

Mathematical generalization from the articulation of advanced mathematical thinking and knot theory

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ABSTRACT

Background: The professors of Topology and modern algebra expressed interest in the need to create a space that allows deepening the process of Mathematical Generalization from the articulation of some concepts of Theory of Knots with the development of Advanced Mathematical Thinking (PMA) skills. **Objective:** To offer students an additional space for disciplinary training that allows them to deepen the process of Mathematical Generalization. **Design:** The methodology used has a qualitative approach, as a strategy we take action research from the Whitehead (1991) proposal from three phases. **Setting and participants:** students of the Bachelor of Mathematics program who take the third to sixth semester. **Data collection and analysis:** we emphasized in the second phase (intervention), since it allowed us to articulate the holistic scheme of knot theory with the PMA as shown in Tables 2, 3 and 4 (results section). **Results:** The result was the creation of a syllabus and subject guide for an elective seminar, which is offered to undergraduate students. **Conclusion:** since 2019 this elective seminar is offered to students, which awards 3 credits.

Keywords: Articulation; Mathematical generalization, Knot theory; Advanced mathematical thinking

Generalización matemática desde la articulación del pensamiento matemático avanzado y la teoría de nudos

RESUMEN

Antecedentes: Los profesores de Topología y algebra moderna manifestaron interés en la necesidad de crear un espacio que permita profundizar el proceso de Generalización Matemática desde la articulación de algunos conceptos de la Teoría de Nudos con el desarrollo de habilidades del Pensamiento Matemático Avanzado (PMA). **Objetivo:** Ofrecer a los estudiantes un espacio adicional de formación disciplinar que les permita profundizar el proceso de Generalización matemática. **Diseño:** La metodología utilizada tiene un enfoque cualitativo, como estrategia asumimos la investigación-acción desde la propuesta de Whitehead (1991) desde tres fases. **Entorno y participantes:** estudiantes del programa Licenciatura en Matemáticas que cursan de tercer a sexto semestre. **Recopilación y análisis de datos:** enfatizamos en la segunda fase (de

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intervención), dado que nos permitió articular el esquema holístico de la teoría de nudos con el PMA como se muestra en las tablas 2, 3 y 4 (sección resultados). **Resultados:** El resultado fue la creación de un sílabo y guía de asignatura para un seminario electivo, que se oferta a los estudiantes de la licenciatura. **Conclusión:** desde el 2019 se oferta a los estudiantes este seminario electivo que otorga 3 créditos.

Palabras clave: Articulación; Generalización matemática; Teoría de nudos; Pensamiento matemático avanzado.

Generalização matemática a partir da articulação do pensamento matemático avançado e da teoria dos nós

RESUMO

Antecedentes: Os professores de topologia e álgebra moderna manifestaram interesse na necessidade de criar um espaço que permita aprofundar o processo de Generalização Matemática a partir da articulação de alguns conceitos da Teoria dos Nós com o desenvolvimento das habilidades do Pensamento Matemático Avançado (PMA). **Objetivo:** Oferecer aos alunos um espaço adicional para treinamento disciplinar que lhes permita aprofundar o processo de Generalização Matemática.

Desenho: A metodologia utilizada possui abordagem qualitativa, como estratégia adotamos a pesquisa-ação da proposta de Whitehead (1991) em três fases. **Cenário e participantes:** alunos do curso de Bacharelado em Matemática do terceiro ao sexto semestre. **Coleta e análise dos dados:** enfatizamos a segunda fase (intervenção), uma vez que ela permitiu articular o esquema holístico da teoria dos nós com o PMA, como mostra as Tabelas 2, 3 e 4 (seção de resultados). **Resultado:** O resultado foi a criação de um plano de estudos e um guia de disciplinas para um seminário eletivo, oferecido a estudantes de graduação. **Conclusão:** desde 2019, este seminário eletivo é oferecido aos alunos, com três créditos.

Palavras-Chave: Articulação; Generalização matemática; Teoria dos nós; Pensamento matemático avançado.

INTRODUCTION

The theory of knots as a disciplinary field of topology does not simply underlie the purely disciplinary mathematics processes. This theoretical field can be approached from a problem of the didactics of mathematics, allowing future mathematics degree holders to delve into this disciplinary area from another academic perspective. Linking some concepts of knot theory with the development of advanced mathematical thinking skills allows us to expand the process of mathematical generalization aiming to strengthen the range of didactic strategies that guide the mathematics teaching and learning that enables, reflectively, and innovatively, the interaction with various backgrounds and training levels.

This paper is divided into seven sections: the first describes the background. The second section presents the normative framework of the research, describing the conceptual areas that allow dealing with the investigative work related to the process of mathematical generalization, the skills of advanced mathematical thinking, and some concepts of knot theory. The third, presents the methodology used. The fourth section

brings the analytical results of the articulation between the categories proposed and the skills of advanced mathematical thinking, and the incorporation of some elements of the knot theory that allowed us to build the syllabus, the subject guidelines and the validation of the elective seminar. In the fifth, we present some conclusions. In the sixth some recommendations, and finally the references used.

BACKGROUND

This research aimed to identify how some basic concepts of the knot theory make possible the development of advanced mathematical thinking (PMA in Portuguese acronym, AMT from here onwards) skills from the process of mathematical generalization with students of a mathematics degree course, attending the third to the sixth semester of a non-state university of the city Bogotá D.C., through the planning, design, and implementation of an elective seminar called: “Una Aproximación a la Teoría de Nudos, y su incidencia en el proceso de generalización matemática, desarrollando habilidades del PMA/An approach to the knot theory, and its impact on the process of mathematical generalization, developing skills of the AMT”.

The research problem established is defined under the conceptual elements of the knot theory and its incidence in the process of mathematical generalization based on the theoretical position of Mason et al. (1999), stating that three substantial aspects must be considered in said process: seeing, expressing and saying. In this order of ideas, the knot theory, through aspects of logical-mathematical thinking and the teaching of elementary algebra, will allow us to delve into each of the aspects established for the development of the AMT skills from the process of mathematical generalization in a transversal way, and articulated with its substantive elements.

As a relevant aspect of the research, we evidenced that few investigations are addressing the AMT skills and that integrate them with the process of mathematical generalization; besides, there are no other research works that relate them to some concepts of the knot theory. This incidence accounts for the contribution of this research in the field of the didactics of mathematics.

THEORETICAL FRAMEWORK

Mathematical generalization process

Mason et al. (1999) define and characterizes mathematical generalization as a process that can be approached from three phases: *Seeing*, *Saying*, and *Recording*. These become evident when carrying out inductive methods from regularities, which in generic terms converge with each other according to the circumstances that arise when interacting with a particular problem of regularity. Although the dynamism of this process allows to enhance the abilities for the compression of particular mathematical objects and to

strengthen the recognition of algebraic symbolization, Mason et al. (1999) tacitly show that mathematical generalization must be a process transversal to the mathematical activity in the classroom, as it is a transversal element that impacts the content taught there.

Mason et al. (1999) define the three phases described as follows: *Seeing*: “(...) relates to the mental identification of a pattern or a relationship (seeing a pattern can occur after a time working with some particular examples)” (p. 17). *Saying*: “(...) can take place both out loud, to other people, and in words that are said ‘in the mind’” (p. 21). *Recording*: “It is making language visible, which requires a movement towards symbols and written writing” (p. 17). They mention that Recording may involve various mechanisms, such as drawings; word-supported drawings; just words, and some symbols; just symbols; algebraic notation (p. 23).

Like the theoretical elements mentioned by Mason et al., (1999), the Azarquiel Group (1993, p. 31) defines 3 stages of the generalization process: “The vision of regularity, the difference, the relationship. (*Seeing*); Verbal exposition (*Describing*); Written expression, in the most concise way possible. (*Writing*)”.

First, *Seeing* is related to how the elements of a particular situation or regularity are perceived and distinguished, that is, to observe the situation in a different way, from a new perspective. “It is about distinguishing between what is proper to each situation, each example, and what is common to all of them; what does not vary” (Azarquiel, 1993, p. 31). Stage two, *Describing*, focuses on characterizing and detailing regularity perceived through oral expressions, in other words, it tries to communicate and describe what has been seen, how it is done, and the elements subtracted from regularity, the symbolic expression is more or less exact (Azarquiel, 1993, p. 37). In the development of *Describing* group work, “it facilitates the exchange of ideas and opinions because communication with others fosters the joint testing of conjectures, the reformulation of hypotheses, the gradual approach to increasingly adjusted solutions” (Azarquiel, 1993, p. 38). The third stage, *Writing*, aims to record in writing the ideas that arise from the regularity characterization. In this sense, “Recording does not necessarily mean writing a symbolic expression. The symbolic expression is only a way of doing, and not exactly the most natural one (...)” (Azarquiel, 1993, p. 37). We observe in both proposals presented a systemic articulation between seeing, saying (describing), and recording (writing).

Sessa (2005) presents in a methodological way the process of mathematical generalization, through seven stages to conceive the generalization of a regularity. 1) Give a pattern and ask the guiding question about the generality. 2) Ask guiding questions about the occurrence of the generality. These first two stages are carried out individually. 3) Group meeting and discussion on the behavior of the regularity. 4) General discussion (all groups) on the fundamental characteristics of the generality. 5) Each group is asked to present in writing a formula that represents the regularity. 6) The different formulas are presented to all the groups and they reflect collectively on which is the most appropriate. 7) Useful alternatives of the formula are proposed to the group in general. When articulating these theoretical references, we observe that the process of generalization, in a convergent way, contains three broad categories (*Seeing*, *Describing* and *Writing*) that are implicitly

associated with a¹ specific action of the process, thus allowing full recognition and characterization of the interior of each category.

Advanced Mathematical Thinking Skills (AMT)

Elementary mathematical thinking (EMT) Garbín (2005, p. 142) “is considered as a preliminary stage, as the first level, of the AMT. It is a stage and an intellectual moment in which the mathematical contents do not require a previous formalism”. In this sense, this type of thinking is closely related to basic school education. The AMT is primarily related to the teaching of mathematics in high school and university, however, as Belmonte (2009) mentions, citing Dreyfus (1991, p. 58):

There is no obvious distinction between many of the EMT and the AMT processes, even though advanced mathematics focuses largely on the abstractions proper to definition and deduction (...) It is possible to think about advanced mathematics topics in an elementary way and there is also advanced thinking about elementary topics. A differential feature between the EMT and the AMT is the complexity and its manipulation (...).

We recognize that the AMT has particular ways to approach mathematics teaching. Besides the elements described by Garbín (2005), some cognitive abilities or capacities relate to it. Azcárate, Camacho, and Sierra (1999, p. 284) indicate that: “(...) progressive mathematization implies the need to abstract, define, analyze and formalize. Among the cognitive processes with a psychological component, besides abstracting, we can highlight representing, conceptualizing, inducing and visualizing”. However, although “(...) abstraction is not a characteristic of higher mathematics, nor is analyzing, categorizing, conjecturing, generalizing, synthesizing, defining, demonstrating, formalizing, it is evident that these last three gain greater importance in higher courses (...)” (Azcárate, Camacho, & Sierra, 1999, p. 284).

Knot theory

The knot concept (*Figure 1*) is the mathematical abstraction that arises from the traditional notion, which was used to tie a stone to a piece of wood to form an ax or braid lianas to build ropes, which were later knotted to build fishing nets (Cisneros, 2011). The knot theory seeks to establish a specific characterization that allows the compression of its elements from a mathematical perspective, hence, “it is necessary to highlight that the study of the knots is carried out thanks to the use of very deep techniques that come from different branches of mathematics such as geometry, algebra and analysis” (Vendramin, 2014).

¹ The associated actions were determined from the elements that were evidenced in each of the examples proposed by the authors mentioned.



Figure 1. Example of knots. Image taken from the KnotPlot software

Xiao (2012) defines the “knot as a subset in \mathbb{R}^3 homeomorph² an S^1 , that is, a connected, compact and borderless curve within a three-dimensional space” (p. 7). Molina (2011) indicates that: “The subset $K \subset \mathbb{R}^3$ is a knot if there is a homeomorphism of the unit circle S^1 in \mathbb{R}^3 whose image is K . Where S^1 is the set of points (x, y) on the plane \mathbb{R}^2 that satisfy the equation $x^2 + y^2 = 1$ ” (Molina, 2011, p. 8).

The graphic representation of the knots is closely related to the studies carried out by Peter Tait at the end of the 20th century, classifying them with ten cuts or intersections (Berenguer & Gil, 2010), generating a new structure for the classification and understanding of the knots. The graphic representations proposed by Tait were a significant input to graphically understand some characteristics and particularities of the knots. Besides, synthetically we can verify that a knot can be represented in a plane or three-dimensional way as long as its particularities and theoretical aspects are respected.

For this work, we present some types of knots with their respective characterization: *Trivial Knot*: It is simply an untied rope (Xiao, 2012, p. 7). *Prime Knot*: “Given two knots, a connected sum (sum or composition of knots) can be made, which consists of eliminating an arc in each knot that does not pass through any intersection and joining the extreme points of those arches through paths that do not intersect each other.” (Xiao, 2012, p. 13). *Figure 2* presents this type of knot:



Figure 2. Prime knot. (Xiao, 2012, p. 14)

Equivalent Knot: One knot is equivalent to another, “(...) if and only if it can be passed from one to the other by a finite number of type I, II and III transformations (...)”, *Figure 3*.

² Intuitively two objects are homeomorphic if one is obtained from the other after a non-traumatic deformation, that is, without ruptures or opening of holes. (López, 2016, p. 1).

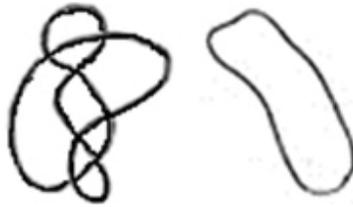


Figure 3. Equivalent knot. (Xiao, 2012, p. 9)

A knot can have multiple equivalences and shapes, however, different correspondences can be obtained from each other, by carrying out a series of transformations known as “*Reidemeister Move*”. According to Livingstone (1993), Reidemeister’s theorem ensures that two knots are equivalent if their diagrams can be converted into one another by a sequence of moves. Reidemeister’s theorem establishes three types of moves (Xiao, 2012, p. 9), (*Figure. 4*):

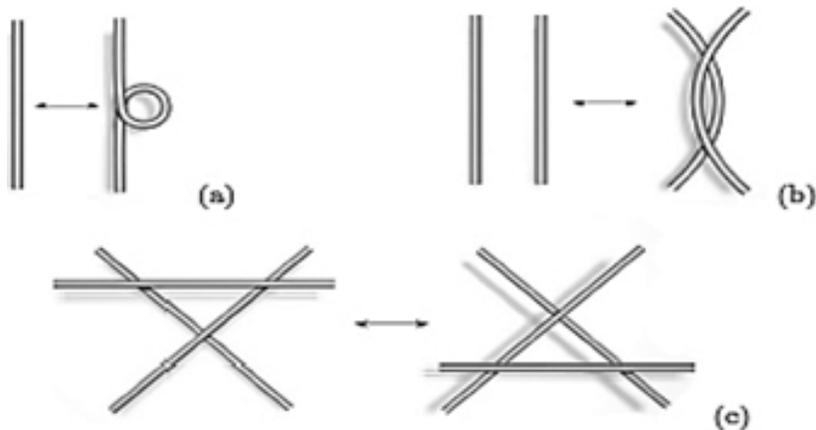


Figure 4. Reidemeister moves (a) Type I (b) Type II (c), (Xiao, 2012, p. 9)

Molina (2011) establishes a definition for each type of Reidemeister moves. Type I: add or remove a curl, type II: add or remove two consecutive over (under) crossings, and type III: triangular move. In short, Reidemeister’s three moves demonstrated that “two knots (or links) in the space can be deformed into each other if and only if their regular diagrams can be transformed into each other by the three moves” (Molina, 2011, p. 30). Thus, the use of these moves when manipulating the knot can allow the recognition of equivalent or trivial knots as the case may be.

As the understanding of the everyday notion of knot is deepened from various mathematical aspects, it becomes necessary to look for a mathematical notation that significantly represents their particular characteristics. Gauss established the most

representative notation, which consisted of transforming the graphic aspect of the knot into a Lxn matrix. Molina (2011) describes this notation as follows:

This notation was based on an oriented knot diagram. We select a point other than a crossing arbitrarily, and follow the path according to the orientation up to the first crossing, which is then labeled with the number 1; then, we follow the path until the next crossing, which, if not labeled, is assigned the following number; if it is already labeled, we go on to the next crossing until we reach the point we selected initially. Once the diagram has been labeled, we follow the path of the knot from the point selected, and we write either positive or negative on each one of the labels we go by, according to the crossing: if it is over, it will be attributed a positive sign, otherwise it will be attributed a negative sign (p. 32).

METHODOLOGY

We developed the research with students attending from the third to the sixth semester of a mathematics degree course at a non-state university in the city of Bogotá, during two academic semesters in 2019. We choose a qualitative approach, and as a strategy we adopted the action research to address the goals of the research proposed. The approach considered three phases: exploratory, intervention, and results (*Figure 5*). Table 1 shows the division considered to address each of the proposed phases.



Figure 5. Phases of research and intervention

Table 1
Description phases of the investigation

Phase Name	Description
I. Exploratory phase	<p>This phase was oriented to the definition of the research problem, to the recognition of the diverse perspectives and theoretical bets that are related to the categories of analysis that we defined.</p> <p>This phase gave us a descriptive overview of what tools we should use: diagnostic test, validation, evaluation rubric, validity process of the seminar proposed, and analysis of the information, which allowed us to systematically understand the research problem.</p> <p>We approached this phase from the Whitehead's (1991) proposal, cited by Suarez (2002), on the action-research cycle. In the framework of this methodological phase, the diagnostic test intervention processes, instrument validation, and validity of the elective seminar we proposed were developed.</p>
II. Intervention phase	<p>According to the results obtained in the diagnostic test, a preliminary version of the syllabus and subject guidelines was designed with its respective piloting, which was subsequently validated through the completion of the elective seminar during the periods 2019-I and II (16 weeks, each) at a non-state university in the city of Bogotá.</p> <p>The moments of the process of piloting and validation of syllabus and subject guidelines allowed decisions for the final consolidation of the syllabus of the elective seminar.</p>
III. Results phase	<p>In this phase, each of the results of the diagnostic test, piloting, and validation of the elective seminar (intervention phase) was evaluated and analyzed. Likewise, some conclusions, recommendations, and inquiries of the research study were provided in light of the objectives stated.</p>

We focus our work on the intervention phase from five moments that allow us to integrate the skills of the AMT with the knot theory as an innovative axis of the process, aiming to strengthen the process of mathematical generalization. In Table 2 we show the holistic scheme that we established to articulate these research purposes.

Table 2
Articulation between the process of mathematical generalization with the skills of the AMT

Research phases	Intervention Phases	Activities
I. Exploratory		<ul style="list-style-type: none"> ❖ Definition of the research problem. ❖ Definition of the categories of analysis. ❖ Recognizing the various perspectives and theoretical bets (Background-Theoretical Framework). ❖ Recognition of information gathering tools

Research phases	Intervention Phases	Activities
	Feeling or experiencing a problem and imagining the solution to the problem	<ul style="list-style-type: none"> ❖ Defining the research problem. ❖ Recognizing the various perspectives and theoretical bets (Background-Theoretical Framework).
II. Intervention	Implementing the solution imagined	<ul style="list-style-type: none"> ❖ Designing and validating information collection instruments. ❖ Designing, validating and piloting the elective seminar
	Evaluating the results of the actions taken and modifying the practice in light of the results	<ul style="list-style-type: none"> ❖ Results of the information collected. ❖ Analysis of the information collected.
III. Results.	<p>Analysis of results</p> <p style="text-align: center;">Triangulation</p> <p style="text-align: center;">Description and qualitative intervention Description and quantitative intervention</p> <p style="text-align: center;">↓</p> <p style="text-align: center;">Conclusions, recommendations, and limitations of the research</p>	

RESULTS

For this work we define three macro-categories (*Seeing, Describing and Writing*), articulating the contributions of Mason et al. (1999), Azarquiel (1993) and Sessa (2005), which allowed us to highlight, in the observed students, the potentization of some of the skills of the AMT (abstracting, defining, analyzing and formalizing), for the comprehension of specific mathematical objects (knots), where they managed to strengthen the recognition of the algebraic symbolization of a knot (conceptualizing, inducing and visualizing).

During the intervention phase, in the piloting of the instruments created, we realized the students had serious difficulties, as they did not understand the structure of the knot, nor could establish some kind of generalization in it. It was the first time that they faced elements of the knot theory, figures such as those proposed in the diagnostic instrument, trivial, prime, and equivalent knots impaired seriously their understanding. Therefore, it was necessary to readjust the diagnostic tool created and establish relationships and comparisons between the polygonal numbers as a first element that would bring them closer to visualizing, analyzing, and synthesizing this type of graph. The students were expected to be able to formalize, represent, categorize, synthesize and conjecture

regularities in this type of structure so that they could relate them to the graphs of the knots proposed in the first instrument.

As a result, we could determine which AMT skills could relate to each of the categories of the mathematical generalization process. Table 3 presents the association between the categories of the mathematical generalization process, its associated actions, and the respective AMT skills.

Table 3
The AMT skills and mathematical generalization process

	Categories	Actions associated	Advanced Mathematical Thinking Skills
Process of mathematical generalization	Azarquiel Group (1993) - Mason (1999)	Azarquiel Group (1993) - Mason (1999) - Sessa (2005)	
	Seeing - Seeing	❖ Recognizing the sequence	❖ Analyzing
		❖ Drawing a pattern	❖ Defining
		❖ Identifying a pattern	❖ Viewing
		❖ Analyzing the regularities	❖ Representing
		❖ Intuiting visual regularities	❖ Categorizing
	Describing - Saying	❖ Showing results found from the generality to the group	❖ Inducing
		❖ Saying and talking about what happens to the regularity	❖ Representing
		❖ Describing orally	❖ Synthesizing
		❖ Discussing in the classroom about what is found in the regularity	❖ Conjecturing
		❖ Using symbols, drawings, etc. to describe the relationship found.	
	Writing - Recording	❖ Recording and/or describing in writing the relationship found.	❖ Representing
❖ Expressing in writing		❖ Formalizing	
❖ Expressing in writing with symbols		❖ Synthesizing	
❖ Using algebraic symbols to express the evidence in the regularity.		❖ Conceptualizing	
❖ Coming up with a formula for the regularity,		❖ Inducing	
❖ Validating the formula.	❖ Demonstrating		

The actions associated with the process, described in the previous table, do not present in any way a sequence between its parts, on the contrary, they are activities that can be distinguished simultaneously in each of the categories of the generalization process,

where we relate the following skills with the AMT: defining, analyzing, formalizing, conceptualizing, inducing, visualizing, and demonstrating since we find the implicit development of these skills related to representing, categorizing, synthesizing and conjecturing.

When we validated that the students visualized, analyzed, induced, and formalized the polygonal numbers, we presented them with a second instrument with trivial and some equivalent knots, seeking to achieve a mathematical generalization from the application of the AMT skills described above. To do so, we confront them with the proposal put forward by Whitehead (1991), quoted by Suarez (2002, p. 38) about the action-research cycle: “feel or experience a problem; imagine the solution of the problem; implement the imagined solution; evaluate the results of the actions taken, and modify the practice in light of the results “.

We found that the first two referred to recognizing the problem through specific theoretical elements, such as the type of polygonal number, then the type of knot, the regularity present in each of them, and previous work on the subject, that is: identifying the existing regularity, analyzing it, formalizing it, and trying to conceptualize and express said regularity in a formula that allows it to be generalized. These elements could be reached facially with the polygonal numbers and gradually with the trivial knots, and with some equivalent knots, a situation that cannot be validated with prime knots, given the complexity of the graph.

The third moment, putting the solution imagined into practice, aimed at first to establish the type of instruments to collect information. Secondly, to design the syllabus and subject guidelines of the elective seminar and, later, validate it. Finally, at evaluating the results of the actions undertaken and modifying the practice in light of the results, the definitive syllabus for the seminar proposed was established, which is currently offered to students as an elective subject within their training as mathematics degree holders.

CONCLUSIONS

With the results obtained, the final version of the syllabus and the subject guidelines of the elective seminar was designed. We developed an evaluative rubric in which the categories of the process of mathematical generalization, its associated actions, and the AMT skills are integrated. For each of these aspects, specific questions and indicators were established³, according to the concepts of the knot theory⁴ that were defined; elements summarized in table 4.

³ The rubric has 25 indicators distributed in each of the categories of the mathematical generalization process (seeing-seeing; describing-saying; writing-recording). The indicators set are consistent with AMT skills exposed. The rubric designed served as evaluative guidance of the diagnostic process and to validate the elective seminar.

⁴ The basic concepts of knot theory that were contemplated are the graphic representation of knots; types of knots; Reidemeister moves and knot notation.

Table 4

Characteristic aspects of the evaluative rubrics of the syllabus of the elective seminar proposed

N. of the pilot session	N. of subject guideline session	Evaluative instrument ⁵	Categories of the mathematical generalization process to be evaluated	AMT skills	Evaluative indicators ⁶	
1	Session 1	Socialization and explanation about the process of mathematical generalization.				
2	Session 2	Individual workshop: a practical case of the process of mathematical generalization through a regularity particularity 1.	Seeing – Seeing; Describing – Saying; Writing-Recording	Visualizing, analyzing, representing, categorizing, defining, inducing, synthesizing, formalizing, and synthesizing.	11	
3	Session 3	Individual workshop: a practical case of the process of mathematical generalization through a regularity particularity 2.	Describing – Saying; Writing-Recording	Inducing, representing, synthesizing, formalizing, conceptualizing, and demonstrating.	10	
4	Session 7	Individual workshop: knot notation.	Seeing – Seeing; Writing-Recording	Visualizing, analyzing, representing, categorizing, defining, inducing, formalizing, and synthesizing.	7	
5	Session 8	Individual workshop: our knot notation.	Writing - Recording	Representing, formalizing, synthesizing, conceptualizing, and inducing	6	
6	Does not apply	Questionnaire of the final perception of the elective seminar		Does not apply		

Once the syllabus and subject guideline of the elective seminar were consolidated, with the support of the direction board of the mathematics degree program of the university chosen, we managed an academic space for a group of students attending the third semester to take it during the first and second school terms of 2019. From this, we found it convenient to initially approach the process of mathematical generalization, changing the regularity of knots for pentagonal numbers but retaining the same structure, since this transition allows students an approximation and subsequent appropriation of the concepts of the knot theory. To date this seminar is offered to students of said degree.

We contemplated the formative purposes of the seminar on the characterization of the process of mathematical generalization. Compared to this session, it was evident that in the *Seeing-Seeing* category of the process of mathematical generalization, students achieve skills to visualize, analyze and represent patterns, elements that allow them to

⁵ Each evaluative instrument responds to some specific categories of the mathematical generalization process and AMT skills.

⁶ According to the 25 indicators that were established in the general rubric, the number of indicators per evaluation instrument for the validation process of the elective seminar was determined, according to the theme proposed in the subject guidelines and the AMT skills.

overcome difficulties in categorizing and defining regularities. The seminar applied showed that the students reached an average level. Regarding the *Describing-Saying* and *Writing-Recording* category, the seminar allowed students who presented certain difficulties to describe in writing the regularity proposed in a trivial, prime, or equivalent knot, to achieve an average development of AMT skills: inducing, synthesizing and formalizing. Regarding the *Describing-Saying* and *Writing-Recording* category, students describe correctly the regularity presented.

We highlight that for *Describing-Saying* the students often use some symbols to interpret what occurs in the regularity, however, their rationale for their description is not clear and maybe bring ambiguities, by which we can infer that it is not possible to determine students' actual development of the inducing, representing and synthesizing skills for this category. Pending elements to be considered in future research.

Regarding the *Writing-Recording* category, the seminar allows us to identify that the students manage to make a description of the formalization of the pattern, however, the explanation they offer does not support the work done. This type of analysis and the assumption that the formalization of knowledge is a development strategy and contribution to the cohesion processes of learning mathematics is of particular interest to this work. However, most of the students managed to develop a mathematical formula of regularity without problems, besides verifying it, corroborating that this was correct, which validates the position mentioned before.

We evaluated the seminar through a perception survey conducted with the students who took it during the two academic semesters of 2019, highlighting that: 1) the seminar raises interesting topics that transcend and contribute to the formation of the mathematics degree holder, such as skills to visualize, analyze, represent, categorize, define and induce a regularity. 2) They consider that the conceptual elements of the knot theory s can be more easily addressed from the process of mathematical generalization, demystifying the conceptual complexity of the topology that is worked in the last semesters of the undergraduate degree in some mathematics degree courses, strengthening skills such as formalizing, conceptualizing and demonstrating.

Therefore, to implement the process of mathematical generalization and the development of AMT skills, it is necessary to consider at least:

- a) An approach to the processes of mathematical generalization from the identification of common regularities with polygonal numbers. At this stage students expected to be able to conjecture algebraically about aspects of the regularity.
- b) Formalization of generalizations and regularities. The student must be able to identify regularities from different records.
- c) Theorizing about what a knot is, classes of knots, and some characteristic elements of the knot.
- d) Mathematical generalization from an approximation of the knot theory.

It is essential to recognize that the structuring of an evaluative rubric with defined criteria and guidelines was an essential guarantor to recognize those starting elements that allowed the consolidation of the elective seminary. This is because the results presented in the diagnostic phase showed that the students evaluated had difficulties in some specific AMT skills in each category of the mathematical generalization process. This incidence implies that the proposed subject guideline could use more time in strengthening the development of these skills so students can have a conceptual and practical approach to each one of them. In other words, that the student knows and knows how to do, that is, that he be mathematically competent.

We find that mathematical generalization is a process that requires the development of specific skills such as visualizing, analyzing, representing, categorizing, defining, and inducing a regularity; however, the skills of formalizing, conceptualizing, and demonstrating are presented as weaknesses. This shows the need to enhance and develop the process of mathematical generalization from the first years of schooling.

From our experience with this research, we propose an approach to the definition of mathematical generalization as a systemic and dynamic process that allows recognizing the particular characteristics of the regularity to establish a general law. For this, it is necessary to be able to express efficiently what was observed either through verbal, written, or algebraic communication.

We present in Table 5 the overall outline of the seminar designed and implemented.

Table 5
General outlines of the seminar.

GENERAL INFORMATION OF THE ELECTIVE SEMINAR						
Program that offers	Mathematics Degree Course					
Title of the Seminar	Didáctica de las Matemáticas: Una Aproximación a la Teoría de Nudos/ Mathematics Teaching: An Approach to Knot Theory					
Type of subject	Elective					
Number of credits	2 -3					
Type of credit	Theoretical / 1: 2 ratio					
Distribution of credits	Hours of direct work with the professor	32-48	Hours of autonomous work	64-96	Total Hours	96-144
Prerequisites	None					

⁷ In the results of the satisfaction evaluation of the elective seminar, we concluded that to achieve the expected learning results, it is necessary to have more time, both for direct work with the professor and for students' autonomous work. The implication above aims at the flexibility of increasing the elective seminar by one academic credit, going from 2 to 3, in terms of hours, increasing from 96 to 144. However, if the group of students presents an advanced level on average in the process of mathematical generalization, it can be left with two credits with their respective workloads.

RECOMMENDATIONS

We suggest that the professor in charge of the seminar previously gives specific theoretical-practical feedback on the conceptual aspects of the knot theory planned for each session, aiming at assuring students for proper management of the skills of formalizing, conceptualizing, and demonstrating, which often appear as weaknesses in young people.

We consider it important that the professor in charge carries out a permanent follow-up to lead the students on the objectives of each of the activities proposed, given that, if the student cannot correctly visualize, analyze, represent and induce regularity, this will be an obstacle that will impair their progress in the mathematical generalization process.

On the other hand, to carry out the elective seminar, it is necessary the permanent support of infrastructure, as well as the university's didactic-methodological educational means that will allow for the time and space organization that are requested for the development each of the sessions proposed.

During the development of the elective seminar, the participating students may present some difficulties to recognize regularities in three dimensions. Here, the professor in charge must have the technological means (mathematical software), a support that allows them to carry out the respective constructions so that students can visualize, analyze, represent, categorize, define and induce the regularity presented.

The performance of various activities in two and three dimensions strengthens the skills of formalizing, conceptualizing, and demonstrating, typical of the AMT, which are regularly presented to students as a weakness. Here the professor in charge must recognize the characteristics, knowledge, and skills that the population that wishes to take the elective seminar has to strengthen, empower the group of students so that they achieve optimal development of the skills mentioned above. In this same direction, the professors in charge must have a longer time of direct teaching and autonomous work for the construction of a particular notation of the knot.

Based on the experience obtained, we consider that the seminar may apply to other educational contexts and training levels, for example, in elementary and high school education.

AUTHORS' CONTRIBUTION STATEMENT

EMN directed the research project, planned: background, question, objectives, theoretical framework, methodology, triangulation of the information collected, results of the syllabus, subject guideline of the elective seminar, and the formal structure of this document. CARJ developed the research project, and carried out the fieldwork, collecting information on site, helped to design, apply instruments; apply the phases of the methodology, plan the results, conclusions, as well as organize the syllabus and subject guidelines of the elective seminar.

DATA AVAILABILITY STATEMENT

The data supporting this study will be made available by the corresponding author (EMN), upon reasonable request.

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