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A Learning Trajectory to the Understanding of the Curve Length Concept

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ABSTRACT

In this article, we present results of a research study focusing on the analysis of a hypothetical learning trajectory carried out with students taking a mathematics teaching degree. The aim of this study was to examine students' understanding of the concept of curve length. The qualitative research was carried out with nine students participating in a course on Differential and Integral Calculus discipline of a private university in which that content was approached. The data were obtained through records of the students' worked out solutions, notes from observation recorded in the teacher's field diary and audio recordings made during the course development. From the analysis of the results, it can be inferred that the students showed gaps in their previous knowledge and difficulties on how to use that knowledge in the construction of new concepts; however, evidence was observed that the planned hypothetical learning trajectory facilitated, in part, the understanding of the concept of curve length.

Keywords: Curve Length; Concept image; Concept Definition; Hypothetical Trajectory of Learning.

Uma Trajetória de Aprendizagem para a Compreensão do Conceito de Comprimento de Curva

RESUMO

Neste artigo, apresentamos resultados de uma investigação tendo como foco a análise de uma trajetória hipotética de aprendizagem realizada com alunos de um curso de formação inicial de professores de Matemática que tem como objetivo analisar a compreensão dos alunos sobre o conceito de comprimento de curva. A pesquisa de cunho qualitativo foi realizada com nove estudantes matriculados na disciplina de Cálculo Diferencial e Integral de uma Universidade Particular em que esse conteúdo foi abordado. Os dados foram obtidos por meio dos registros das resoluções dos alunos, das observações anotadas no diário de campo da professora e das gravações realizadas durante o desenvolvimento das aulas. Da análise dos resultados pode-se inferir que os

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alunos apresentam lacunas quanto aos conhecimentos prévios e dificuldades de como utilizar esses conhecimentos na construção de novos conceitos, porém, observaram-se evidências de que a trajetória hipotética de aprendizagem planejada facilitou, em parte, a compreensão do conceito de comprimento de curva.

Palavras-chave: Comprimento de Curva. Imagem de Conceito. Definição de Conceito. Trajetória Hipotética de Aprendizagem.

INTRODUCTION

The History of Mathematics has shown that the definition of some concepts, as they are presented in textbooks by students, has been suffering reformulations and many mathematicians have given their contributions so that the definitions have the written rigour and explicitly as formulated nowadays. The concept of plane curve length needs, by the comprehension, the notion of limit that, according to Cornu (1991), is one of the concepts from Differential and Integral Calculus necessary to the development of advanced mathematical thinking. For the author, the concept of limit is complex and presents many obstacles for the students' comprehension. Possibly, one of the obstacles is in the way the concept is presented to the students in the classroom. The same author highlights that intuition is a factor that favours the comprehension of the concept, and this does not happen only with the formal presentation of this mathematical definition.

About the construction of the Differential and Integral Calculus Concepts, many researchers have done their researches with this focus. In his research, Rezende (2003) refers to the way concepts are worked by the teachers, highlights that "the excessive algebraization of the limits' operation characterises well what we want to say with the 'prevalence of ethics' about the meaning". The way the concept is built in the classroom, with the vast number of new concepts the student needs to understand in the Calculus subjects, can be obstacles for the development of advanced mathematical thinking.

When talking about the National Curricular Guideline (Brasil, 2015) for the Mathematics Teaching Courses, it is recommended for the mathematical concepts be worked contemplating the analytical and graphical aspects and their multiple applications. What is observed nowadays is that, in relation to the Calculus basic concepts, there is still a valuation of technical procedures in detriment of the exploitation of intuitive capacity and graphic representation. Research results as in Vinner (1989), Tall (1994), Meyer and Igliori (2003), Bossé and Bahr (2008), Nasser (2009), Cury (2009), Karatas, Guven and Cekmez (2011), Silva (2011), Rasmussen, Marrongelle and Borba (2014), among others, point to the difficulties in the teaching process of Calculus and the obstacles for the comprehension of the concepts. Yet, they indicate that the students have a better performance when doing activities in which there is the prevalence of exercises that focus on operational and technical aspects.

In this work, our purpose is to analyse the comprehension of students from a Math Course about the concept of the plane curve length. This is a concept that involves three basic ideas, the notion of limit, the notion of measure, and the notion of Riemann sum, which is the reason for its complexity. The theoretical referential is based on the theory of Tall and Vinner (1981), about the concept image and concept definition, and on the ideas of Star and Stylianides (2013) about the conceptual knowledge and the procedural knowledge. We tried to understand how students build the concept of plane curve length from the experience of a Hypothetical Learning Trajectory (Simon 1995; Clements & Sarama 2004) and, in this context, we analysed the concept images built by the students.

THEORETICAL REFERENCE

The discussion about the conceptual comprehension in Mathematics, and from the desirable balance between this comprehension and the procedural knowledge have occupied researchers and mathematical educators for many years (NCTM, 2000; Rittle-Johnson, Schneider & Star 2015; Sierpinska 1994; Star & Stylianides 2013). The multifaceted nature from the biggest part of the mathematical concepts students find during their academic life has equally received extra attention about the systems of mathematical representation and about the mathematical and extra mathematical connections that are part of the conceptual map, exclusively in topics from the Calculus (Bossé & Bahr, 2008).

When it comes to teaching and learning of Mathematics, it is essential to comprehend the meaning of the terms "conceptual knowledge" and "procedural knowledge" because the absence of balance between them can take to gaps in the formation of the students. Bossé and Bahr (2008) highlight the following aspects: Mathematics, including school math, is a net of interrelated concepts and procedures; the knowledge of how applying the mathematical procedures does not necessarily follow the comprehension of the underlying concepts; the knowledge of mathematical concepts contributes more efficiently to the comprehension of mathematics than for the procedural knowledge. If the students learn mathematical concepts before they learn the procedures, they can more effectively understand the procedures when asked; however, when they learn the procedures first, it is less likely that they learn the concepts.

From these considerations, the question that arises is, how do we process the acquisition of mathematical knowledge? When trying to answer it, Tall and Vinner (1981) developed a theory based on the notions of "concept image" and "concept definition". For the authors, the concept image

> [...] describes the cognitive structure associated with the concept that includes all mental figures and associated properties. It is developed along the years, through experiences of all types, changing while the individual finds new stimulus and gets mature. (Tall & Vinner, 1981, p.2)

According to the authors, the concepts can only be mechanically memorised by students. Either, they can be meaningfully built and understood as they have opportunities to create different images about the mathematical concept.

The concept definitions are described by Tall and Vinner (1981) as the conjunct of words used by students to explain their ideas having as reference the concept images built by themselves. The concept definition is not something static, but changes as the student modify his/her concept images about specific content and can coincide with the language used by academic texts or can be explained with its own words.

For Tall (1994), there is an interrelation between the student's success in Math and the creation of concept images. According to the author, the process of teaching, centred only in algorithms and rules, emphasising technical and analytical aspects, that is, promoting the procedural knowledge and not promoting reasoning and the use of visual representation, can be one of the reasons for the failure of Mathematics.

This idea of how working with math concepts promoting only the procedures is an obstacle for the comprehension, mainly concerning the graphics features, and can contribute to the creation of restrict concept images. For the comprehension of a particular concept to happen, it is necessary before anything that meaningful concept images be built by students.

On the research of Bossé and Bahar (2008), when concerning about the teaching and learning of Math, the authors affirm that both types of knowledge, conceptual and procedural, are fundamental and might be valued and seen as complementary, even though each one has its focus. For the authors, it is necessary to understand that to learn new concepts and acquire new abilities in Math is necessary to establish a connection between the concepts. The procedural knowledge, as well as the concept image, is understood as an element that is part of the learning process and is not a starting point for the study of new content.

In this work, we analyse the comprehension of the concept of the plane curve length, developed by students of a degree in Mathematics. As a presupposition of this study, a Hypothetical Learning Trajectory was put into practice during classes of Differential and Integral Calculus. It is our intention to look for evidence for the concept comprehension by empirical data as the ones obtained in the resolution of de activities developed by the students because we admit that the comprehension is possible of being shown in the resolutions as well as in oral manifestation.

According to Simon (1995), a hypothetical learning trajectory comprises three aspects: the learning objectives, the activities used to promote the students learning, and the hypothesis about the learning process referring to the mathematical concept studied. Brunheira (2017, p.34), based on the studies from Clements and Sarama (2004), highlights that "a learning trajectory comprises a learning objective, a theoretical perspective (or pattern) about the progression of students' thinking and learning in a determined dominion, and a sequence of activities".

According to the authors Simon and Tzur (2004, p.93), the following assumptions are underlying in a hypothetical learning trajectory:

- The construction of a hypothetical learning trajectory is based on the comprehension of the current knowledge of the students involved;

- A hypothetical learning trajectory is a vehicle for the learning planning of specific mathematical concepts;

- The mathematical activities provide the tools to promote the learning of specific math concepts and, therefore, are fundamental in the teaching process;

- Because of the hypothetical nature and the inherent incertitude of the process, the teacher should review each aspect of the hypothetical learning trajectory systematically.

Although all researchers recognise the three elements that constitute the hypothetical learning trajectory, each one uses and interprets the notion with different ways and objectives.

Ivars, Buforn and Llinares (2016) highlight that there are different conceptions about the learning trajectory, but according to the authors,

> A common characteristic is that it provides a learning progression pattern of particular mathematical concepts (more and more sophisticated progression levels of mathematical thinking) linked to a learning objective obtained from the results of previous investigations and a conjunct of possible activities that can support this learning progression.

In this work, the hypothetical learning trajectory presented by Brunheira (2017) will be used, based on studies by Clements and Sarama (2004). The three components are, therefore, considered. First, the learning objective, which in this research is the comprehension of the concept of the plane curve length. Second, the theoretical reference, which here is the theory based on the concept image and concept definition described by Tall and Vinner (1981), the ideas from Star and Stylianides (2013), and Bahr and Bossé (2008) about conceptual knowledge and procedural knowledge. Lastly, an activity sequence, which was elaborated observing some students previous knowledge and the different mathematical representations of the plane curve length concept.

It is believed that the identification of these elements and the establishment of relations among them, through the different concept images built in a learning trajectory, is a way of characterising if there was a “comprehension progression” of the concept studied.

METHODOLOGY

The methodology used for the development of this research if of qualitative nature (Bogdan & Biklen, 1994) and is based on the construction and development of a learning trajectory. The study follows the design of a learning experience (Brown, 1992; Cob, Confrey, Disessa, Lehrer, & Achauble, 2003) that happens in the natural classroom scenery, with students from the Differential and Integral Calculus subject. The data is produced in the classroom context o and during the realisation of the activities proposed by the teacher.

The participants of this research were nine students, here named as E1, E2, E3, E4, E5, E6, E7, E8 and E9. Students formed three pairs and one group of three. Groups were named as G1, G2, G3 and G4. All students accepted participating in the research, and for this purpose, they signed the Informed Consent Form.

The data survey was done through the collection of students' reports done in their field diaries, from the activities resolutions, the notes from the teacher's filed diary and the audio recordings. Na audio recorder was given to each group and the students' speech, as well as their comments about the resolutions, were transcribed.

The learning trajectory, which focused on the plane curve length, was structured in three steps detailed further on and developed by the first author, totalling nine classes in the first semester of 2018.

At first, a conjecture to guide the work developed was elaborated. This conjecture had the following outline: "*A work, based on a sequence of activities that privileges the aspects concerning the measurement of line segments, sum of Riemann and the notion of limit in a sum, can provide the creation of concept images about the plane curve length and the students' comprehension about this mathematical content as they build the meanings shared in classroom.*"

To infer if there was the students' comprehension about the plane curve length concept, we analysed the concept images built by them during the development of the teaching activities. Domingos (2003) classifies complexity levels established from determined mathematical activities made for students in order to characterise the way they understand the concepts taught before. The author identified three different levels of concept image: incipient concept image, instrumental concept image and relational concept image. According to Domingos (2003),

> The main objective when establishing these levels was to distinguish different types of concept images that can coexist in the student's mind. In front of teaching determined concept, some students show image concepts very close to some exclusive from the elementary mathematics (incipient concept image), while others show conceptions closer to the ones that characterise advanced mathematics (relational concept image). (Domingos, 2003, p.129)

For the analysis of the results, the concept images built by students were classified according to the three levels defined by Domingos (2003).

There were considered incipient concept images the answers given by the students using only their elementary and little relation between the mathematical concepts involved; there were considered instrumental concept images when the student is capable of doing some procedures but does not establish links among concepts or the link are weak. There were considered relational concept images the answers in which the student described with a precise mathematical language the different steps of curve length calculus, did not make mistakes and knew how to relate the mathematical concepts. In the last category, there are some relations among the various representations such as the algebraic and graphic ones, and the mathematical processes are more complex. Therefore, in this level, there is the chance from the elementary to the advanced concept, modifying the students' cognitive structure.

In order to evaluate if there was learning progress, the following levels of reasoning were taken into consideration:

a) The intuitive knowledge of approximating the length of a curve by line segments to the representation of the curve length.

b) The construction of the meaning of each calculus procedures.

To analyse the first level, there were considered the concept images built from the graphic representation of curve approximation by the line segments with the help of Geogebra, the use of the sum from Riemann and the chance to the limit to obtain the length of a curve.

In the case of the meaning construction of procedures, there were analysed if the students calculated well, if they knew how to justify it graphically and if they understood how the algorithm for the calculation of the curve length works, by explaining its functioning clearly.

DATA ANALYSIS AND RESULTS

We present, as follows, the data analysis and the results referring to the activities developed. For that purpose, it was done a description of the way the activities were developed in the classes, the way the students solved the questions and then, it was analysed how the students took the comprehension of the concept of the plane curve length and if the activities that compose the teaching trajectory facilitate the comprehension. The learning trajectory was structured as follows:

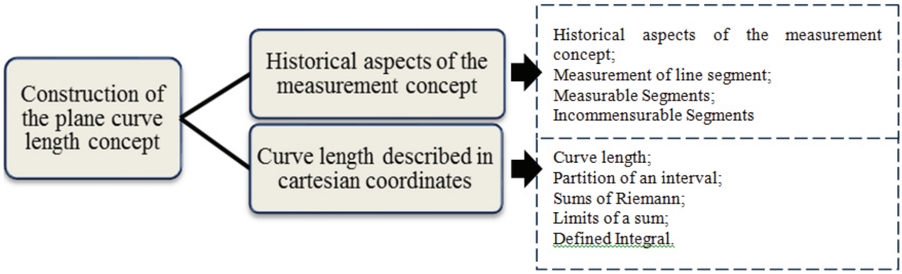

Figure 1. The structure of learning trajectory.

The first step of the learning trajectory aimed at recovering historical aspects of the measurement concept, of measurable and incommensurable segments, beyond identifying students' previous knowledge about the notion of length measurement.

On the second step, the purpose was to construct the curve measurement concept when it is not built only by line segments and/or is a representation of the graphic of a continuous function and a defined derivative in a determined interval.

In this work, the activities developed and the answers given by students to show the way the concept was built will be presented.

Activity 1 – a graph paper was distributed to the students, and the following was asked:

– *Draw three possible paths to go from city A to city B.*

The groups talked and represented possible paths to go from one city to another in this activity.

Figure 2, as follows, presents the possibilities designed by the four groups.

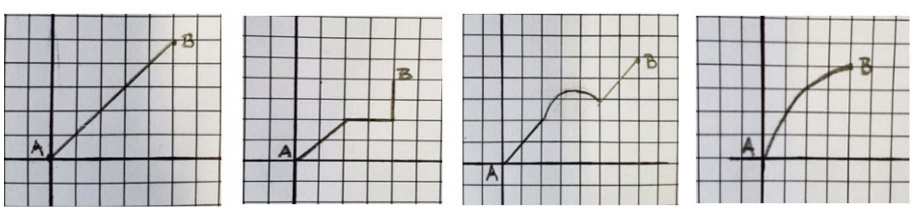

Figure 2. Paths from city A to city B designed by groups G3, G2, G1 and G4, respectively.

We observed that all groups located city A in the origin of the Cartesian system. Students from group G3 said the straight line was the easiest path, and group G2 drew a path with line segments. When asked about this issue, students answered, "These are the easiest ways to represent that". We noticed that when building the path, the students from group G3 thought about the shortest distance between both cities. Considering that, these students have a crucial previous knowledge: that this path has a minimum length. Students from group G2 design the paths represented by line segments, group G3 drew a path with line segments and a curve, and group G4 represented the path with a curve.

After taking all the representations from the groups, the teacher asked:

– Is it possible to measure the length of each of these paths?

All students tried to calculate the curves' length. In the case of the first two drawings, students did not have difficulties. As they were working with the graph paper, they used the side of the square as the measuring unit to make the calculation. On the following two cases, appeared the first obstacle. Students commented:

E5 – *In the case of having only segments was easy, but with the curve…*

E2 – *The difficult here is to measure the piece in curve… if it was a circumference, we would know*... (The student possibly referred to the circumference, once they knew how to calculate its length).

E4 – *I have an idea... we can try to straighten this curve... let's calculate it was it was a line...*

Teacher – How can you approximate it by a line? Try to represent it on the drawing…

Two groups united it by a line segment and excluded the curve from the drawing, but groups G1 and G4 discussed between themselves the possibilities and approximated the curve by two segments as showed in the drawing from Figure 3.

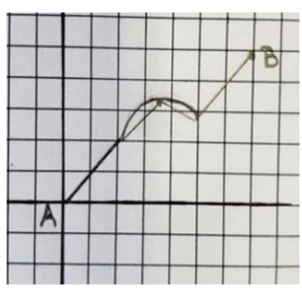

Figure 3. Representation elaborated by groups G1 and G4.

As students were working with graph paper, they used the side of the square as the measuring unit to obtain the segments' length. The teacher asked, "*Can we use the classmates' ideas to calculate the curve length of the path described by group G4?"*

E8 – *Let's try... it seems to be easy.....*

Figure 4 shows the drawing done by group G4.

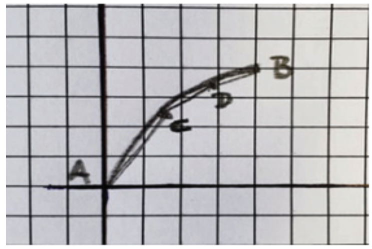

Figure 4. Representation elaborated by group G4.

Students described that the curve length could be closer by the sum of the length from segments AC, CD and DB. We can conclude, in this case, that students understood that the more segments they inserted, the closer they would get from the curve length. From the speech and the drawings done by groups G1 and G4, there are signs that these students built concept images classified in instrumental levels because they were able to approximate the curve graphically by line segments and use the procedure of calculating the length of each segment and sum them. The other groups were capable of approximating the curve by line segments but always waited for the teacher's explanation to make the calculations. We can infer that these students still build incipient images concerning the concept of curve length.

The second part of the activity was built with the help of Geogebra.

Students did the curve approximation, considering part of an interval constituted by four points, and calculated the segments' length. Figure 5, as follows, shows the construction of group G1.

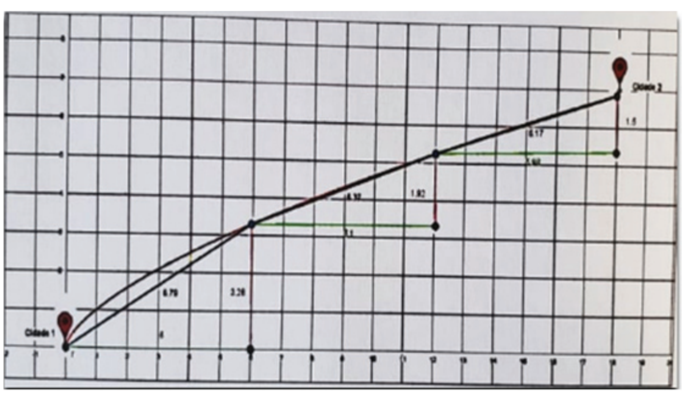

Figure 5. Construction elaborated by group G1.

The line segments were designed by L1, L2 and L3, and the students calculated the sum of the lengths obtaining:

$$\sum_{k=1}^{3} L_k = L_1 + L_2 + L_3 \cong 6{,}79 + 6{,}3 + 6{,}1 \cong 19{,}19.$$

Refining the partition and considering the one constituted by the seven points, the students from group G2 represented the approximation of the curve by line segments as showed in Figure 6.

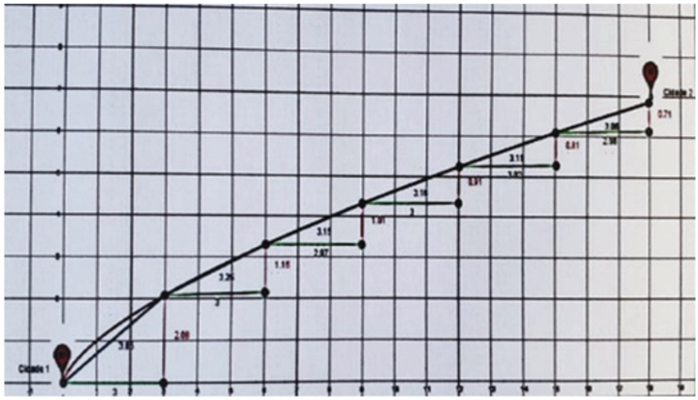

Figure 6. Construction elaborated by group G2.

The sum of the segments' length obtained was

$$\sum_{k=1}^{6} L_k = L_1 + L_2 + L_3 + L_4 + L_5 + L_6 \cong 19{,}38.$$

Refining once more, considering now the partition with 13 points, group G2 represented the approximation as showed in Figure 7, as follows.

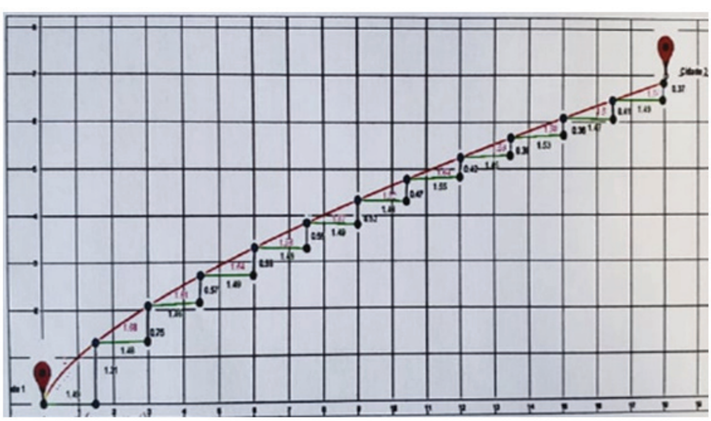

Figure 7. Construction elaborated by group G2.

In this case, the sum of the segments' lengths was:

$$\sum_{k=1}^{12} L_k = L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 + L_9 + L_{10} + L_{11} + L_{12} \cong 19{,}41.$$

Students observed that the values of the sums were very close and commented:

Students: we thought that as we fragmented the curve in more line segments, the sum of the measures would approximate more from the curve… let's fragment the curve until we find the length…

We observed that in this case, there was the comprehension from students about the process of approximating the curve length by the sum of the line segments' length and that refining the partition provided a better approximation. It is possible to conclude, then, than the students who had more difficulties, could develop this activity and that Geogebra helped them in this resolution. Thus, we can infer that the concept images built can be classified at the instrumental level, also evidencing, by students' speech, that some students built concept images at the relational level.

Activity 2 – Draw on Geogebra, the graphic of the function $f(x)=\frac{2}{3}x^{3/2} - 1$ on the interval $0 \leq x \leq 5$ and calculate the approximate curve value, inserting (refining the partition) two points; eight points and twelve points, respectively.

This task was done by all groups, once it did not present many difficulties, and the students knew how to work with Geogebra. In Figure 8, we have the construction of first length approximation, inserting two points in the partition and obtaining three line segments. For this first approximation, the students obtained the value L1=8,98 as the approximate length. On the second step, they inserted eight points, obtaining the approximate value of L2=9,02 and, to finish, inserted twelve points, obtaining the value L3=9,01for the curve length.

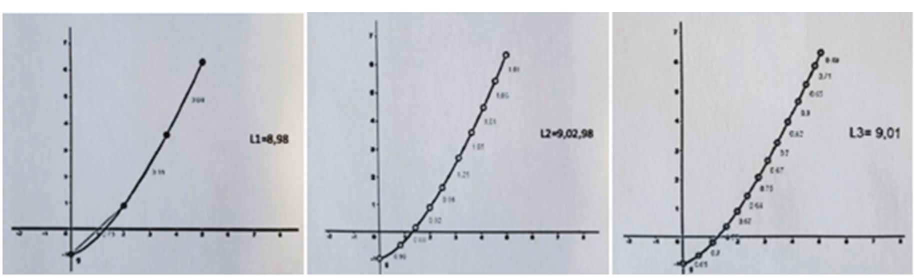

Figure 8. Approximations of the curve length elaborated by students on Geogebra.

When developing this activity, students noticed that as in the previous case, the more points they inserted in the partition, the better would be the approximation. We can infer this perception from their dialogue.

E5 – *On the second approximation, the value raised, but on the third, when we put twelve points, the value got a little lower. Shouldn't it increase? Or is its length smaller?*

This observation aroused the curiosity of all classmates.

Teacher – *Can we discover if this value will increase or decrease?*

E6 – *Can we try to calculate it, inserting more points.... And calculate...*

Teacher – *This is a possibility, but, and what if we do not know the law of the function? Let's try to calculate the length of a curve in a general case?*

Students – *In a general case, you mean for any function?*

Teacher – *Yes, we will take a function that is continuous and defined in a limited interval, for example, in the line interval [a,b]. Are you capable of drawing graphics of continuous functions?*

The students draw different curves. We observed that they had very clear concept images referring to continuous functions. None of them drew curves with points of discontinuity and, although the teacher has not talked about the function derivability, all of them drew continuous functions "without points"; the curves were smooth.

Teacher – *Let's take the path linking city A to city B, designed by group G4. We do not know which function originated this graphic. How can we calculate the length of this curve?*

Students from groups G1 and G4 immediately started to draw and calculate. They represented the points by $(x_0,f(x_0))$, $(x_1,f(x_1))$, $(x_2,f(x_2))$,...,$(x_b,f(x_b))$. In order to calculate the L length of the curve, it was necessary to make a sum of the lengths of the segments. The teacher asked, "*How do we calculate the distance between two points?*"

The students discussed among themselves because it was necessary to remember the previous contents already studied and necessary to answer this question. They tried to remember the process by calculating the distance between two points with numerical coordinates and tried to transfer for a general case. This was one capacity developed by students, to associate concept images already built in previous subjects and transfer them to this new context. Vinner (1989) says that the definition of a concept can keep inactive, but the concept images can always be evoked to solve other problems.

Evoking previous knowledge, in the general case, the length of the k-th segment was written as:

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

Summing the lengths of the line segments, the students had the L length of the curve:

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

At this moment, the teacher observed that she could use the Mean Value Theorem, so, it was necessary that the function was derivable to guarantee the existence of a x_k^* point between x_{k-1} and x_k, such as the following equality would be true:

$$\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}} = f'(x_k^*) \text{ or } (x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k.$$

The teacher conducted this explanation until students obtained a representation of the curve length by the sum,

$$L \approx \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\Delta x_k.$$

The passage to the limit when n grows, that is, the extensions of subintervals tends to zero, and the acquisition of the integral that defines the curve length was also a process conducted by the teacher.

$$L = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}dx.$$

It was necessary to ask students to evoke the images referring to the calculus of limits and the sums of Riemann, images that were built during the study of the defined integral, to comprehend the meaning of a curve by a defined integral. This step, of changing the elementary knowledge for a piece of advanced mathematical knowledge, was one of the tasks that presented more difficulty. Only students from groups G1 and G4 could build concept images classified in an instrumental level, once the use of a mathematical language needs concepts already worked in Calculus, and those aspects were the ones that raised difficulties. Some students also had difficulties in the resolution of integrals; they presented little abilities to use the methods of resolution. According to Bahr and Bossé (2008), conceptual and procedural knowledge should be complementary so that the learning happens. In addition, the little ability to calculate can have as a reason the construction of restrictive concept images about the content needed for the resolution of the questions. From the categories listed, none groups built the relational concept images, although some students had meaningfully evoked the previous knowledge needed for the problem solutions.

FINAL CONSIDERATIONS

We presented in this article the partial results of a research that aimed at analysing the comprehension of the curve length concept by students taking a mathematics teaching degree and, for such, it was elaborated a hypothetical learning trajectory. In order to analyse if there was the comprehension of the concepts by the students, concept images built by the students were examined. The concept images were built when students graphically represented the curves approximation by line segments, using the construction process of Riemann sum, and the passing to the limit to obtain the defined integral. We observed that the last items were difficult to be understood, and the process could be done with the help of the teacher. We can infer, from the resolutions' analysis and the dialogue established among the students themselves and between the teacher, that at least the participants from groups G1 and G4 could build concept images at the instrumental level, and in some moments, concept images at the relational level, once they could establish some links between the concepts. In this case, we can imply that procedural knowledge was prevalent, but it also evidenced characteristics from conceptual knowledge. The other groups could with the teacher's explanations, represent the curves' approximation graphically by the line segments using Geogebra. In this situation, the use of a technological resource contributed to the comprehension of the process to obtain the plane curve length. These groups created concept images that are between the incipient and instrumental levels. We observed that the hypothetical learning trajectory allowed a progression in the learning process of students, even though not everybody created concept images at an instrumental level and little did in relational level. The results obtained in this research leads us to reflect concerning the potentiality of using hypothetical learning trajectories for the teaching of Calculus. It also showed the need for redesigning this trajectory, to favour students' learning process.

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AUTHORS CONTRIBUTIONS STATEMENTS

This article was conceived by the three authors, being the theory developed by E.B and V.B. E.B.R adapted the methodology, performed the activities and collected the data. E.B.R. and E.B. analysed the data. All authors discussed the results and contributed to the final version of the manuscript.

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