Use and Abuse of Calculators: Implications for Mathematics Education

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ABSTRACT

Fostering mathematical understanding and sense for numbers is an objective for current mathematics education. This paper presents a historical vision of the use of numbers and of calculation processes and reflects about the current need of electronic calculators. The coexistence of two different mathematical systems, one based on the Greek axiomatic mathematics and another, the floating point system, used in electronic calculators, can lead to some confusions that which affects the teaching of mathematics, especially in the numerical sense development. The paper also presents common mistakes caused by the missing verification of the results from calculations or lack of awareness about the limitations of calculators. Finally, the implications those elements have on the mathematical education are presented together with some conclusions.

Keywords: Number sense. Calculators. Mathematics education. Scholar calculation.

Uso e Abuso de Calculadoras: Implicações para a Educação Matemática

RESUMO

Promover a compreensão matemática e a noção de número é um objetivo da educação matemática atual. Este artigo apresenta uma visão histórica do uso de números e de processos de cálculo e reflete sobre a necessidade atual de calculadoras eletrônicas. A coexistência de dois sistemas matemáticos diferentes, um baseado na matemática axiomática grega e outro, o sistema de ponto flutuante, usado em calculadoras eletrônicas, pode levar a algumas confusões que afetam o ensino de matemática, especialmente no desenvolvimento do sentido numérico. O artigo também apresenta erros comuns causados pela falta de verificação dos resultados de cálculos ou falta de consciência sobre as limitações das calculadoras. Finalmente, as implicações desses elementos na educação matemática são apresentadas, juntamente com algumas conclusões.

Palavras-chave: Noção de número. Calculadoras Educação matemática. Cálculo escolar.

HISTORY

Knowledge of numbers, when they refer to a collection of real objects, has always been considered a characteristic that animals possess. A bird will protest if the number of eggs in the nest is less than what was there before. Jokingly, Professor Sancho Guimerá, one of the promotors of the Faculty of Mathematics at the University of Salamanca,

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would always use his dog, Pinto, as the example of a good mathematician: if there were three sausages on one plate and two on another, Pinto would always go for the former. However, the concept of cardinal numbers is not something tangible at all, but rather abstract: it is not possible "*to have the number 3 in your pocket*", although you can "*have 3 sweets in your pocket*". That is why this concept has not been linked naturally to human intelligence, even though it has always been associated with us. The truth is that it took many years, thousands of years, for this concept to become rooted in the human mind and accepted by it, and more complex numerical sets, such as whole numbers, took even longer. For example, the possibility of comparing different collections of objects is one of the references of human evolution. Bertrand Russell explained "it must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number two". However, when humans managed to abstract the concepts of number and the different number systems, two different mathematical systems were developed (Mora, Mora-Pascual, García-Chamizo, & Signes-Pont 2017; Nagar, 2018).

One of these was based on Greek axiomatic mathematics, in which numbers are "geometrized" and their properties studied for that purpose. The appearance of Pythagoras' theorem for right-angled triangles led to the discovery of strange numbers: if the two smaller sides of this type of triangle measure 1 cm each, the longer side has a length of $\sqrt{2}$ cm, an amount that cannot be expressed in the form of a fraction. This led to the concept of irrational numbers, numbers that cannot be expressed as a quotient (in other words, those whose denominators are not powers of 10). This posed a new situation since $\sqrt{2}$, and similar numbers, could not be measured, running counter to the precision that other known numerical systems allowed. This circumstance led to the consideration of two groups of number sets: on one hand, the logical set of rational numbers, containing natural numbers and fractions, and on the other, the illogical set of irrational numbers, containing roots such as $\sqrt{2}$ and other numbers such as π and e. This classification has entailed diverse everyday arithmetic problems, which we shall take a look at later.

The other mathematical system is that of floating point numbers, originally developed in 499 AD in India by the mathematician Aryabhata. In this system, numbers are obtained by measurement, and the strange ones, such as $\sqrt{2}$, were approximated to any level of accuracy required by taking as the value an expression in powers of 10. For example, $\sqrt{2}$ = 1414/1000 would be a close enough approximation for the building of houses or temples. This is the floating point system, the one used in electronic calculators and computers. For example, $\sqrt{200000}$ would be written as 44721/100, that is, 4.4721x10^-2. The word "floating" is used because the -2 index signals the level of precision taken: if we needed greater precision, we could write $\sqrt{200000}$ = 4.472136x10^-4. In this system, it is not possible to reach total precision but neither is it necessary for practical objectives.

Matching up these two systems is somewhat problematic given the current use of electronic calculators. And this is even more important in the case of Mathematics teaching at elementary level, an aspect that constantly comes up if the objective is to adapt the education of all citizens to their future daily life. Although readers will be familiar with many other examples, the ones we describe here are used to illustrate what we mean. It seems that, from the perspective of Greek mathematics, electronic calculators produce strange results. How does this circumstance affect the teaching and learning of mathematics, the resolution of arithmetic problems or the development of numerical sense?

THE CURRENT SITUATION

a) In real life:

Example 1 (outside): One day, a housewife was affably conversing with Pete, the neighbourhood butcher, whose shop she visited frequently. She bought similar things there most of the time, and thus the price never varied much; she usually spent around 25 Euros. On this day, however, when buying an amount that was not unusual for her, she had the following exchange with Pete:

'That'll be €29.58'.

'But Pete, it can't be! I've bought what I always buy. You must have made a mistake!'

'Well!' said Pete, pointing to the electronic scale with a calculator. 'Calculators don't make mistakes!'

'Then you must be right – it's me who's confused. I must have bought more of something than I usually do'.

But when she got home and looked at the receipt, she saw that Pete had pushed the button on the calculator twice when weighing the chicken breasts and therefore €4.22 appeared twice! Even though she was sure he had not done it on purpose, and she was also certain that the calculator had not made a mistake, an error had taken place. This happened in a Spanish butcher shop but it could have happened anywhere.

Example 2 (the student): In a classroom, a student is working with mixed numbers but considers that 2 ¼ plus 3 2/9 are, respectively, 2 ¼ = 2 × ¼ = ½ and 3 2/9 = 3 ×2/9 = 6/9 = 2/3. A result of the anarchy in Mathematics conventions! This happened in a British classroom but could have happened anywhere.

Example 3 (the teacher): In a state Mathematics examination for 16 year olds, the examinees were told to take 22/7 as the value of π. A subsequent question asked the students to identify which numbers, from among the following, were rational: {2.4; π; 27/8; √2}. Which one or ones should the students have chosen? This happened in the UK but could have happened anywhere.

b) With the calculator:

In Dynamic Geometry, computers obtain results that are impossible with Euclidean Geometry. Something similar occurs with electronic calculators, which also produce strange outcomes. Some examples are given below [using a SHARP model EL-509A electronic calculator]:

Example 1: Use the calculator to find the following series of square roots:

$\sqrt{4} = 2$ $\sqrt{2} = ?$ $\sqrt{\sqrt{2}} = ??$ $\sqrt{\sqrt{\sqrt{2}}} = ???$ $\sqrt{\sqrt{\sqrt{\sqrt{2}}}} = ????$

It is easy to observe that $\sqrt{4} = 2$

$$\sqrt{2} = 1.4142136$$

$$\sqrt{1.4142136} = 1.1892071$$

$$\sqrt{1.1892071} = 1.0905077$$

$$\sqrt{1.0905077} = 1.0442738.$$

However, if we write this same number on the screen and perform the reverse operation, we have

$$(1.0442738)^2 = 1.0905078$$

$$(1.0905078)^2 = 1.1892072$$

$$(1.1892072)^2 = 1.4142138$$

$$(1.4142138)^2 = 2.0000005.$$

That is, the calculator is not making correct calculations in the two cases because different results are obtained: a number close to 2, above or below it, is obtained, but it is not exactly the number 2. Thus, what we have is only an approximation that goes against many elementary mathematical principles.

Example 2: Calculate (1/3 + 1/5) + 2/3.

This can be done two ways. One of them is to enter the fractions directly in the calculator as their decimal equivalents, that is: 1/3 = 0.333333; 1/5 = 0.2; 2/3 = 0.666666. The sum of these gives an answer of 1.2.

But it can also be calculated indirectly by manually entering these same decimal representations first without referring to the fractions they represent (that is, 0.333333; 0.2; 0.666666). In this case, the answer given is 1.1999999.

Although this may seem a bit contrived, the use of truncated limitless decimal representations has a significant impact on all of Greek mathematics.

Example 3: Calculate $(\pi$^$1.2)$^$(5/6)$.

Since the calculator being uses does not allow fractions to be raised to a power, the value of 5/6 is considered and stored in the memory. Then π ^1.2 is calculated, and elevated to the number stored in the memory. The answer it gives is 3.14159. However, the answer should coincide with the calculator's value of π: 3.1415927.

Therefore, the properties of the indices are not fulfilled.

RECOGNIZED POINTS OF VIEW

For a long time it was mistakenly thought that mathematical intelligence was exclusive to the human species and that animals were only guided by instinct. But in relatively simple circumstances, a domesticated animal such as a dog, cat, monkey or elephant rapidly perceives if an object disappears from a small familiar set. In a certain number of species, the behaviour of mothers shows that they know when one of their offspring has been snatched away.

This situation shows that this kind of behaviour is conscious and that the notion of number is not totally foreign to animals. They have a natural disposition that allows them to recognize when a small number of objects, perceived a second time, has undergone a change. More specifically, this ability has been observed in certain birds subjected to prior training? Many ingenious experiments have shown that a goldfinch, taught to choose its food from between two little piles of grain, generally learns to distinguish three from one, three from two, four from two, four from three, and six from three. Even more striking is the case of nightingales, magpies and crows, which, without previous training, are clearly able to recognize specific amounts ranging from one to three or four. Let us look at a famous example:

"A man decided to kill a crow that had made its nest in the watchtower of a castle. He'd tried several times to surprise the bird, but every time he got close, the crow would leave the nest and perch on a nearby tree, returning when the man had left. One day the man decided to resort to cunning: he made two other men enter the tower and after a short time one of them left and the other remained. The crow, far from being taken in by this trap, waited until the second man left before returning to the nest. The next time three men entered, and two of them left after a few minutes, while the third one stayed behind as long as he wanted waiting for an opportunity to trap the crow. The crow, however, turned out to be more patient than he was. This happened several more times, always without success. Finally, the trick worked using four or five people, because the crow was unable to visually recognize the presence of more than three or four humans at the same time" (Ifrah, 1998, p.38).

In another vein, Duea, Immerzeel, Ockenga and Tarr (1980) explained an experiment in which students were given 20 minutes to solve problems similar to ones appearing in their textbook. After collecting their papers, the teacher was surprised to observe that almost all the students had completed more than 26 of the 40 problems posed. Even more surprising was the fact that more than 90% of the answers were correct. This outcome was quite different to what had happened in other classrooms in similar circumstances. What had changed? Why were these students so successful in solving these problems? The answer: each of them had used a calculator!

When adults have to do any kind of important arithmetic they use some type of calculator. Likewise, students can use it to do calculations without the inherent difficulties of working with pencil and paper, and thus can concentrate on the problem solving process. In this way their use of a calculator is not an end in itself, but rather a means to be able to place more emphasis on the process than on the final outcome. This allows them to be unconcerned about accessory difficulties and therefore everyone can do it! (Koay, 2006, Lee, 2016; Sánchez & García, 2009).

IS IT ENOUGH TO VERIFY WITH CALCULATION?

It is thought-provoking that today computers and calculators are considered important tools for doing calculations and it seems that many jobs and occupations could not be carried out as they are now without them. The general public consider that these machines know the most about mathematics and never make mistakes. People who use them in their work seem unable to live without them. People who use them in their work could not live without them. Nowadays, all mobile phones incorporate calculators among their basic applications and users often use it for to perform any calculations, however simple it might be. How can this be if computers do not really know numbers? Of course, this is not completely accurate because they do know the natural numbers, whole and rational numbers, but they do not know real numbers. Also, possible programming errors are not being taken into account, as in the case of the first batch of Pentium computers, errors which nobody would have known about if it had not been for a mathematician who discovered them by chance when using large numbers to verify certain properties. What is certain, however, is that they are instruments whose results cannot be taken as infallible by the general public.

They have also become an everyday working tool for many mathematicians. It has been possible to demonstrate different theorems thanks to the help of powerful computers working over a certain period of time to arrive at a solution (Paulson, 2018; for example, by Apple and Haken, in 1976, to approach the whole range of possibilities and solve the four-colour theorem or the mathematician who discovered the highest prime number ever known and explained that he has done it while he slept). Are the proofs absolutely valid and rigorous even if they are not the most elegant and brilliant? That is, is verification sufficient or is a deductive proof also needed?

Let us take a look at the following investigation for students inspired by John Wallis (17^{th} c.):

$(0 + 1)/(1 + 1) = ½$

$(0 + 1 + 2)/(2 + 2 + 2) = ½$

$(0 + 1 + 2 + 3)/(3 + 3 + 3 + 3) = ½$

In the classroom, one verification can be sufficient when carried out in different situations. However, proving it for 10, 100 or even 1000 cases is not enough in the world of modern mathematics; the way of proceeding is different to the empirical deduction of the natural sciences. In the latter, a particular series of observations of a certain phenomenon makes it possible to establish a general law that governs all the possibilities of the phenomenon. For example, based on a finite number of observations it has been verified that in the morning the Sun comes up in the East, and therefore tomorrow it will also come up in the East. However, in the mathematical sense it is not possible to confirm a general law after verification of a finite number of cases, large as that number may be, and even if no exception whatsoever is known. It would still be a hypothesis subject to possible modifications.

We think that the discrepancies between working with numbers in a theoretical way as opposed to with a calculator come from using knowledge that in turn derives from different origins and training: the Indian and the Greek traditions. The differences in the epistemological orientation of these two traditions coincide directly with the question of which Mathematics should be taught: the Mathematics that requires a rigorous mathematical proof or the one that only needs a verification of calculation. Furthermore, there are "differences of knowledge" between these two traditions: in Indian mathematics there is no conflict between the visual proof and verification by calculation, on the one hand, and the proof by deduction, on the other (Joseph, 2015). We consider that this reflection can be of value for mathematical educators and philosophers.

On the other hand, today a profound debate is going on in the educational world as to the need or not to carry out proofs in the maths classroom. For example, the official Spanish curriculum advises that they should not be addressed during the stage of compulsory secondary education (up to age 16), which clashes with an educational tradition based on the teaching-learning of mathematical theorems, accompanied by their own proofs, and concepts. This has provoked different reactions of perplexity and doubt among teachers, at the least, which links in with muffled resistance in the classroom, in some cases, or with submissive obedience without personal considerations, in others, and, in general, wide disagreement. As a result, the teaching of much of the contents is done intuitively and by approximations. Students often run into difficulties when they move on to the Upper Secondary stage, which works with rigorous mathematical proofs.

The large number of proofs that aspiring professional mathematicians have to understand, study and in many cases reproduce when completing their university degree, and even until not so long ago, when finishing upper secondary education, can lead us to consider whether or not this background of knowledge is necessary for a citizenship or professional education in the case of all students. Undoubtedly, for professional mathematicians, knowing the proofs of many theorems can help them to understand the historical process in which they were created, go deeper in to the concepts themselves and work on other contents in a similar way, even though this does not always happen. However, this can be questionable in the case of pre-university students, who have been forced to learn a large number of theorems together with their rigorous proofs, in many cases entirely through memorization with no understanding of the content, abetted by the general permissiveness of the teachers. As a result, the learning does not remain with the student. This rote learning, completely detached from the students' interests, was sorely in need of some kind of change.

Another important circumstance is that many students start university with a profound ignorance of the differences between proof and verification, and generally, of what mathematical reasoning is all about. We believe that these aspects are fundamental for any person's all-round education and future life as a citizen, whatever that may be. That is why we assume that the official recommendations are more in the line of not working on "finished" rigorous mathematical proofs of theorems, knowledge of which is neither useful nor sensible even for many professional mathematicians. This does not mean drawing away from rigour and logical reasoning but rather the opposite, it should be coupled with an individual formalization of everything that is addressed in the mathematics classroom. If each student can manage to reason out the knowledge that he or she is constructing, personal and more meaningful learning will be possible. This is not utopian thinking if we take into account what Thornton (1998) said based on a dialogue between a mother and her two year old son: that reasoning can be done at almost any age (more examples in Chamoso, González, Hernández, & Martín, 2013):

> Child (very offended): Jack broke my car!
>
> Mother: Oh, I'm sure he didn't...
>
> Child: He did do it! He did! Harry didn't go there (to the playroom) – Jack broke my car!

It is also important to remember that the use of proofs in the mathematics classroom varies from one level to another and in different situations, but its main objective is to help students understand that it is necessary to confirm through reason the different types of knowledge that are being addressed in that classroom. Using proofs demonstrates to the students that what they are learning makes sense and has a logical development, something that can often be extrapolated to daily life.

A POSSIBLE SOLUTION

How can we get electronic calculators to be used correctly in society? Perhaps the solution can be found in history. Before they became available, human calculators did non-trivial arithmetic calculations: a "calculator" symbolizes a person who had learned how to do precise arithmetical calculations. In the Middle Ages, the Persian mathematician Al Kashi wrote the book entitled "The Key to Calculators", which collected the principles of calculation that were used in the Hindu number system, including square roots. Subsequently, Simon Stevin, in the Netherlands, introduced the decimal number system into Europe in the seventeenth century. A calculator had to master and control the calculation techniques that were in these books. That person would not have been concerned with rigorous precision because, if he had been, he would have known that some quantities do not have an exact form in the decimal system. A calculator who only followed the dictates of Greek mathematics would only be useful for theoretical Geometry. In short, the recipe for a good mediaeval calculator would be: familiarity with the principles of calculation and a certain independence from the hegemonic principles of Greek mathematics.

Let us consider this idea of calculator as a solution to the calculation problems found in society today. To do so we suggest a new updated version of the book "The Key to Calculators". This key not only takes into account the principles of arithmetic calculation, but also the development of number sense and how it transforms under different actions. We therefore propose, first of all, a modern key to calculators that not only consists of learning the principles of arithmetic calculation but also includes something more important: the development of number sense and the knowledge that numbers can change according to how they work in different arithmetic situations. Clearly, Pete in our example above did not have number sense because he was not aware that the number 4.22 was entered into the electronic calculator in a strange way. It is obvious that this number sense needs to be inculcated at an early age in the same way language is. The logical thing would be to develop this number sense in students before they begin to handle electronic calculators, for the simple reason that the meaning of the result of each operation carried out by the calculator can be confirmed by the user if he or she has number sense.

This proposal is framed within our idea that teaching mathematics should not resemble a transmission of knowledge in the form of an oracle. Calculators would be accepted in this way if their results were accepted as dogma, with no critical reactions or questions from students. However, the most up-to-date ideas on teaching encourage a more personal and meaningful mathematics, such that each result is the final point in a process of individual construction which is meant to be individual even when others have gone through a parallel process at one time or another.

Let us take a look at the legacy left by the Greeks. It is accepted that current mathematics is still influenced by traditional Greek mathematics, as is school maths in some countries. This must be kept in mind when considering mathematics teaching. However, when teaching mathematics in school we differentiate between theory and pragmatism, that is, it is necessary to differentiate between abstract mathematics as a discipline and the practical maths needed for coping in society. What do we mean by this? Perhaps the following considerations can help to make sense of these ideas: the area of a triangle is obtained using a simple and well-known formula, but how many pieces of land are in the form of a perfect triangle? Approximate methods of measurement are needed. The square root of 2 is irrational and cannot be expressed exactly, but for the usual financial calculations it is enough to have an approximation of its value (it really is not possible to extend the tools).

It must also be understood that number sense cannot be acquired solely by memorizing rules. And this sense will be difficult to retain when handling concepts that throw things out of joint, such as fractions, ratios or percentages, as we have seen. In fact, in Mathematics there are very few independent concepts. That is why a teaching based on the establishment of connections between apparently disparate concepts is fundamental for better and more meaningful learning. The equivalency among the following expressions would be an example of what we are talking about:

$3/5 \times 10 = 30/5$ 3:5 of 10 is 60% of 10 = 6

$9 + 3 = 12$ $9 - - 3 = 12$ $3 - -9 = 12$

Thus, enabling students to understand the connections among the different arithmetic concepts can make it easier for them to acquire number sense in an integral and homogeneous way.

Another aspect that should be given importance is the way in which an arithmetic expression is entered into the calculator. For example, if we consider 4^3/2, the answer is 8 but if we enter it directly into the calculator we obtain 32 (4 raised to the 3rd power and divided by 2). This is due to the hierarchy of mathematical writing which, in many cases the calculator does not permit. This hierarchy of arithmetic operations is expressed with several mnemonics in different countries, for example, BODMAS for remember Brackets, Order, Division/Multiplication, Addition/Subtraction in the UK, India and Australia, BEDMAS for Brackets, Exponents, Division/Multiplication, Addition/Subtraction in Canada and New Zealand or PEMDAS for Parentheses, Exponents, Multiplication/Division, Addition/Subtraction in the United States (Bay-Williams & Martinie, 2015).

These rules seem to have been forgotten in the face of the rapid and widespread use of calculator technology, but an effort must be made to avoid errors such as the one mentioned above. Moreover, when teaching students how to use calculators, emphasis must be placed on their limitations so that users can be aware that the answer they get may not always be correct (especially when working with fractions and irrational numbers, examples of which we have seen above).

Thus, since universities differentiate between the pure mathematics of Greek influence and the applied mathematics of everyday life, we should also make this distinction in the school curriculum. Nobody doubts that all schoolchildren need a mathematical training that they will use in their ordinary life, some of them, in addition, will need formal mathematics for further scientific training. This is one of the recommendations that have been given for years now in guiding the different school curricula reforms in different countries (Hayes, 2017). But these indications tend to remain on the theoretical plane and no guidelines are offered as to how to put them into practice. This may be the result of an overall philosophical shortcoming in the general directives for teaching mathematics. It generally comes to light in particular interpretations seeking to improve numeracy (Yasukawa, Rogers, Jackson, & Street, 2018). In most cases, instead of fostering understanding, the latter are based on strengthening the development of routines in an attempt to improve exam outcomes, and number sense cannot be developed in this way. This development in the teaching of mathematics without a general philosophical foundation is widespread in most countries with one exception: Hungary.

Since the beginning of the twentieth century, most Hungarian schools have followed a philosophy of social construction of mathematical knowledge in the classroom based on a high level of student interaction. This teaching model is applied in most mathematics classrooms in Hungary. Furthermore, according to the University of Plymouth's Centre for Innovation in Mathematics Teaching (CIMT) (England), the use of specific materials to explain concepts in Hungarian Mathematics classrooms at all levels means that their students attain more advanced learning than in other countries, although more research about it would be necessary. This helps to understand why Zoltan Dienes, the most famous maths teacher in Hungary, wanted children to be taught in this way from a very early age: an advanced number sense could facilitate the future use of the usual concepts of infinitesimal calculus.

FINAL CONSIDERATIONS

Mathematics are understood as a social construction influenced by the changes in society (Restivo, 2017). This is why the tremendous technology development in the last decades shall have repercussions on how mathematics is taught (Prodromou & Lavicza, 2017). Calculus is perhaps one of the most affected fields. This is an important aspect because computers are based on floating point system, which strongly differs from the numerical system used in traditional pure mathematics.

The benefits of technology cannot be disregarded and therefore calculators and computers are here to stay. They have great advantages, one of lesser importance being that of facilitating rapid calculations in complicated operations. These instruments also entail disadvantages, such as the fact that if they are not used critically they can breach the most elemental mathematical laws. Their disadvantages, however, should not impede future generations from taking an effective and positive use of the new technologies only due to an insufficiently tailored Mathematical Education. Otherwise, we might witness a metaphorical repetition of the burning of Parliament.

After many attempts, the British Parliament finally accepted the Hindu-Arabic number system. This took place in the eighteenth century. At that time, the “tally sticks” used to count and which accumulated in the accounting offices were burnt in furnaces located in the basement of the Palace of Westminster. One time the fire got out of control and the Houses of Parliament burnt down (Dantzig, 1955, p.23). Perhaps in our day it is another kind of “tally stick” that is attaining great influence.

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