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The Consolidation of Rules of Signs and Stages of the Scientific Spirit in Bachelard

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ABSTRACT

The purpose of this paper is to analyze the historical path of the consolidation of the rule of signs from Gaston Bachelard's epistemological perspective, as well as explore the epistemological obstacles still present in the teaching and learning processes of such rule nowadays. The consolidation was a slow and surprising process, marked by advances and setbacks. We suggest herein the presence of three scientific stages of mind: concrete, concrete-abstract, and abstract. It is possible to realize that the two first stages related to the development of the rule of signs are still very present in pedagogical activities and teaching. However, some studies have been indicating that teaching such rule formally, i.e. avoiding metaphors related to concrete examples, can stimulate the transfer from the concrete to the concrete-abstract spirit, and later on to the abstract state of scientific spirit.

Keywords: Steps of the Scientific Spirit. Rules of Signs. Teaching and Learning Negative Numbers.

A Consolidação das Regras de Sinais e as Etapas do Espírito Científico em Bachelard

RESUMO

Este artigo analisou a trajetória histórica da consolidação da regra de sinais na perspectiva epistemológica de Gaston Bachelard, bem como os obstáculos epistemológicos que ainda persistem no processo de ensino e aprendizagem dessa regra na atualidade. A sua consolidação foi um processo lento e surpreendente, marcada por avanços e retrocessos. Propusemos, nesse movimento, a presença dos três estados do espírito científico: concreto, concreto-abstrato e o abstrato. Veremos que os dois primeiros estados relacionados ao desenvolvimento das regras dos sinais ainda são muito presentes nas ações pedagógicas e no ensino dessas regras. Entretanto, estudos vêm apontando que o ensino dessa regra pela via formal, evitando as metáforas presas a exemplos concretos, pode favorecer a passagem do espírito concreto para o concreto-abstrato, alcançando, finalmente, o estado abstrato do espírito científico.

Palavras-chave: Etapas do Espírito Científico. Regra de Sinais. Ensino e Aprendizagem dos Negativos.

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INTRODUCTION

Teaching rules of signs of relative integer numbers faces problems which resounds throughout the school life of students. The difficulty faced by students in learning the multiplication of two negative numbers has challenged us to investigate the obstacles found by mathematicians during the historical process of consolidating this rule and which still remain in current teaching. This epistemology was chosen, among many, because Bachelard had studied in depth the origin of the epistemological obstacles found in the early stages of knowledge and their relations with pedagogical practices.

Knowledge is conceived by Bachelard (1996) as a reform of an illusion, that is, what we know always occurs against a previous knowledge, so for him there are no eternal truths. The truths are transitory and build from errors, therefore, it is in a constant dynamic process. From this perspective, error plays an important role in the construction of scientific knowledge.

Bachelard (1996) introduces the concept of epistemological obstacles and shows that they stop the advance of the scientific spirit. Both, common knowledge grounded in the real given, in the empiricism of first impressions, as the scientific knowledge supported by rationalism, whether radically taken, works as an epistemological obstacle to scientific knowledge. The scientific spirit must be dialectical and it is important and indispensable that an alternation between empiricism and rationalism takes place, because these doctrines are linked, they complement each other. For Bachelard (1978), to think scientifically is to place oneself in the intermediate epistemological field between these two poles.

We intend on this article to analyze the movement of the historical process of consolidation of rules of signs relating it with the three states of the scientific spirit which, according to Bachelard (1996), are: the concrete state, concrete-abstract and abstract. Regarding to the process of teaching and learning these rules, it remains a great challenge, as many epistemological obstacles remain in the present time for our students. The teaching of these kind of numbers bounded on concrete examples are very comfortable to address the addition operation, however, this same and practical example falls apart, when multiplying two negative numbers is shown: a debt plus other one becomes even more debt (– + – = –). However, how to explain that one debt multiplied by another becomes credit (– × – = +)?

GASTON BACHELARD EPISTEMOLOGICAL ELEMENTS

For Bachelard (1996), the scientific spirit, in its individual formation, necessarily passes through concrete, concrete-abstract and abstract states. In the concrete state, the mind is concerned about the first images of the phenomenon and is supported by a philosophy that exalts nature. In the concrete-abstract state, the spirit adds geometric schemes to the physical experience. Finally, in the abstract state, the spirit receives information disconnected from intuition, immediate experience, and to some extent in contradiction with the first reality. Thus, the path that guarantees the human knowledge progress goes not only through induction, but is a construction of man's mind and tends to become increasingly rational and abstract.

It is within the act of knowing that the slowness and the troubles of scientific practice appears. Knowledge of the real is never immediate and full, it becomes clear when the arguments are established. The truth is found by recovering the past of errors in a genuine intellectual repentance, it is no longer thought as instances definitively reached. There is no more truth, but there are several, plural, historical truths. "Basically, the act of knowing takes place in confront with a previous knowledge, destroying badly established knowledges, overcoming what, in the spirit itself, is an obstacle to spiritualization" (Bachelard, 1996, p.17). In this way science is definitely opposed to opinion. You cannot build anything based on the view before, it needs to be unutilized. This is because: "Opinion thinks badly; do not think: translate personal needs into knowledge. When opinion designating objects by its utility, opinion prevents itself from knowing these objects" (Bachelard, 1996, p.18). Thus, the opinion must be one of the first obstacles to be overcome.

The notion of epistemological obstacle involves aspects of the historical development of scientific thought. Bachelard (1996), studying the concept of epistemological obstacle within the history of science, realized that some knowledges even impede the progress of learning, creating barriers and obstructing scientific knowledge. Faced to this situation, the French philosopher seeks to break away from the shackles of the knowledge pre-scientific and boost scientific development, contributing significantly, with abstract scientific thought. Its repercussion can be perceived in the most varied lines of scientific research in the close relationship between epistemology and mathematical education, especially.

The history of mathematics has been used in teaching contexts. However, the idea of analyzing mathematical knowledge from a historical perspective, in order to gain some insight into how the process of knowledge construction takes place, is still very timid (Igliori, 2008, p.125). It is with this "shy" look that we propose, now, to analyze the historical trajectory of rules of signs consolidation, thinking under Bachelard's perspective of the scientific spirit formation and its concrete, concrete-abstract and abstract states.

A BACHELARDIAN LOOK AT THE HISTORICAL PROCESS OF RULES OF SIGNS CONSOLIDATION

The historical trajectory of the negative number concept was marked by much hesitation in a slow and surprising process (Glaeser, 1981). The origin of the rule of signs "$- \times - = +$" is usually attributed to Diophantus of Alexandria, still in the 3th century AD, who makes no reference to relative numbers, however, in his Arithmetic – Book I, he mentions: "Negative multiplied by negative is positive and negative by positive is negative" (Diofanto, 200, p.22).

The context of the consolidation of this rule presents itself in different ways in the global scenario. It was a great challenge for many ancient people and it has been dragged on until the 17th century. Throughout of the scientific spirit development route, many obstacles have been overcome. The first one, according to Bachelard (1996), was the concrete state in which the spirit is attached to the first images, concerned with practical examples that can be found in nature, as we can see on the explanations given by Simon Stevin in 1634. This mathematician proposed in his "*Arithmetic*" a demonstration of rules of signs. Let's see:

> Positive multiplied by positive gives positive product, & negative multiplied by negative gives positive product; & positive multiplied by negative, or negative multiplied by positive, gives negative product. Explanation of the data: Let's suppose 8 – 5 multiplied by 9 – 7 as follows: –7 multiplied by –5 is +35 (+35, because, as the theorem says, – multiplied by – is going to result +). Next, –7 multiplied by 8 results –56 (–56, because, as theorem says, – multiplied by + gives –). And similar, 8 – 5 multiplied by 9, & will give products 72 – 45; then add +72 +35, which is 107. Then, adding –56 –45, it's 101; and subtracting 101 from 107 remains 6, for the product of such multiplication. Explanation of the required. It must be demonstrated by the data, that + multiplied by + gives positive, & that – by – gives +, & that + by –, or – by + gives –. Demonstration. The number to multiply 8 – 5 is 3, & the multiplier 9 – 7 is 2. But multiplying 2 by 3, the product is 6. So, the above product also 6, is the true one. But the value found by multiplication, where we said that + multiplied by + gives product +, & – by – gives product +, & + by –, or – by + gives product –, so the theorem is true.

$$\begin{array}{rrr} & 8 & -\ 5 \\ & 9 & -\ 7 \\ \hline & -\ 56 & +\ 35 \\ 72 & -\ 45 & \\ \hline & 6 & \end{array}$$

(Glaeser, 1981, p. 312)

The history of Ancient Greek mathematics tells us that the concept of negative number was not recorded in this period. According to Pontes (2010), one of the characteristics of Ancient Greek mathematics was its persistence with rigorous demonstrations, reaching an independent existence. The strong attachment that the Greeks had to geometry made it impossible for them to dare consider negatives as numbers, because

> [...] for whom geometry was a pleasure and algebra a necessary demon, they rejected the negative numbers. Unable to adjust it [negative numbers] in their geometry, unable to represent it by figures, the Ancient Greeks regarded negatives not exactly as numbers. (Kasner & Newman, 1968, *quoted in* Medeiros & Medeiros, 1992, p.51)

In Europe, as trade expanded, interest in mathematics in the Middle Ages spread out slowly. Practical men were interested in counting, arithmetic, and computing, with desires directly influenced by the growth of commercial cities (Struik, 1992). The use of negative numbers became accepted with the expansion of financial relations in trade, which favored the arising of a credit structure. The idea of taking 7 out of 5 was miraculous, so

> [...] it was necessary to wait by the forthcoming of a banking system with an international credit structure, such as it appeared in the cities of northern Italy (particularly Florence and Venice) during the fourteenth century. The seemingly absurd 5 minus 7 subtraction became possible when new bankers began allowing their clients to withdraw 7 gold duchies while their deposits were only 5. (Singh, 1972, *quoted in* Medeiros & Medeiros, 1992, p.52)

In this context, the negative number was eventually used for accounting purposes. However, according with Medeiros and Medeiros (1992), though useful, the idea of the negative, associated with a debit, was unsatisfactory and did not meet the requirement of mathematical metaphor, especially regard of rules of signs.

In all these demonstrations for rules of signs, the spirit was bound by concrete examples that are usually sources of deception, of errors. However, scientific knowledge, in Bachelard's view, is structured in overcoming these errors which collaborated for the scientific spirit could have advanced in the rupture of what was thought to be known. Thus, at the post-concrete stage some signs of generalizations begin to appear, but the mind is disturbed and confused in a process of detachment from the real and the first impressions.

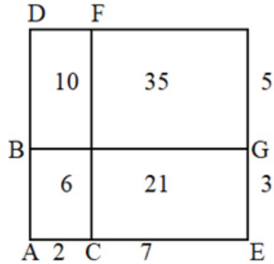
In the concrete-abstract state the geometric schemes were added to the experiments. This action of geometry, for Bachelard (1996, p.7), is "midway between concrete and abstract," starting from what is visible to what is invisible and achieved only by formal reasoning.

> When a geometrical law can be formulated, it is performed a surprising spiritual inversion, vivid and smooth as a conception; curiosity is replaced by the hope of creating. Since the first geometrical representation of phenomena is essentially an ordination, this first ordination opens us to the perspectives of an alert and conquering abstraction [...]. (Bachelard, 1996, p.8)

But at this stage the spirit is still in conflict. This can be seen in the geometric demonstration presented by Stevin:

> Let's suppose AB 8 - 5 (namely AD8 - DB5). After AC9 - 7 (namely AE9 - EC7) its product will be CB: that is, according to the previous multiplication ED72 – EF56 – DG45 + GF35, which we will prove to be equal to CB in this way. Of all ED + GF, subtracting EF, & DG, remains CB. Conclusion. Thus, positive multiplied

by positive gives positive product, & negative multiplied by negative, also gives positive product, & positive multiplied by negative, or negative multiplied by positive, gives negative product; what we wanted to demonstrate.



(Glaeser, 1981, p.312)

Stevin's example shows us how geometry supports arithmetic, helping to prove that the rule works. However, we note that in this historical period the "– × – = +" rule is only used as a transitional procedure.

Laplace also was unsuccessful in his justification for rules of signs, which, in some ways, resembled the Euler one. And, also, cannot propose a formal extension to the relative numbers. Let's see how Laplace presents the justification of rules of signs:

> (rules of signs) presents some difficulties: we just have to conceive that the product of –a by –b is the same as that of a by b. To make this identity sensitive, we observe that the product of –a by +b is –ab (since the product is –a repeated so many times when it has units in b). We then observe that the product of –a by b –b is null because the multiplier is null; thus the product of –a by +b is –ab, the product of –a by –b must be opposite or equal to +ab to destroy it. (Glaeser, 1981, p.333)

Another example that can be presented as an illustration of the disturbance caused by the spirit in the concrete-abstract stage is the resolution of the cubic equations. Always that the three roots of an equation are real and non-zero, the formula of Tartaglia-Cardan (1501–1576) leads to the calculation of a square root negative. Then a new type of number begins to appear: the negatives. In this context appears the figure of an Italian algebraist, Rafael Bombelli (1526–1573), who came up with the idea of conjugated imaginary. Nevertheless, Bombelli's observations did not contribute to the effective resolution of the cubics, however, Bombelli pointed to the important role that the conjugated imaginary would play in the future (Boyer, 2010).

At the end of the seventeenth century, according to Glaeser (1981), it started a change in the question of negative numbers acceptance marked by Colin MacLaurin. He comes to understand the number as an action and no longer as a state and presents a justification for rules of signs, using the distributivity of multiplication in relation to addition. His deduction contributed to the beginning of a formalism until then nonexistent. His explanation was based on the following idea: If $+a -a = 0$, then, if we multiply this expression by a positive number $+ n$, we will have the first term $+n$, and as the second term $-n$, for the product is also expected to be zero, then, $-n +n$ should also be zero. And when the expression $+a -a$ is multiplied by a negative number, that product must also be zero. So, if we multiply the expression $+a -a$ by $-n$, we get $-na$ as the first term and $+na$ for the second term, because the two terms need to be canceled.

Therefore, he enunciates rules of signs by stating that the product of two numbers with different signs is negative and the product of two numbers with the same sign is positive. Despite the important contributions of MacLaurin about the relative numbers, he was not able to present the theory of relative numbers, but his studies were used as a reference by later mathematicians.

In this context, some aspects of abstract scientific thinking are glimpsed on the moment in which first impressions started losing ground to generalized thinking.

> [...] we would like to show this grand destiny of abstract scientific thought. For this we must prove that abstract thinking is not synonymous of bad scientific conscience, as the trivial accusation seems to say. We should prove that abstraction detangling, relieves and streamlines the spirit. [...] And to better show that the process of abstraction is not uniform, we will not hesitate to use sometimes a controversial tone, insisting on the character of obstacle that presents the experience, estimated concrete and real, assessed natural and Immediate. (Bachelard, 1996, p.8-9)

We share Bachelard's idea when he states that the process of abstraction constitutes itself as a destabilization of the opinion's obstacle that is bound to the first and real experience. This can be noticed during the scientific practice of rules of signs. While the demonstration of rules of signs were based on concrete thought, the spirit was in conflict. Once the difficulties that prevented the correct abstractions begin to be overcome, the spirit becomes untangled, for the abstraction reassures and stimulates it. Thus, in the concrete-abstract state, even the spirit is still living in a paradoxical situation, it already feels itself stronger on the process of abstraction.

However, the path that can guarantee the progress of human knowledge goes through the disconnection of intuition. In the abstract state, the spirit, far from the first reality, builds scientific knowledge that tends to become increasingly rational and abstract. One of the mathematicians who set out on this path was George Peacock, who published in 1830 the "*A Treatise on Algebra*". In this work, he shows the distinction between arithmetic algebra and symbolic algebra. This view, by applying to symbolic algebra the same rules as arithmetic algebra, provokes a true evolution for the formation of the theory of relative numbers and for the formation of the scientific spirit.

As a consequence of Peacock's contributions, the German Hermann Hankel established in 1867 the principle of the permanence of formal laws. Hankel proposes the multiplication of relative numbers as an extension of the properties of positive real numbers to reals and states the following Theorem: "The only multiplication over R, which extends the usual multiplication over R+, respecting the distributions (left and right), is according to rules of signs"(Glaeser, 1981, p.338). And it presents a fairly simple demo:

> Demonstration:
> $0 = a \times 0 = a \times (b + \text{opp } b) = ab + a \times (\text{opp } b)$
> $0 = 0 \times (\text{opp } b) = (\text{opp } a) \times (\text{opp } b) + a \times (\text{opp } b)$
> From where
> $(\text{opp } a) \times (\text{opp } b) = ab$
> (Glaeser, 1981, p.338)

Hankel's revolution, refusing to look for a good model, according to Glaeser (1981), is to approach numbers from another perspective. We can no longer look for practical examples that explain relative numbers by analogies, for these numbers are no longer discovered, but invented, imagined. It is only in the abstract state of mind that rules of signs can be consolidated.

> It is not possible to speak so fiercely against such a widespread view that these equations [rules of signs] can never be proved in formal mathematics; they are arbitrarily established conventions to preserve the formalism already existing in the calculations. [...] However, once defined, all other laws of multiplication derive from them by necessity. (Schubring, 200, p.6)

Hankel, unlike Laplace who sought an explanation in nature for rules of signs, approaches the multiplication of relative numbers as an extension of properties of the positive real numbers to the reals. Thus, rules of signs are a convention which purposes to keep the internal consistency of mathematics itself.

This ability that civilized man has to make generalizations and abstractions, unlike primitive man and even some philosophers who perceived numbers as impregnated in nature, Caraça calls "principle of extension". In his words:

> [...] the man has a tendency to generalize and understand all the acquisitions of his thought, whichever way these acquisitions are obtained, and to seek the greatest possible yield from these generalizations by the methodical exploration of all their consequences. All man's intellectual work is, at its core, guided by certain norms, certain principles. That principle by virtue of which the tendency just mentioned is manifested, we shall call the extension principle. (Caraça, 1963, p.10)

The man's intellectual work, guided by certain norms and principles, of which Caraça speaks to us, has led to the expansion of numerical sets and its operations. Thus: "Operations on relative numbers are defined by the immediate extension of operations with the same name studied on the real field" (Caraça, 1963, p.100).

However, throughout history, we can see the difficult acceptance of negative numbers, and this refusal was still present for a period, even after the revolution achieved by Hankel. Schubring (2007) shows an example of warm academic debates in the math teacher community about the hesitation of relatives. We will mention an excerpt from Hoffmann (1884, quoted by Schubring, 2007, p.17) who sets a picture of horror and drastic consequences for mathematics teaching if teachers have to tell students that rules of signs are a convention: "I'm going to be afraid to see the eyes of surprise and amazement of students. Smart students would survive with questions: Is this truly arbitrary? Can't it be demonstrated? This hesitation to accept rules of signs disengaged practical examples is one characteristic thought pre-scientific that is before an obstacle not yet overcome. "You have to give up your own habits. The scientific spirit has to combine the flexibility and the rigor. [...] Abandoning common sense knowledge is a difficult sacrifice. No wonder the ingenuity that is accumulated in the first descriptions of an unknown world" (Bachelard, 1996, p.277).

This difficult detachment, that the pre-scientific mind has to give up its habits, can be perceived on teachers' resistance in the past (and perhaps at present) to accept that explanation for rules of signs cannot be seen in nature, which "– × –" needs to be more to preserve the existing mathematical formalism.

ASPECTS OF BACHELARD'S EPISTEMOLOGY IN LEARNING PROCESS AND TEACHING RULES OF SIGNS

Students, in modern times, still faces many obstacles that have been encountered and overcome by mathematicians in the past. The permanence of these obstacles often goes unnoticed in the process of teaching and learning the rule of signs "– × – = +" (negative multiplied by negative is going to result positive). In our view, this problem is directly related to the teaching and learning processes and the construction of knowledge, therefore, presents epistemological and pedagogical elements.

In the current education system, it is clear that the negative numbers are still addressed with practical examples and stuck to nature. Textbooks, used by the PNDL-2011[1] of the Brazilian educational system, destinated to the middle school 7th degree were analyzed by Hillesheim (2013), which points out that 60% of books used the arithmetic model to address the addition of integers. In other words, the authors were based on problem situations that related the positive number to the idea of gain/profit/credit and the idea of negative number linked to loss/debt/loss. This obstacle of early experience, in

[1] PNDL means *Plano Nacional do Livro Didático*, in portuguese, or National Textbook Plan.

which negative numbers are strongly related to the business model, to concrete models bound to nature, is at the concrete stage of the scientific spirit.

In order to facilitate understanding of integers, the approach used for teaching negative numbers is done with practical examples, in contexts of commercial use, temperature, sports score results, and so on. However, this fact reinforces the mistaken and confusing idea when it comes to justifying that "– × – = +". How does one debt multiplied by another debt turn it into a gain? "[…] Metaphors cannot be so easily confined in the realm of expression.. However much one does, metaphors seduce the reason. They are particular and distant images that, insensibly, become general schemes" (Bachelard, 1996, p.97).

In the para-didactic book entitled "*História de sinais*" (History of signs), Ramos (2006) presents throughout history several situations in which the characters live with real situations of concrete application of relative integers. When the character Alexander explains the multiplication of two negative numbers to Milena, he uses the idea of opposition. Thus, $(-7) \times (-4) = -(+7) \times (-4) = -(-28) = +28$. However, the character Milena was not completely satisfied with the explanation. See their dialogue:

> – (Milena) How can multiplying two negative factors give a positive result?
>
> – (Alexandre) Ah, is that it? Do you know I was also weird the first time I thought about it?
>
> – So, tell me where the logic of that is ...– she asked.
>
> The boy thought for a moment before answering:
>
> – The most concrete example I could give you is this: I went out with my T-shirt inside out. I will then turn inside out the inside out...
>
> – (Milena) But the inside out is the right one.
>
> – (Alexandre) Yes, it is! It is as if the right side were represented by the idea of positive and the reverse side by the idea of negative. (Ramos, 2006, p.60)

Understanding the negative numbers by the pre-scientific spirit seeks to realize the abstract (the idea of negative number), making use of analogies (gain/loss, credit/debit, wrong/right) that empty all scientific content. That is, from abstract phenomena a person generates concrete images that block the epistemological rupture of common sense for the comprehension of abstract aspects of the multiplication of two negative numbers.

Rules of signs, in the conception of the character Milena, needed to be used and applied in concrete situations. In this context, one notices the difficulty that the concrete state of the scientific spirit has in conceiving the idea of a useless phenomenon. For the character, the rule of signs "– × – = +" only makes sense when related to a situation that can be experienced. This shows the need that the spirit of science in the concrete state has to cling to examples of everyday life, causing disturbances and preventing the mind reaches the abstract state, that this case concerns the understanding of multiplication between two negative numbers. This example, found in the literature, can serve as an illustration of what happens to our students on the classroom.

We can also think that these difficulties may be related to the phenomenon of semantic non–congruence highlighted by Duval:

> Two expressions may be synonymous or referentially equivalent (they may 'mean the same thing', they may be true or false at the same time) and not semantically congruent: in this case, there is a cognitively important cost for understanding. (Duval, 2012, p.100)

Generally, when a semiotic representation passes to another system spontaneously, semantic congruence is said to occur. However, when this passage does not occur immediately, these representations are not congruent with each other. Thus, the problem of congruence or semantic non-congruence of two presentations of the same object is the cognitive distance between these two representations. As greater the cognitive distance, as greater the cost of moving from one semiotic representation to another, and as greater the risk that the mathematical process will not be performed or understood by students.

> Discursive problems that are semantically congruent with the mathematical expression but not referentially equivalent lead to a very low success rate; similarly with problems that are referentially equivalent but not semantically congruent. Solving problems that require a discursive record to be moved to an arithmetic or algebraic record requires referential equivalence. (Moretti, 2012, p.705)

In this sense, the teacher should be aware that semantic congruence does not always lead to successful results in solving mathematical problems. It is no coincidence that students have so much difficulty in understanding the operations with relative numbers as indicated by research of Pontes (2010). In our view, the teaching of negative numbers, attached to the business model, empties the scientific content of rules of signs and further aggravates students' difficulties in situations where there is no semantic congruence.

The research conducted by Pontes (2010) points out that the current teaching and learning process of negative numbers and the rule of signs still faces epistemological obstacles that have already been overcome by mathematicians of the past. It is not our purpose here to use the history of the rules of signs consolidation to explain the problems related to it in the present, but, yes, we think epistemologically, taking the facts as ideas, parsing it the documents collected by the historian and judging it under the point of view of reason. In this research, Pontes (2010) presents a suggestion for teaching the multiplication of integers relative as indicated, according to the author, the Notebook 9 of the NCTM Mathematical Themes Collection entitled system of integers.

> In this justification, it is suggested to multiply the sequence of the integers from 4 to –4 by the positive number 5. Starting with the product of $4 \times 5 = 20$ and following

> until the product of 0 × 5 = 0 it can be observed that the first factor decreases from one by one, that the second factor remains the same and that products are decreasing every five units. Therefore, continuing to make the products up to –4 × 5 = –20, we get the following sequence:
>
> | × | +4 | +3 | +2 | +1 | 0 | -1 | -2 | -3 | -4 |
> |---|---|---|---|---|---|---|---|---|---|
> | +5 | **+20** | **+15** | **+10** | **+5** | **0** | **-5** | **-10** | **-15** | **-20** |
>
> Thus, we conclude that the product of a negative number by a positive number is a negative number. Then the same strategy is used to multiply the whole numbers from 4 to –4 by the negative number –5, initially from 4 to 0, so that the sequence of products is perceived and then from –1 to –4, which generate the products:
>
> | × | +4 | +3 | +2 | +1 | 0 | -1 | -2 | -3 | -4 |
> |---|---|---|---|---|---|---|---|---|---|
> | -5 | **-20** | **-15** | **-10** | **-5** | **0** | **+5** | **+10** | **+15** | **+20** |
>
> According to the previous sequence, we conclude that the product of a negative number by a negative number is a positive number. (Pontes, 2010, p.104)

We realize in this suggestion, indicated by Pontes (2010), that the rule of signs for multiplication is presented midway between the concrete and the abstract, exposing a certain degree of abstraction between the visible and the imperceptible. This geometrization leads to other domains of knowledge, representing the initial step for the scientific thought formation. In this context, the scientific spirit is in the concrete-abstract state, in the incessant search for a possible generalization. The spirit is comfortable with the arithmetic example, but glimpsing himself about algebraic ideas, emerging abstract thinking. According to Bachelard (2004, p.14), in order to grasp the real "one has to have the courage to place it at its oscillating point, where the spirit of refinement and the geometric spirit are mixed."

In the analysis of the PNDL-201 middle school 7th degree Textbooks, Hillesheim (2013) points out that none of the books presented the multiplication of integers using the algebraic model, i.e. the formal way, as presented by Hankel. However, this rule can only be thought of by extending the properties of Positive Reals (R+) to Reals (R). It is a convention aimed at maintaining the internal consistency of mathematics itself.

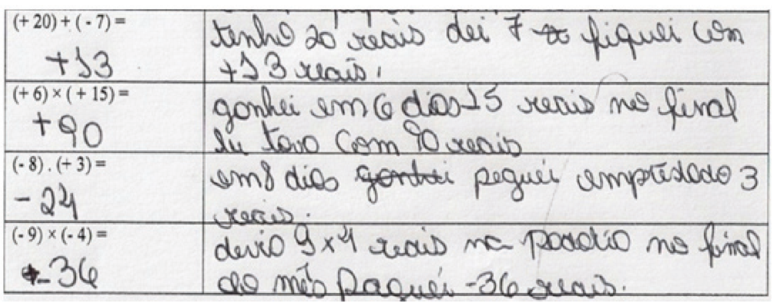
> There is no doubt it would be simpler to teach only the result. But the teaching of science results is never a scientific teaching. Without explaining the spiritual production line that led to the result, one can be sure that the student will associate the result with his or her most well-known images. It needs 'that he understands'. You can only keep what you understand. The student understands his way. Since they did not give him the reasons, he adds to the result personal reasons. (Bachelard, 1996, p.288-289)

Thus we can see that simply memorizing the rule "negative multiplied by negative is positive" does not guarantee your meaningful learning. The conceptual introduction of relative integers, using the simple and concrete ways, can contribute to the formation of gaps regarding the understanding of rules of signs. We need to think about ways that promote and guarantee the understanding of their production process. If the student does not understand the path of formally consolidating rules of signs, he or she will most likely seek to understand it in his own way, attributing the results to his personal experiences.

The conception of the negative number linked to the business model (gain/loss) is unabated in the spirit. Because,

> [...] as it is concrete and it makes it much easier to understand the relative at the beginning of their learning, students adopt it and want to use it while it is no longer adapted: not only does it explain nothing more, but it represents more nothing, it no longer works at the symbol level. (Coquin-Viennot, 1985, p.183)

The concrete state of the scientific mind generates a comfortable and secure takeover, unable to be shaken even in conflicting situations, as in the case of justifying the "– × – = +" rule with practical and metaphorical examples. One of the findings of Hillesheim's (2013) research was to be able to show how the teaching of relative integers, via the business model, brings serious harm to understanding the multiplication of negative numbers. In the application of its didactic sequence, one of the questions requested the resolution of the operation, together with the justification for the result found. Below is the answer presented of Hillesheim's (2013) research by a class student to this question:



(+ 20) + (- 7) = +13	tenho 20 reais dei 7 ~~e~~ fiquei com +13 reais.
(+ 6) × (+ 15) = +90	ganhei em 6 dias 15 reais no final eu tava com 90 reais
(- 8) . (+ 3) = -24	em 8 dias ~~ganhei~~ peguei emprestado 3 reais.
(- 9) × (- 4) = ~~+~~-36	devo 9x4 reais na padaria no final do mês paguei -36 reais.

(Hillesheim, 2013, p.155)

From top line to bottom: 1. I have BRL$20.00, give 7, still have BRL$13.00; 2. Won in 6 days BRL$15.00, at the end, I have had BRL$90.00; 3. In 8 days got BRL$3.00 borrowed; 4. I owed 9 x 4 on the bakery, at the end of the month have payed BRL$ –36.00.

This student was successful in all his calculations and justifications so far that he encountered a multiplication between two negative numbers. So the business model that was so well-suited so far ceased to function and led to an incorrect outcome. Realize we the conflict that this student was at the time of termination of the operation: first he had put the positive sign on the result of "(–9) × (–4)" but as this result would not make sense (as a debt multiplied by another debt could result in a gain?) He erased the positive sign and put the negative sign in response to the operation. In this regard, Coquin-Viennot points out that:

> This conception (on the concrete basis) cannot function in a multiplicative structure; it needs to be reversed in order to continue learning, but it is well-established and comfortable to solve the additive problems encountered so far, which itself constitutes a real obstacle to the installation of level IV. (Coquin-Viennot, 1985, p.184)

Level IV that Coquin-Viennot mentions is precisely the conception of multiplication of relative numbers. In other words, the concrete model that works very well for teaching additive properties is a barrier to teaching the multiplicative properties of these numbers. According Hillesheim (2013), although this student taking part in a didactic sequence conducted by the formal route, he was not able to have their design brought from previous years experiences shaken, since the student was enrolled in the middle school 7th degree for the second turn.

For Bachelard (1996), the commitment of the scientific philosophy is "[...] to psychoanalyze the interest, to overthrow any utilitarianism, however disguised as it may be, to turn the spirit from real to artificial, from natural to the human, from representation to abstraction" (Bachelard, 1996, p.13). As long as the teaching of negative numbers is conducted by the spirit in its concrete state, the student is unlikely to be able to detach himself from the real situation and achieve abstract and formal thinking. This is because scientific experience is opposed to immediate experience. We agree with Michelot (1966) when he states that "the notion of negative number can only be correctly defined by the level of formal thinking" (p.238).

We believe that the teacher, by introducing the negative number by real situations, only seeks to facilitate learning of additive operations for these numbers. However, with this scheme, would not be like "[...] introducing a false didact contract using a concrete model to present all the relative numbers?" (Coquin-Viennot, 1985, p.183), once this conception just works for the addition? In our view, this attachment to concrete situations, although comfortable at first, becomes a hindrance to understanding the multiplicative properties of these numbers.

> 'In order to admit the notion of negative number, it is necessary to completely free up obstacles to conscious thinking' (Michelot, 1966, p.228). This formula remains true for all domains where relative ones are exercised; The difficulties encountered by students in the face of multiplicative problems clearly manifest that 'such obstacles of concrete thought have not fallen', why would they fall

> when these same students use relative in an additive situation? (Coquin-Viennot, 1985, p.184-185)

These resistances are understandable from a pedagogical point of view, because the presentation of negative numbers by commercial means works very well when it comes to relating these numbers to everyday situations such as temperatures, soccer results, accounting situations, sea level, etc. However, the justification proposed by the student, for multiplying two negative numbers, shown by Hillesheim (2013), illustrates well the estrangement caused by this view when he/she, the student, is faced with the multiplication of two negative numbers.

Many teachers in the teaching process of relative numbers forgot to take into account the middle school 7th degree student, when studying the integers, brings their empirical knowledge already made up by their experiences "[...] it is not therefore about acquire an experimental culture, but to change it, to bring down the barriers already sedimented by everyday life" (Bachelard, 1996, p.2-3). Bachelard seeks to illustrate his position through his testimony as a teacher:

> Little by little, I try to gently release the students' spirit from their attachment to privileged images. I lead them on the path of abstraction, striving to arouse the taste for abstraction. Anyway, I think the first principle of science education is, in the intellectual kingdom, this asceticism which is abstract thinking. Only it can lead us to master experimental knowledge. (Bachelard, 1996, p.292)

The Hillesheim (2013)'s research presents relevant results in order to conduct the teaching of relative integers, avoiding the association of the positive number with a gain and the negative number with a loss and envisaging the possibility of promoting the passage of the scientific spirit in the concrete state to the concrete-abstract and finally for the abstract. Studies realized by Hillesheim (2013), Hillesheim & Moretti (2013) have pointed out new perspectives and possibilities of teaching rules of signs preventing the business model. The didact sequence organized and applied for Hillesheim (2013) on students of the middle school 7th degree, indicates positive results for the teaching and learning process of rules of signs by via formal, avoiding the business model, favoring the formation of abstract thought and stimulating the development of generalizations. After application of the didactic sequence, Hillesheim (2013) found that most of the students in the class had already conceived the unification of the number line and additives problems were solved on integers. The addition operation, being presented by displacements on the numerical line, contributed to the negatives not being treated separately from the positives. This way of conducting the teaching of addition eventually contributed to students realizing that the set of integers is a union between zero, positive and negative. However, the subtractive problems were not achieved by the students, as they could not break free from the conception of subtraction linked to the idea of drawing as conceived in the naturals, and broaden this conception in the relative ones where the subtraction of these numbers means working with negative operators that operate position transformation. The assimilation of the multiplicative problems was achieved to a small group of students, however almost 80% of the group has been succeeded to calculate the multiplication $(-7) \times (-4)$.

We believe that the application of didactic sequence of Hillesheim (2013) contributed to the development of the three states of the scientific spirit presented by Bachelard, culminating in "The abstract state, where the mind takes information voluntarily removed from intuition of the real space, voluntarily switched off of immediate experience and even in open controversy with the first reality, always impure, always formless ". (Bachelard, 1996, p.11)

FINAL CONSIDERATIONS

The present work is an attempt to show the epistemological obstacles that were present in the three states of the scientific spirit in the process of consolidation of the "$- \times - = +$" rule of signs. The epistemology of Bachelard (1996) argues that scientific knowledge progress with the action of a constant questioning and to achieve the science thought it is necessary to accept a sudden change in the spirit, which contradicts the past. In this constant movement of the spirit, truths are provisional and built upon errors.

The three states of scientific spirit pointed out by Bachelard (1996) can be perceived all of them in the process of formalization of rules of signs by mathematicians, and in the process of teaching and learning about this rule in present times. While mathematicians intended explaining the rule with examples attached to nature, the scientific spirit was in the concrete state. It was only when the spirit came off these concrete and concrete-abstract examples, moving to the abstract state, that the scientific spirit could extend the concept of negative numbers, consolidating, thus, the rule of signs for these numbers.

In the teaching process and learning negative numbers, specifically the rule of signs, we realize that there are currently pedagogical obstacles that facilitate face those difficulties found by students during their learning process. The conceptual approach of these numbers by concrete, tied to practical examples, causes conflict and discomfort when students are confronted with the multiplication of two negative numbers. It is necessary to explicit the processes of production and systematization of the rule of signs for students to understand it. Only what is understood can be preserved in memory. Rules of signs of the addition or multiplication of integers need not be memorized by students, it can and must be understood by them.

We agree with Hillesheim (2013) to argue that the conceptual introduction of these numbers should be made by the formal route, avoiding metaphors, which, in order to facilitate the structure understanding, lead students to misconceptions formation and confusing regard those numbers. It is necessary to conduct the process of teaching and learning these numbers for the training of the scientific spirit, and, to this end, it is necessary detachment to concrete examples.

Bachelard states that: “In its individual formation, the scientific spirit would necessarily pass through the following three states [...]: concrete, concrete-abstract and abstract” (Bachelard, 1996, p.11). Thus, in our view, the didactic intervention proposed by Hillesheim (2013), under a Bachelardian view, seems to have enabled the liberation of the students’ spirit from the first impressions, attached to concrete examples, moving to the pre-scientific spirit stage, through geometrization, contributing to the formation of generalizations, culminating in the abstract stage.

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AUTHOR CONTRIBUTION STATEMENT

Both authors M.T.M. and S.F.H. produced the final text by sharing ideas and writings.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article because no new data was created or analyzed in this study.

REFERENCES

Alexandria, D. (2007) *La aritmética y el libro sobre los números poligonales.* Tres canto: Nivola Libros Ediciones.

Bachelard, G. (1978) *A filosofia do não; O novo espírito científico; A poética do espaço;* seleção de textos de José Américo Motta Pessanha; traduções de Joaquim José Moura Ramos. (et al.). São Paulo: Abril Cultural.

Bachelard, G. (1996) *A formação do espírito científico: contribuição para uma psicanálise do conhecimento*. Tradução Estela dos Santos Abreu. Rio de Janeiro: Contraponto.

Bachelard, G. (2004) *Ensaio sobre o conhecimento aproximado.* Rio de Janeiro: Contraponto.

Boyer, C. (2010) *História da matemática.* São Paulo: Blucher.

Caraça, B. J. (1951) *Conceitos fundamentais da matemática.* Lisboa: Bertrand.

Coquin-Viennot, D. (1985) Complexité mathématique et ordre d'aquisition : une hierarchie de conceptions à propos des relatifs. *RDM, 6*(2.3), 133-191.
Duval, R. (2012) Diferenças semânticas e coerência matemática: introdução aos problemas de congruência. *Revemat,* Florianópolis, *7*(1), 97-117.
Glaeser, G. (1981) Epistemologie des nombres relatifs. *RDM, 2*(3), 303-346.
Hillesheim, S. F. & Moretti, M. T. (2013) Modelo Comercial: um entrave persistente à aprendizagem da regra de sinais. *REnCiMa,* São Paulo, *4*(2), 37-56.
Hillesheim, S. F. (2013) *Os números inteiros relativos em sala de aula: perspectivas de ensino para a regra de sinais* (216 f.). Dissertação de Mestrado (Programa de Pós-Graduação em Educação Científica e Tecnológica) Universidade Federal de Santa Catarina, Florianópolis.
Hoffmann, J.C.V. (1884) Zwei wichtige Fragen über das Negative, beantwortet vom Herausgeber. *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 15, 580-582.
Igliori, S. B. C. (2008) A noção de "Obstáculo Epistemológico" e a Educação Matemática. In: Machado, S. D. A. (org.) *Educação matemática: uma (nova) introdução.* EDUC, São Paulo, 49-75.
Kasner, E; Newman, J. (1968) Mathematics and the Imagination. Middlesex, Pelican Books.
Lopes, A. R. C. (1996) Bachelard: o filósofo da desilusão. *Caderno Catarinense de Ensino de Física,* Florianópolis, *13*(3), 248-273.
Medeiros, A. & Medeiros, C. (1992) Números negativos: uma história de incertezas. *Bolema,* Rio Claro, *7*(8), 49-59.
Michelot, A. (1966) *La notion de zero.* Paris: Vrin.
Moretti, M. T. (2012, abril) A regra dos sinais para a multiplicação: ponto de encontro com a noção de congruência semântica e o princípio de extensão em matemática. *Bolema,* Rio Claro, *26*(42B), 691-714.
Pontes, M. O. (2010) *Obstáculos superados pelos matemáticos no passado e vivenciados pelos alunos na atualidade: a polêmica multiplicação de números inteiros* (157 f.). Tese de Doutorado (Programa de Pós-Graduação em Educação), Universidade Federal do Rio Grande do Norte, Natal.
Ramos, L. F. (2006) *História de sinais.* São Paulo: Ática.
Schubring, G. (2007) Um outro caso de obstáculo epistemológico: o princípio de permanência. *Bolema,* Rio Claro, *20*(28), 1-20.
Singh, J. (1972) *Mathematical Ideas: their nature and use*. London: Hutchinson and Co. Ltd.
Struik, D. J. (1992) *História concisa das matemáticas.* Lisboa: Gradiva.