ISSN: 2178-7727
DOI: 10.17648/acta.scientiae.5716

Development of Variational Thinking for the Teaching of Preliminary Notions of Calculus. A Class Experience in Basic Education

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Received for publication on 1 Mar. 2020. Accepted after review on 26 Mar. 2020
Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Background: This work incorporates results that help to understand how to favour the development of variational thinking and language in students of middle school education. **Objective**: to relate variational thinking with some preliminary notions of calculus. **Design**: a sequence of activities is presented to analyse variational aspects of the function concept and the results of its implementation. **Setting and participants**: the experience takes place in a middle and high school education institution with a sample of 40 students attending the 9th grade. **Data collection and analysis:** a didactic sequence (DS) was elaborated from the results of a diagnostic test and applied to the students' group. The information collected from the application of the DS was submitted to an inductive analysis. **Results**: we focused our attention on the way in which the study variation processes allowed students to build meaningful approaches to understanding and using functions as models of change situations. **Conclusions**: despite the institutional limitations and the difficulties of the students, the development and emergence of variational arguments is observed in young people, based on previous knowledge that they relate to experiences of daily life.

Keywords: Variational thinking; Co-variation; functions, Notions of infinitesimal calculus

Desarrollo del Pensamiento Variacional para la Enseñanza de Nociones Preliminares de Cálculo. Una Experiencia de Aula en la Educación Básica

RESUMEN

Contexto: Este trabajo incorpora resultados que ayudan a comprender cómo favorecer el desarrollo del pensamiento y lenguaje variacional en estudiantes de educación básica secundaria. **Objetivo**: relacionar el pensamiento variacional con

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algunas nociones preliminares del cálculo. **Diseño**: Se presenta una secuencia de actividades para analizar aspectos variacionales del concepto de función y los resultados de su implementación. **Escenario y participantes**: la experiencia se desarrolla en una institución de educación media con una muestra de 40 estudiantes que pertenecen al grado noveno. **Recolección y análisis de datos**: se elaboró una secuencia didáctica (SD) a partir de los resultados de una prueba diagnóstica y se aplicó al grupo de estudiantes. La información recopilada de la aplicación de la SD se sometió a un análisis inductivo. **Resultados**: Centramos la atención en la forma en que el estudio de procesos de variación permitió a los alumnos construir acercamientos significativos para la comprensión y el uso de las funciones como modelos de situaciones de cambio. **Conclusiones:** A pesar de las limitaciones institucionales y las dificultades de los alumnos, se observa en los jóvenes desarrollo y surgimiento de argumentos variacionales, partiendo de los conocimientos previos que relacionan con experiencias de la vida cotidiana.

Palabras clave: Pensamiento variacional; Covariación; funciones, Nociones de cálculo infinitesimal

Desenvolvimento do pensamento variacional para o ensino de noções preliminares de cálculo. Uma experiência de sala de aula na educação básica

RESUMO

Contexto: este trabalho incorpora resultados que ajudam a compreender como favorecer o desenvolvimento do pensamento variacional e da linguagem em alunos do ensino fundamental II. **Objetivo**: relacionar o pensamento variacional com algumas noções preliminares de cálculo. **Design**: Uma sequência de atividades é apresentada para analisar os aspectos variacionais do conceito de função e os resultados de sua implementação. **Cenário e participantes**: a experiência se passa em uma instituição de ensino fundamental e médio com uma amostra de 40 alunos do 9º ano. **Coleta e análise de dados:** uma sequência didática (SD) foi elaborada a partir dos resultados de um teste diagnóstico e aplicada ao grupo de alunos. As informações coletadas com a aplicação da SD foram submetidas a uma análise indutiva. **Resultados**: focamos nossa atenção na maneira como o estudo dos processos de variação permitiu aos alunos construir abordagens significativas para compreender e usar funções como modelos de situações de mudança. **Conclusões**: Apesar das limitações institucionais e das dificuldades dos alunos, observa-se nos jovens o desenvolvimento e surgimento de argumentos variacionais, a partir dos conhecimentos prévios que relacionam às experiências do cotidiano.

Palavras-chave: Pensamento variacional; Covariação; funções, noções de cálculo infinitesimal.

INTRODUCTION

The work was framed in the line of mathematical variational thinking and language, included within variational calculus [1], which studies the articulation between research and social practices that give life to the mathematics of variation and change in didactic systems. We designed a teaching sequence to be applied in class with 40 middle school students (attending the 9th grade) from a state public educational institution in Bogotá, Colombia. The activities conducted allowed us to analyse various scenarios of variation (which, how, and how much magnitudes change), characterise variations between magnitudes, differentiate them from variables, and explore the different registers of representation that students use. These elements favour the development of visualisation as one of the cognitive processes involved in mathematical thinking.

We set out to determine the levels of covariational reasoning that students develop in a common classroom environment, given that the teachers who work in the institution realise that, on the one hand, students find it difficult to "model"[2] when one variable changes as a function of another, and, on the other, to interpret the different representation registers (graphic, tabular, symbolic, and natural language). This situation is corroborated by the results students obtained in the internal assessments associated with the component of variational thinking and the low performances obtained in the state assessments called Saber[3].

We sought to integrate the group of students to understand different phenomena of nature where manifestations of variation and change appear, such as management of longitudinal units and magnitudes, increase or decrease in temperature throughout the day, and filling containers of different heights, but with a constant volume. The different magnitudes that characterise these phenomena are intimately related so that some of them are completely determined by the values of the others. Dolores (2000) states that, historically,

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[1] Variational Calculus deals with the mathematical formulation of functions, of maximising and minimising them.

[2] In this work, modelling constitutes an analogous structure of the real world or imaginary situation, event or process that a person constructs in the mind when reasoning. In other words, represent or show ideas and relationships [in this case mathematics through objects, illustrations, graphs, equations, among other methods. (Moreira and Rodríguez, 2002)].

[3] The report delivered to the Educational Institution by the Colombian Institute for the Promotion of Higher Education (ICFES, 2016) in the Saber 2015 Tests indicated that 55% of the sample for 9th grade did not use or relate different representations to model situations of variation. In 2016, the figure was 72%.

these relationships were fundamental in the search for the general laws governing change, considering that this type of relationship gave rise to the notion of function.

We introduced the concept of function from problem situations related to the context of the population, enabling the use of students' daily language in the dialogue of knowledges, and then institutionalise it in the formal mathematical language. After the intervention, the strategies that had a noticeable advance were: identify regularity patterns and draw up tables of values, identify fixed quantities and variable quantities that intervene in a situation, indicate maximum and minimum values and range of variation of a quantity in a problem situation.

BACKGROUND

Mateus-Nieves and Hernández (2020) show that with the development and formalisation of infinitesimal calculus, the concept of function acquires a high level of abstraction that a middle education student cannot achieve easily. Current teaching models have assumed this abstract concept causing the understanding of the concept of function to depend on the understanding of notions - also abstract - of set, ordered pair, and correspondence, among others, which do not allow the students to grasp the ideas of variation and change that underlie this concept, making difficult the connections with other sciences. In this regard, López and Sosa (2008) state:

> The way the concept is usually transmitted in school leaves aside the process of constructing the concept of function. Learning experiences in class do not favour appreciating the nature and functionality of the concept to understand, model, and explain phenomena of a variational nature, causing students' learning difficulties and misconceptions (p. 309).

Recognising that the concept of function has been linked in history to the modelling of processes of variation, Carlson, Jacobs, Coe, Larsen, and Hsu (2002), Dolores and Salgado (2009), Tall (2009), and Mateus-Nieves and Hernández (2020) focused their attention on how variation can become a fundamental axis for teaching and learning this concept. However, we recognise that the study of functions from a variational perspective is related to experimentation, reflection, construction of meanings, and ways of expressing generality as a result of the mathematical modelling processes. This context becomes a tool that allows the analysis and modelling of various phenomena of variation coming from mathematics itself, from everyday life, or from the natural and experimental sciences that, in a broader sense, positively influences the development of students' variational thinking, as well as their variational language, as long as they can communicate their ideas.

One of the fundamental concepts for the understanding of modern mathematics is the concept of function. Traditionally, functions are approached from elementary models (linear, affine, quadratic), starting from the algebraic expression to be represented in another representation register, usually graphic or tabular(Deulofeu, 1991), obviating that numerical writing systems, symbolic notations, algebraic expressions, and Cartesian graphics constitute semiotic systems of expression, which must be differentiated from the mathematical object they represent (Duval, 1999). Despite the importance of function compression, to identify variables involved in a dynamic phenomenon and the dependency relationship between them, students who complete high school education emerge with a weak understanding of that mathematical object (Carlson et al., 2003). Therefore, it is possible to affirm that the preliminary notions of calculus are directly related to the learning of the rate of change, the identification of variables, and the covariation that occurs between them.

The learning of mathematics in recent times and especially in basic education has had a didactic leap. Vasco (2002) highlights that there have been several recommendations that have been made to move from the explanation of concepts, theorems and definitions, and the memorisation of formulas or recognition of function graphs to the development of mathematical - numerical, spatial, metric, random, or probabilistic and variational – thinking.

Muñoz (2015) mentions that the development of variational thinking is structured in the analysis of change phenomena, since it plays an important role in solving problems that are based on the variation, change, and modelling of everyday life processes related to the development and interaction with the other types of mathematical thinking. This thinking also contributes to developing conceptual systems, the mathematisation of situations, identification of variables, and establishment of relationships that expand themselves among other thoughts (Vargas, Reyes, & Cristóbal, 2016).

Cabezas and Mendoza (2016) carry out a didactic analysis of student productions related to variational thinking. They consider that "one of the purposes of variational thinking is to articulate research and social practices that give life to the mathematics of variation and change in didactic systems and situations involving variation in contexts" (p. 15). The authors characterise and categorise different forms of manifestations of variational thinking, affirming the importance of mastering basic elements of calculus for the generation of mathematical models and the use of different representation registers (written language, graphics, algebraics, tables or others), as necessary elements for the development of cognitive skills such as visualisation, argumentation, representation, and communication among others.

As part of mathematical thinking and variational language, we relate the study of strategies and actions that students use when faced with situations that require the analysis of change. Cantoral and Farfán (2003) express that:

> Variational thinking and language studies the phenomena of teaching, learning, and communication of mathematical knowledge typical of variation and change in the educational system and the social environment that accommodates it. It emphasises the study of the different cognitive and cultural processes with which people assign and share senses and meanings using different variational structures and languages (p. 185).

We share Mateus-Nieves and Devia's (2021) position, that the development of variational thinking and language is a slow process. Significantly building the notion of change requires mastery and integration of different concepts, some elementary, others more advanced, such as the processes of abstraction, justification, visualisation, estimation, and reasoning. This work aims to identify the characteristics of variational thinking and language and the way it develops. Cabrera (2009, p. 55) states that "this type of thinking is characterised by proposing the study of situations and phenomena in which change is involved, based on students' intuitions and conceptions," which are executed and evolved through problem situations. In this way, the notions of variation and change, in variational thinking and language, are focused on the way in which the phenomena studied change from one state to another, identifying the changes, quantifying them, and analysing how they occur.

Concerning the cognitive processes involved, the situations must be such that students do not only need to resort to memory to respond to them but also guide, validate, modify, or construct arguments for them. The treatment and conversion between different representations will be of fundamental importance for the understanding of situations of variation.

THEORETICAL FRAMEWORK

We base this work on two elements: variational mathematical thinking and variational language.

Variational Mathematical Thinking

We place it within advanced mathematical thinking, by the topics we deal with: function, variation, covariation, since it understands the relationships between the mathematics of variation, change, and thinking processes. In this context, a primary concept is change, mathematically modelled by difference. Differences show how much the variable changes in a variation process. The incorporation of variational elements and the granting of meaning to the different elements related to the variation of a variable in a function will favour the construction of that function. In a broader sense, they will positively influence the development of students' variational thinking, and their variational language, as long as they can communicate their ideas (Dolores & Salgado, 2009).

Within the development of thought and the production of mathematical knowledge is the use of notions associated with numerical, graphic, algebraic, and verbal registers. Duval (1999, 2006) notes that the integral understanding of an object is based on the coordination of at least two representation systems belonging to different registers. In general, the tasks of conversion between different systems of representation are minimised, and this produces limitations in the understanding and development of one of the thinking styles: visual.

In this context, we investigated the notions related to the concept of function that students build when they interact with activities articulated around the idea of variation and change, which favour the treatment and articulation of different representation systems. The first benchmark at the national level on variational thinking was established by the mathematics standards of the Ministry of National Education (MEN, 2014) where they consider that:

> [...] variational thinking combines the cognitive and the didactic to promote its genesis, empowerment, and development. In this order of ideas, it is proposed that variational mathematical thinking should be considered as the basis on which the mathematical curriculum is structured, since this is a pillar and axis of the other mathematical thinking (numerical, spatial or geometric, stochastic, metric) (p. 15).

From this guideline, we can affirm that variational thinking favours the development of other thoughts as it articulates, connects, and interrelates them. Likewise, the study of the processes of variation and change in the field of mathematics education resumes different mathematical objects and favours the modelling of everyday life processes (MEN, 2014). Based on these contributions of MEN, Vasco (2002) states that the curricular guidelines at the national level are not clear compared to what should be understood by variational thinking, since, when is no formal definition, this resonates on difficulties to interprete it. Precisely, he himself proposes an approximation to what could be understood by this mathematical object "variational thinking gives the possibility of distinguishing what changes from what remains constant and the possible regularities that can be generated" (Vasco, 2002, p. 20).

Dolores and Salgado (2009) agree that: "Variational thinking and language is the field in which the phenomena of teaching, learning, and communication of mathematical knowledge typical of variation and change are studied" (p. 65). Cabezas and Mendoza (2016) present a perspective in which they include within variational thinking:

> [...] the elaboration of strategies, forms of reasoning, elements and linguistic structures that allow the study and analysis of change and variation to be communicated, the objectives of variational thinking are oriented to develop thought structures that allow the identification, analysis, and interpretation, in a natural way, of situations related to change and, in turn, model and transform them into simpler ones" (p. 15).

We conjecture that: perceiving, identifying, and characterising variation in different contexts is the fundamental object of variational thinking in accordance with contributions presented by Dolores and Salgado (2009). On the other hand, variational thinking can be approached from covariation when interpreting dynamic phenomena and reasons for change. For Vasco (2002), "the object of variational thinking is, therefore, covariation between quantities of magnitude, mainly the variations over time" (p. 4). Assuming this thought from this perspective involves a substantial change in the way we approach it; it implies considering these situations as two-way systems where the magnitudes are correlated in a quantified and qualified way.

Carlson et al. (2002) indicate, on the one hand, that there should be a shift in emphasis from a coordinated image of two variables that change simultaneously to a coordinated image of an instantaneous rate of change with continuous changes in the independent variable for functions associated with dynamic situations and, on the other, that the students' levels when addressing these constructs should be established and, consequently, provided with school content that promotes, if not exceedance, an understanding of what levels they reach.

In the same trend to approach the analysis of dynamic situations from the covariational point of view, Vasco (2002) highlights that: "Variational thinking does not consist in knowing a definition of function, since generally, the definition of function is static" (p. 103), nor does it consist in learning formulas or drawing graphs. He, then, proposes that the object of study of variational thinking is the "covariation between quantities of magnitude, mainly the variations over time, and its guiding purpose is to try to model the patterns that are repeated in covariation" (p. 104). Seen from this perspective, García (2016) presents a definition for covariational reasoning:

> Mental activity involving the coordination of two quantities, which in turn makes it necessary to follow the value of each quantity and, in this way, realise that the other quantity also has a value at each moment (p. 22).

Covariational Reasoning Levels

Five levels of image development for covariation are recognised. These images of covariation are presented in terms of the mental actions supported by each image. Carlson et al. (2002) refer to the understanding of covariation as maintaining in the mind, simultaneously, a sustained image of two values of quantities into which they would deepen. This proposes a significant change to the traditional way of thinking about one variable that is absolutely dependent on the other. They associate this concept with the set of reasoning skills and conceptualisations that are involved in the understanding of dynamic phenomena, identify the cognitive processes related to the development of covariational reasoning, and establish a framework for the construction of images, which include mental actions that influence the interpretation and representation of the functions associated with such events. In Table 1, we present a synthesis of these levels.

Table 1

Covariational Reasoning Levels. (Carlson et al., 2003, p. 129)

Levels	Features

Level 1 (N1). Coordination	At the coordination level, covariation images can support the mental action of coordinating the change of one variable with changes in the other variable (AM1).
Level 2 (N2). Direction	At the direction level, covariation images can support the mental actions of coordinating the direction of change of one of the variables with changes in the other. The mental actions identified as AM1 and AM2 are both supported by N2 images.
Level 3 (N3). Quantitative coordination	At the quantitative coordination level, covariation images can support the mental actions of coordinating the amount of change in one variable with changes in the other. The mental actions identified as AM1, AM2 and AM3 are supported by N3 images.
Level 4 (N4). Average Rate	At the average rate level, covariation images can support the mental actions of coordinating the average rate of change of a function with uniform changes in the input values of the variable. The average rate of change can be broken down to coordinate the amount of change of the resulting variable with the changes in the input variable. The mental actions identified as AM1 through AM4 are supported by N4 images.
Level 5 (N5). Instantaneous Rate	At the instantaneous rate level, covariation images can support the mental actions of coordinating the instantaneous rate of change of a function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate of change results from increasingly smaller refinements in the average rate of change. It also includes the awareness that the turning point is one at which the rate of change goes from increasing to decreasing or the contrary. The mental actions identified as AM1 through AM5 are supported by N5 images.

Table 2 shows how Carlson et al. (2002) associate covariation with the set of reasoning skills and conceptualisations involved in the understanding of dynamic phenomena. They identify the cognitive processes involved in the development of covariational reasoning and establish a framework for the construction of images that include mental actions that influence the interpretation and representation of the functions associated with such events. We highlight this tool because it allows an approach to the characterisation that a student could present in those stages and the required mental actions and externalised behaviours.

Table 2

Mental actions of the conceptual framework for covariation. (Carlson et al., 2003, p. 128)

Mental action	Description of the mental action	Behaviour
AM1	Coordination of the value of one variable with the changes in the other.	Designation of axes with verbal indications of coordination of the two variables (e.g., *y* changes as *x* changes).
AM2	Coordination of the direction of change of one variable as the other variable changes.	Construction of an increasing straight line. Verbalisation of the awareness of the direction of change of the output value while considering the changes in the input value.
AM3	Coordination of the amount of change of one variable as the other changes.	Location of points/construction of secant lines. Verbalisation of the awareness of the direction of change of the output value while considering the changes in the input value.
AM4	Coordination of the average rate of change of the function with uniform increments of the change in the input variable.	Construction of contiguous secant lines for the domain. Verbalisation of the awareness of the rate of change of the output value (concerning the input value) while considering uniform increments of the input value.
AM5	Coordination of the instantaneous rate of change of the function, with continuous changes in the independent variable for the entire domain of the function.	Construction of a smooth curve with clear indications of changes in concavity. Verbalisation of the awareness of the instantaneous changes in the rate of change for the entire domain of the function (the inflection points and the direction of the concavities are correct.

METHODOLOGY

The research approach was qualitative in the research-action context proposed by Elliot (2000). With an inductive process in which we started from - documentary and field - exploration to a description of reality that enabled us to formulate theoretical perspectives, understand contexts, and interpret them. We analysed textbooks reported in the discipline syllabus and that have been commonly used in our environment, compared them with the curriculum proposed for high school in our country, and reviewed the proposed curriculum grid. This led us to propose an intervention that started from the planning, elaboration, and execution of a didactic sequence, where we could lead the student to a first visual and intuitive approach to the concept of function, beginning with the study of variation, covering four fundamental aspects: identifying magnitudes, independent-dependent variables, the change and the rate of change, taking care not to relate them to the derivative, a subject that is addressed at the final stage of high school education in our country.

We developed activities that allowed us to analyse different scenarios of variation, what magnitudes change, how much they change, how they change, that led the student to need to characterise variations between magnitudes, through the calculation of rates of change and which, in turn, make it possible to explore the different registers of representation that students use to understand this variation. We took into account that students alternated the registers: verbal, tabular (numerical and analytical), which required them to treat and convert them (graphic and symbolic).

The research was carried out in four phases: 1) Exploratory: documentary review, approach, and validation of the research problem. 2) Intervention: planning, construction, and piloting. The research was conducted with 40 students from a 9th-grade group. We developed, piloted, and applied a diagnostic test, the results of which allowed us to take actions for the elaboration of the didactic sequence (DS). 3) From intervention: we applied the DS to students' group and collected the information. 4) Analysis and results: through an inductive analysis, we described the information collected, presenting conclusions and recommendations.

RESULTS AND ANALYSIS

Design and implementation of the sequence. The articulation of the didactic sequence with the textbooks and the curriculum grid allowed us to identify that the teaching of functions does not generally correspond to their origins, instead favouring formal logical aspects, such as the register $f(x)$ to express a function: the non-identification of which is the independent and which is the dependent variable when constructing a table of values. Processes that are performed mechanically, without contextualising the student about why one is dependent on the other. We found that a lot of time is spent on algorithm teaching, mechanically, leaving aside the formation of variational ideas. While the introduction of fundamental ideas of calculus is raised since high school, this is not achieved in practice, at least in most institutions in our environment, which allows us to infer that students have their first contact with the mathematics of change only when they reach university.

We took into account that from its origins, the calculus shows an analysis of the variation in dynamic phenomena, characterised by having a fundamentally visual and intuitive component, which we maintained in the design of the DS activities. In the class before the implementation of the DS, activities related to the variational behaviour of related variables were performed, that is, to help the students recognise when one variable is independent and when another obtains results from the independent one, becoming dependent. These data were registered in value tables that allowed students to visualise the dependence and independence relationship between them. Here students reported verbal indication processes, coordination of the two variables (Carlson et al., 2002) reflecting mental actions of level 1 (AM1). With descriptions in "colloquial" language that account for the difference between dependent and independent variables such as that of the student E1: "*the water level changes, because as the tank is filled, the water level increases" or that of* the student E4: "*the capacity of the tank does not change, because the container is solid,*" which allowed us to identify levels of understanding of the magnitudes that vary and those that do not.

Based on this experience, we created the DS proposed from three particular didactic situations: Magnitudes and variables, Registers of Representation, and Covariational Reasoning Levels. For the first, we presented the students daily situations in which they activated previous knowledge about magnitudes; recognised the concept of magnitude, and gave different examples of it; associated the concept of magnitude with that of variable in dynamic situations, to establish the relationship of change between variables. For the second: representation registers, we sought students to make adequate use of the Cartesian plane (location of ordered pairs and establishment of scales); represent values of dynamic situations in data tables; describe a phenomenon of variation from verbal language; use an algebraic expression to construct the tabular representation of the change situation. For the third, we looked for students to identify the levels of covariational reasoning they use to represent dynamic situations; they will register the process of constructing the meaning of covariation from different representational registers.

During the application of the first situation, it was evident that after the intervention, the students showed greater ability to: identify magnitudes present in dynamic situations, determine which remained constant and which varied. We proposed images with containers of different shapes, but of equal volume. We proposed that there were three hose nozzles that emitted jets of water at the same speed and volume. They were asked which container would be filled first, which would need more time to fill. They had to justify their answers. Here, students developed the ability to point out the variables involved in such events, for example, they could identify, in addition to the diameter magnitudes, the jet length, amount of water and time, among others that could be considered: size and shape of the container, defining which data represented the quantitative and which the qualitative variables, differentiating the dependent from the independent variables.

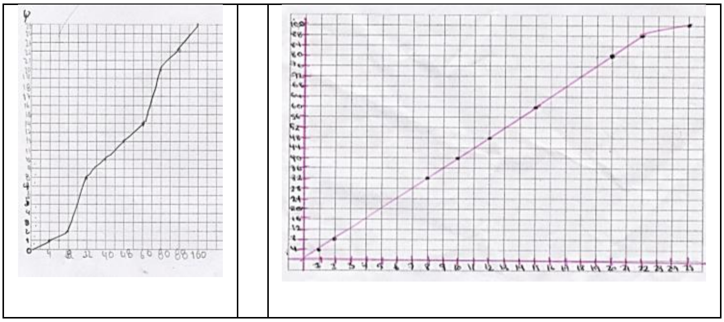
A particular fact drew our attention during the application of situation two, where we proposed a telephone rate: regarding the distance between the scenarios and the duration of the call, the students assigned variables that did not exist in the graphic representations. For example, student E2 indicates distance as a variable represented in the Cartesian plane, which is not correct, since, although the distance is an implicit variable in the situation, the graph only presented magnitudes correlated to the duration of the call and the total cost. After prompting them to reflect, students recognised and differentiated the magnitudes of the unit of measure and established logical dependence relationships such as the case of the student E1 who answers: "*They did not answer*" and E2: "*He did not call or they did not answer*" report this. Here it is evident that the students clearly establish the relationship between the variable total cost of a call and its duration, however, E4 and E5 still confused it and pointed out some units of measurement regardless of the magnitude.

Another relevant aspect was that, by presenting the students with situations of filling oval containers with the same volume as straight cylindrical ones, we contributed to the description of the change of variables in dynamic situations (AM2). Student E1 mentions: "*When it is more closed, the graph is more upright, and when it is wide, the curve gets more horizontal*" and that of E3: "*If the container is narrow, it fills faster and it fills slowly in the wider part*" or E4: "*When it ends like a tip, it is as if a curve went upward.*" With these statements and the graphs made by the students in the Cartesian plane, we can establish that it was possible to recognise the association of changes simultaneously between different variables, evidencing significantly better results, which contributed to the description of change of variables in dynamic situations AM2.

Regarding didactic situation three, levels of covariational reasoning, we assigned students a formula and asked them to give them values to later make the graph in the Cartesian plane. Only 20% easily identified the independent variable, assigning the x-axis. Regarding the axis graduation, 50% took the values from the table obtained and placed them directly on the plane without considering an adequate scale. *Figure 1* shows how E4 takes the values directly from the tabular record and places them in the plane, however, he indicates the *X* and *Y* axes appropriately. E5 adjusts the scale appropriately but in the face of the lack of space, he makes skips the last values, giving as a result an inaccurate graph.

Figure 1

Image of Cartesian graph of E4 (left) and E5 (right)



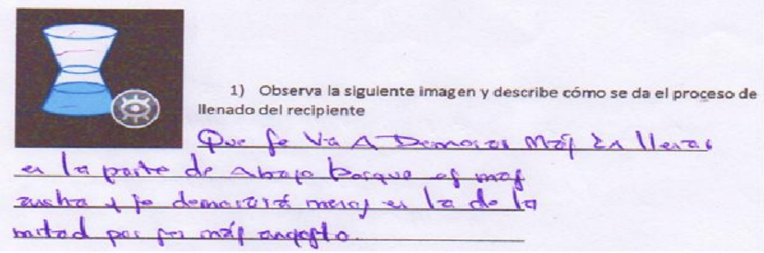
CONCLUSIONS

Together with the students, we reviewed different ways of expressing magnitudes and variables, led them to identify regularity patterns so that they identified how, when, how much, and where they change, making qualitative descriptions, which, then, had to be expressed symbolically through tables of values, where they could identify fixed and variable quantities. This allowed identifying an approximation to level 3, quantitative coordination, where covariation images can support the mental actions of coordinating the amount of change in one variable with changes in the other. In this regard, we validate Villa's position (2012). The author states that levels four and five require developments of "continuous mathematics [limits and derivatives] that allow the student to be aware of where the instantaneous rate of change results from increasingly smaller refinements in the average rate of change" (p. 22), elements that are outside the scope approached in this research (middle school students).

The mental actions identified as AM1 and AM2 were achieved by the majority of the students, since they managed to identify magnitudes present in dynamic situations and determine which remained constant and which varied. We reviewed in detail which students reached level 3 (quantitative coordination). Here, there was a limitation of associating the performance of this level exclusively to the verbalisation of the awareness of the amount of change of the output value while considering the changes in the input value for the situation of tank filling - accelerated or decelerated growth behaviour of the phenomenon -AM3.1- Figure 2, and to the identification of important points of the graph (inflection points and extremes) AM3.2. In the students' terms, this happened only as "faster or slower."

Figure 2

Verbal register of accelerated change awareness in E1



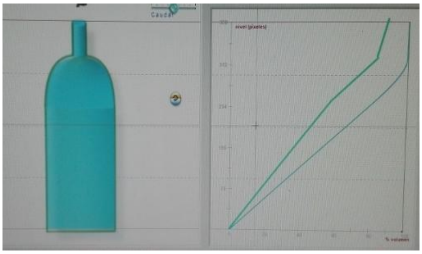
1) Observa la siguiente imagen y describe cómo se da el proceso de llenado del recipiente

Que se Va A Demorar Más En llenar
en la parte de abajo porque es más
ancha y se demorará menos en la de la
mitad por ser más angosto.

AM2 are discriminated around the coordination of the direction of change of the dependent variable in relation to the independent AM2, and the construction of a graphic representation that reflects the direction of change AM2 (*Figure 3*). Faced with the advances noted, the attention to instantaneous changes in other types of dynamic situations did not occur in the same way. For example, when analysing temperature graphs and requesting their description, 100% of the students simply indicated maximum and minimum values or temperature values for different months, which allows us to infer that they remained in the sole verbalisation of the awareness of the direction of the change of the output value as the changes in the input value are considered.

Figure 3

Images of Cartesian graphs made by E1 and E2 against situations of container filling



When we presented the students with magnitudes and variables (independent-dependent) so that they could distinguish and identify which were magnitudes and which were variables, and, among the variables, which were independent and which were dependent, the students resorted, in general, to their previous knowledge, working the idea of change as a variation of something. We observed that they identified the graphic representation with a situation of change in the height of container filling. The greatest difficulty was using representation registers (tabular and graphic). Interpretation, differentiation of the notions of magnitude and variable confused them to the point of not being able to establish in which axis they should locate the independent variable of the dependent. In this regard, their views on aspects related to the concept of function proved to be rather poor.

Students found it challenging to cope with situations involving covariation of two magnitudes and to handle different registers representing those magnitudes (tables, presentation on the number line on the Cartesian plane). They mixed length units with temperature units in the same register. They did not distinguish that they should be treated separately, since they represent different situations (they presented them in the same table and operated them as if they were similar). Likewise, the modelling of dynamic situations in different registers of tabular, Cartesian, verbal, and symbolic representation was quite weak during the application of didactic situations one and two.

In situation three, we integrated representation registers (written, tabular, graphic), inviting students to use formulas that incorporated the variables presented in the tables they had built. We expected them to express this information on the Cartesian plane, looking at what type of graphics it was. However, this is a topic that deserves more work with this type of population. Finally, we invited them to assess the work carried out, so that they could validate, compare, adjust, and, if necessary, correct what they built, to then formalise the topics covered.

During scenario three, we proposed an algebraically defined function for students to work with the changes expressed in the value tables they had built from obtaining the differences (increments) and the quotients between these differences. They found it very difficult to solve this activity, the ideas that they intended to generate arose in several aspects: in the phenomena that change at each instant, it is possible to calculate the filling speed in a given interval, but that result is not enough to determine the precise behaviour of the changes. Students reflected that the filling rate of the tank is not the same as the average and therefore cannot be calculated in the same way. The approach in the numerical register facilitated the calculation of the spaces travelled and the average speeds. The students were surprised at the calculation method proposed in the first item, matching the initial instant with the end. About the interpretation of the table, given the impossibility of using the previous strategy, they found it natural to allocate time variations for filling tanks of different forms but with the same volume.

To establish whether a student reached level 3 (quantitative coordination), there was a limitation of associating the performance of this level exclusively to the verbalisation of the awareness of the amount of change of the output value while considering the changes in the input value - accelerated or decelerated growth behaviour of the phenomenon -AM3.1- (Figure 3), and to the identification of important points of the graph (inflection points and extremes) AM3.2. In students' terms, it happened only as "*faster or slower*."

We share Mateus-Nieves and Moreno's (2021) position as they affirm that "studying the variation of a system or body means exercising our understanding to know how and how much the given system or body changes" (p. 44986). We say that a person uses or communicates arguments and variational strategies when making use of manoeuvres, ideas, techniques, or explanations that somehow reflect and express quantitative and qualitative recognition of the change in the system or object being studied (Cantoral, 2004). Hence, we also considered the observation registers of the teacher and a collaborator present in the classroom during the development of the sequence. This was a valuable contribution to describe the different moments of interaction, the doubts that arose throughout the resolution of the activities, and how students answered the teacher's questions.

RECOMMENDATIONS

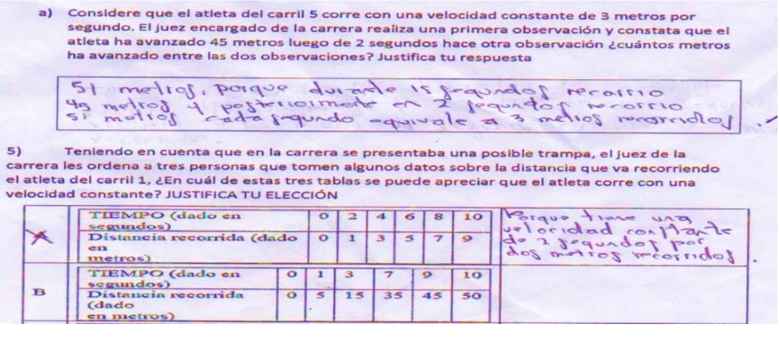
A fundamental element that can be attributed to the best levels of performance of covariational reasoning is associated with the manipulation of different representational registers: verbal, tabular, graphic, and algebraic. According to Duval (1999), the transformation between registers, despite being a "less spontaneous cognitive activity and more difficult to acquire for the vast majority of students" (p. 46), allows the appropriation of the mathematical object insofar as it contributes to the internal coordination of registers, allowing it to be seen as one and not as different objects.

The activities of the didactic situations proposed must allow students a verbal description of the situations proposed. According to Deulofeu (1991) "each of the representations allows us to express a phenomenon of change [however] we must first consider the verbal description, it uses the common language to give us a descriptive and generally qualitative view of the functional relationship" (p. 61). Concerning this subcategory, we recommend to privilege the use of verbal recording to make descriptions of phenomena, starting from there, both to ensure the understanding of the situation and introduce the student to the dynamic event, even though the colloquial language is the one that allows them to understand what they want to express.

It is recommended that students transit among different registers, present activities that involve the use of verbal recording, and move to the tabular and graphic. Figure 4 shows the transit between two registers that one of the students in the sample used in this work.

Figure 4

E27 activity presenting the transit between two representation registers



a) Considere que el atleta del carril 5 corre con una velocidad constante de 3 metros por segundo. El juez encargado de la carrera realiza una primera observación y constata que el atleta ha avanzado 45 metros luego de 2 segundos hace otra observación ¿cuántos metros ha avanzado entre las dos observaciones? Justifica tu respuesta

51 melros. porque durante 15 segundos recorrio
45 metros y posteriormente en 2 segundos recorrio
51 metros cada segundo equivale a 3 metros recorridos

5) Teniendo en cuenta que en la carrera se presentaba una posible trampa, el juez de la carrera les ordena a tres personas que tomen algunos datos sobre la distancia que va recorriendo el atleta del carril 1, ¿En cuál de estas tres tablas se puede apreciar que el atleta corre con una velocidad constante? JUSTIFICA TU ELECCIÓN

A	TIEMPO (dado en segundos)	0	2	4	6	8	10	Porque tiene una velocidad constante de 2 segundos por dos metros recorridos
	Distancia recorrida (dado en metros)	0	1	3	5	7	9	
B	TIEMPO (dado en segundos)	0	1	3	7	9	10	
	Distancia recorrida (dado en metros)	0	5	15	35	45	50	

AUTHORSHIP CONTRIBUTION STATEMENT

EMN directed the research project and the organisation of this manuscript. EMM carried out the research on site and contributed the initial draft of this manuscript.

DATA AVAILABILITY STATEMENT

The data supporting this study will be made available by the corresponding author 1 upon reasonable request.

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