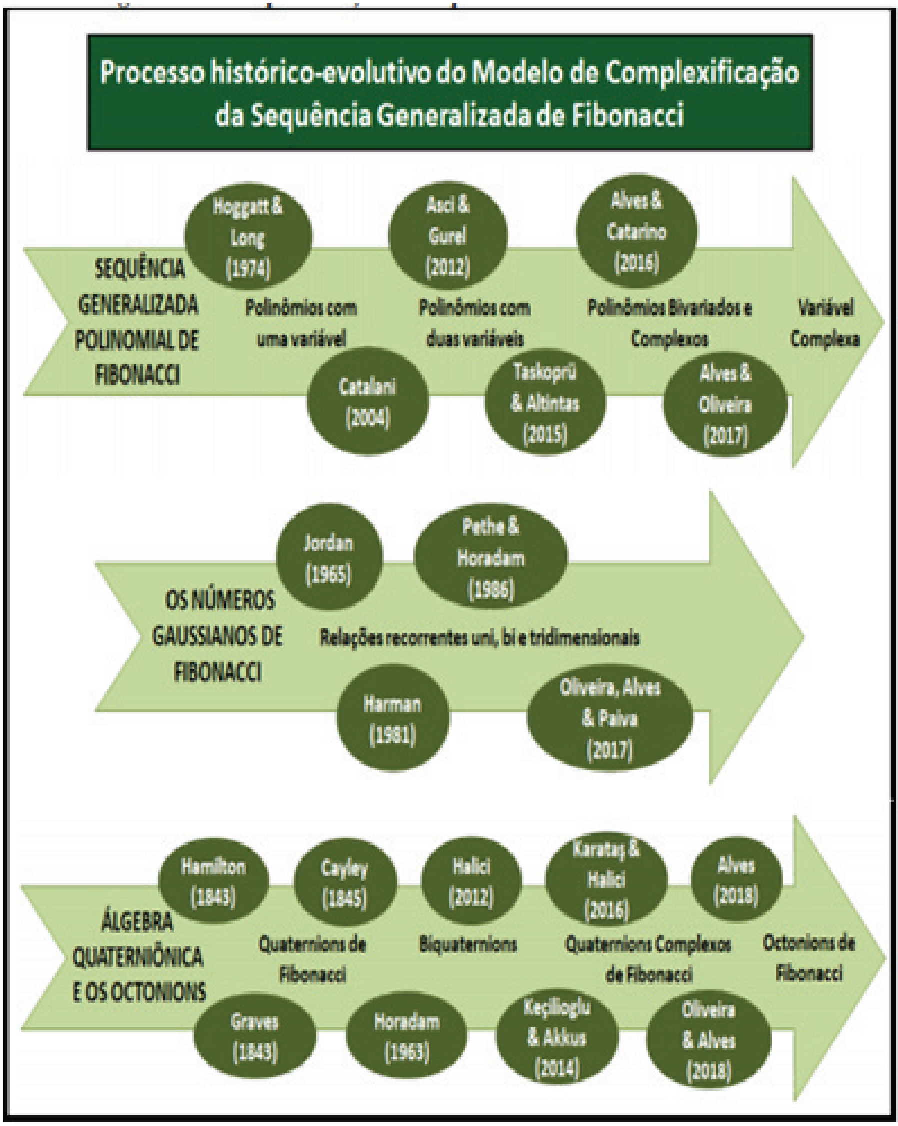


Figure 15. Description of the epistemological, historical and evolutionary process of the Generalized Fibonacci Sequence (Oliveira, 2018).

AUTHOR CONTRIBUTION STATEMENTS

F.R.V.A. oversaw the project. R.R. de O. and F.R.V.A. conceived the idea presented. R.R. de O. developed the theory. R.R. de O. adapted the methodology to this context, created the models, executed the activities and collected the data. F.R.V.A. and R.R.

de O. analyzed the data. Both authors discussed the results and contributed to the final version of the manuscript.

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