

# Operative Invariants of Algebraic Pattern Present in the Strategies of Students of the 3<sup>rd</sup> Grade of Elementary School

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## ABSTRACT

**Background:** One of the great challenges for mathematics education in the 21st century is to alleviate the difficulties of students in the *transition* from arithmetic to algebra. There is already a consensus that there should not be a *transition*, as several experts have indicated algebraic approaches since the early years of schooling. **Objectives:** This study aims to describe and analyze the operative invariants of algebraic patterns present in the strategies of students of the 3rd grade of an elementary public school in the countryside of the state of Rio Grande do Sul. **Design:** The methodology used in this research was the clinical method of manipulation-formalization, created by Jean Piaget and applied in several of his studies. **Setting and Participants:** Students of the 3rd grade. **Data collection and analysis:** Clinical interviews. **Results:** We start from the assumptions of Gerard Vergnaud's theory of conceptual fields to analyze the strategies used by the research participants. **Conclusions:** We identified four operative invariants: the theorems-in-action "count the places each time a table is introduced" and "add two places each time a table is introduced", respectively linked with the concepts-in-action "putting the tables together" and "place at the ends of the tables".

**Keywords:** Operative invariants; Algebraic pattern; Strategies.

## Invariantes Operatórios de Padrão Algébrico Presentes nas Estratégias de Estudantes do 3º Ano do Ensino Fundamental

## RESUMO

**Contexto:** Um dos grandes desafios para a educação matemática no século XXI é amenizar as dificuldades dos estudantes na chamada *passagem* da Aritmética para Álgebra. Já é consenso que não deve haver uma *passagem*, pois vários especialistas indicam abordagens algébricas desde os anos iniciais de escolaridade. **Objetivos:** O objetivo do presente trabalho é descrever e analisar os invariantes operatórios de padrão algébrico presentes nas estratégias de estudantes

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do 3º ano do Ensino Fundamental de uma escola pública localizada no interior do estado do Rio Grande do Sul. **Design:** A metodologia utilizada nesta pesquisa foi o Método Clínico de Manipulação-Formalização, criado originalmente por Jean Piaget e aplicado em vários de seus estudos. **Ambiente e participantes:** Estudantes do 3º ano do Ensino Fundamental. **Coleta e análise de dados:** Entrevistas clínicas. **Resultados:** Partimos dos pressupostos da teoria dos campos conceituais de Gerard Vergnaud para analisar as estratégias utilizadas pelos participantes da pesquisa. **Conclusões:** Identificamos quatro invariantes operatórios: os teoremas-em-ação “contar os lugares cada vez que uma mesa é introduzida” e “adicionar dois lugares cada vez que uma mesa é introduzida”, respectivamente ligados com os conceitos-em-ação “junção das mesas” e “permanência dos lugares nas pontas das mesas”.

**Palavras-chave:** Invariantes operatórios; Padrão Algebrico; Estratégias.

## INTRODUCTION

Algebra is one of the major areas of mathematics that comprises the study of equation solving methods and the more general properties of polynomials. According to Baumgart (1992), the word algebra, come from the Arabic word *al-jabr*, which is part of the title of the book *Al-Kitabal-jabrwa'l Muqabalah*, written by Mohammed ibn-Musa Al-Khwarizmi around 825 A.D.

Many historians date the emergence of Algebra in the 9th century A.D. because they considered Al-Khwarizmi's work as the first text dealing specifically with algebraic methods. Even so, we can say that solving equations is much older, since there are records that Mesopotamians, as well as Egyptians, already solved some types of specific equations long before Arabs and Hindus did (Boyer, 1991).

The difference between the methods of the ancient peoples to the methods introduced by the Arabs is in representation. While the former were more dedicated to describing the resolution of practical problems, expressing the resolution through natural language, the Arabs created a specific language to communicate the most general ideas of their methods. We can think of a parallel concerning the children's' learning of Algebra, that is, at first, they express their algebraic ideas in their natural language, understanding and trying to generalize these ideas as they come in contact with gradually more formal problems throughout their schooling. Hence the importance of understanding how children construct his mental schemes in the face of problems that present algebraic ideas in a less formal language.

The objective of the present work is to describe and analyze the operative invariants of algebraic patterns present in the strategies of students of the 3rd grade of elementary education of a public school in the state of Rio Grande do Sul.

## ALGEBRAIC THINKING IN THE EARLY YEARS

According to data from NCTM (2000), one of the great challenges for mathematics education in the 21st century is to alleviate the difficulties of students in the *transition* from

arithmetic to algebra. There is already a consensus that there should not be a *transition*, as several experts have indicated algebraic approaches since the early years of schooling. In several countries, algebra is already integrated into the curriculum, at all educational levels, from the first grade. This movement to include algebra since the early years has been called *EarlyAlgebra* internationally.

Since algebra was included in the curriculum of the first school years, several researchers have been studying how the children's *algebraic thinking* works, to understand the elements that characterize a situation as algebraic, and also how the children's strategies of representation evolve, until they can understand the formal language of algebra (Blanton & Kaput, 2005; Carpenter, Levi, Franke & Zeringue, 2005; Irwin & Britt, 2006; Canavarro, 2007; Fujii & Stephens, 2008; Stephens & Wang, 2008; Blanton et al., 2015).

Blanton and Kaput (2005, p.413) produced the first important studies on *EarlyAlgebra*, which became a reference for several others, including presenting a classification adopted in later studies, dividing algebraic thinking into functional thinking and algebraic generalization. These authors define algebraic thinking as "the process by which students generalize mathematical ideas from a set of particular cases, establish these generalizations through argumentative discourse, and express them in progressively more formal and age-appropriate ways".

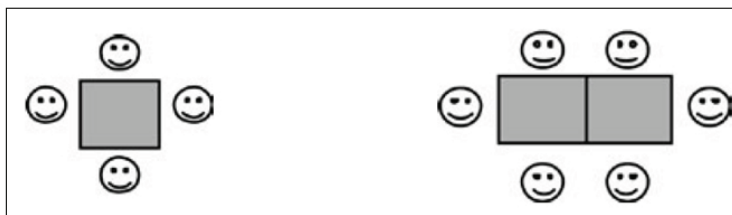
The document of the MEC, *Elementos Conceituais e Metodológicos para Definição dos Direitos de Aprendizagem e Desenvolvimento do Ciclo de Alfabetização (1º, 2º e 3º anos) do Ensino Fundamental* /Conceptual and Methodological Elements to Define the Learning and Development Rights of the Literacy Cycle (1st, 2nd and 3rd grades) of Elementary Education (Brasil, 2012) considers algebraic thinking should be one of the subjects to be studied since the early years of elementary school. It is the first document that refers *EarlyAlgebra* in Brazil.

## METHODOLOGICAL PROCEDURES

The methodology used in this research was the manipulation-formalization clinical method (Delval, 2002), originally conceived by Jean Piaget and applied in several of his studies. The activity proposed in the present work was based on a problem presented by Blanton et al. (2015, p. 85), in which the authors of that work also intended to study the notion of algebraic patterns.

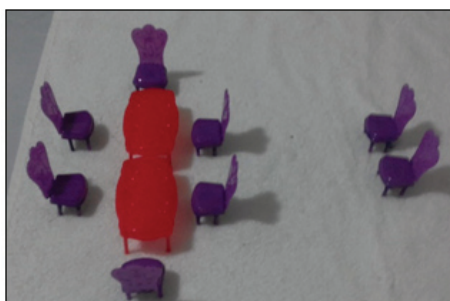
To analyze the notion of the algebraic pattern, Blanton et al. (2015, p. 85, our translation) use the following problem: "Brady invited his friends to a birthday party. He wants to make sure that everyone has a place to sit. It has a square table. He can have 4 seats on a square table, as shown in the figure. If you add another square table to the first, he can have 6 seats. a) If Brady continues to put the tables together this way, how many people will be able to sit at 3 tables? 4 tables? 5 tables? b) Do you notice any relationship in the table? Explain. c) Find the rule that describes the relationship between the number of tables and the number of people who can sit at the tables. Describe the rule in words.

d) Describe your rule using variables. What do your variables represent? e) If Brady puts 10 tables together, how many people will be able to sit? Show how you got the answer". This situation is illustrated in *Figure 1* (Blanton et al., 2015).



*Figure 1.* Representation of the Table with Four and Six Seats. (Blanton et al., 2015)

Data collection was carried out with 24 students from the 3rd grade of elementary school (14 boys and 10 girls). The study was carried out in a public school on the outskirts of a city in the countryside of the state of Rio Grande do Sul. Students in the 3rd grade were chosen because this is the end of the literacy cycle, and the four activities applied in this research were adapted from the work of Blanton et al. (2015), which also uses subjects attending equivalent school level in the educational system of the United States of America. The material used is shown in *Figure 2*, below.



*Figure 2.* Activity Application Material.

We followed the protocols of the manipulation-formalization clinical method (Delval, 2002). For each of the 24 participants, we presented a fictional situation in which a boy, Bruno, invited his friends to a birthday party. He wants to make sure that everyone has a place to sit. At Bruno's home there is a square table with four seats, but he had the idea of bringing another table together. The participant is asked how many friends can sit with two tables together. Then, the number of tables is gradually increased to analyze the participant's generalization capacity, to capture the way he perceives the rule that is formed when the number of tables increases, and it is also possible to identify that there is a relationship between the number of tables and seats.

Data analysis took place in three stages: categorization of the procedures performed by the participants, analysis based on the strategies constructed by the students, and classification of the levels of responses obtained from the interviews.

The research was carried out with the authorization of the school and the teacher regent of the class, so that the institution collected the students' free and informed consent to participate in studies, advertising, use of images and participate in research when they enrolled for the activity.<sup>1</sup>

## THEORETICAL ASSUMPTIONS

Over several decades, the French researcher Gerard Vergnaud studied the process of cognitive construction of the most elementary mathematical operations, such as addition, subtraction, multiplication, and division. Vergnaud (1990) proposes the theory of conceptual fields, defending the hypothesis that the brain does not develop a concept in a unique way, it is necessary that the subject comes into contact with other concepts, so that, in parallel, all of them can be developed. Hence the expression *conceptual field*, coined by Vergnaud.

The initial focus of Vergnaud's research was elementary arithmetic operations (Vergnaud, 1997, 2009). According to the researcher, the addition operation, for example, does not develop if the subject does not have contact with situations that involve the concept of subtraction. For this reason, Vergnaud proposes that the conceptual additive field should be developed, which covers situations related to both addition and subtraction, since the two operations must be understood together, in parallel.

Vergnaud (1990) more precisely defines a conceptual field as the synthesis of three elements: a set of *situations*, a set of *operative invariants*, and a set of *symbolic representations*. The situations are presented to the subject, who needs to create schemes consisting of goals, anticipations, rules of action, inferences, and procedures that can be generalized for a class of situations. Such schemes make up the operative invariants. As the subject needs to represent the concepts and invariants he uses, he makes use of symbolic representations, progressively more complex, as he advances in the understanding of the concepts and the more general characteristics of the conceptual field.

The idea of the conceptual field was originally proposed by Vergnaud, but the operative invariants had already been proposed by his doctoral advisor, Jean Piaget (1971), who proposed before Vergnaud that each situation demands a different type of scheme from the subject. Piaget and Inhelder (1975, 1979) study the relationship between operative

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<sup>1</sup> The study was not submitted to the Ethics Committee, because the research institution had not standardized the research in Human Sciences when the data was collected. The authors understood that the experiment did not offer any psychological and/or physical risk to the research participants. The authors accept total responsibility for and explicitly exempt Acta Scientiae from any liabilities or consequences that may arise from this study. Therefore, authors agree to provide full assistance and compensation due to any possible damage to any of the research participants, per Resolution No. 510 of April 7, 2016, of the Conselho Nacional de Saúde/National Health Council.

invariants and symbolic representations in more depth, concluding that operative invariants are the meanings of concepts, while symbolic representations are the signifiers.

In Vergnaud's conceptual field theory (1990, 1997, 2009), invariants can be of two types: *theorems-in-action* and *concepts-in-action*. Theorems-in-action are propositions considered to be true by the subject without necessarily being tested, generalized, or proven, and which can be reformulated from new situations. Concepts-in-action are characteristics attributed to subjects or objects, which can be used as premises for theorems-in-action.

To characterize operative invariants of algebraic thinking is to describe and analyze the possible concepts-in-action and theorems-in-action, that is, used effectively by individuals who are in the process of forming elementary algebraic concepts, such as, for example, the idea of patterns. Therefore, in this work we start from the assumptions of the theory of conceptual fields to analyze the strategies used by the research participants.

## RESULTS AND DISCUSSION

This section presents the results that were obtained by the application of Activity, focusing on the idea of pattern recognition. Pattern recognition is directly related to the algebraic idea of a variable, which is one of the main algebraic notions, according to Blanton et al. (2015).

### Standardization Procedures

This section presents the results of data collection from the qualitative step for the situation involving the idea of algebraic patterns in sequences of numbers or other representations. In this data collection, the focus was on analyzing whether students were able to recognize a pattern of numerical association between the number of tables and seats, and how they described the pattern found.

The students carried out several types of procedures in this activity. The first category of procedures that we highlighted was *No Rationale*, which is characterized by a reduced presence of causality in the way of handling materials and inferring a rule that relates the increase in the number of chairs to the number of tables. *Table 1* shows some of the students' statements while carrying out Activity.

Table 1  
*Category Without Justification.*

<b>No Rationale</b>	<p>[17] _How many more seats do you think he will have to put, if he wants to put three tables? _<i>Ten.</i></p> <p>[19] _If I wanted to put another table (there were two tables), how many seats do you think there would be? _<i>I think that nine.</i> _Why nine? What account did you do? _<i>I counted two plus four.</i></p> <p>[21] _And if he puts the tables together like that, he put two tables, but what if he wants to put a third table, how many seats will he have to put at the table? (the student thinks for a long time) _<i>Five.</i></p>
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The No-Rationale answers reveal the students find it difficult to understand the expression “the most”, already found in previous studies, as in the work of Beck and Silva (2015). The student [17] answers ten, thinking about the total number of seats, which nevertheless indicates a prediction based on an estimate that does not follow a well-defined rule, and that does not consider the use of the expression “over”. When asked why the answer would be nine, the student [19] explains that he counted “two plus four”, but this account does give the correct result, nor does it reflect the quantities involved in the problem, since there were two tables and six chairs.

The category of procedures that we call *Not-Gathering-Tables-Together* is characterized by the fact that there is an awareness that the number of seats would increase due to the increased number of tables, however, the fact that the tables need to be together is not taken into account. Participants that used this type of procedure just added another table and counted the number of seats. One participant missed the count, counting three more places, and the other one got the count right, however without gathering the tables together, as shown in *Table 2*.

Table 2  
*No-Gathering-Tables-Together Category.*

<b>No-Gathering-Tables-Together</b>	<p>[1] _If he gathers two tables together, he knows that he will be able to invite six people, and if he gets one more table, three tables, how many people can stay? _<i>I think that nine.</i> _How did you think to estimate it? _<i>I counted here, plus the other table that he could put here</i> (placing the chairs around the tables, not leaning the tables against each other).</p> <p>[5] _If he places another table here, how many seats will he have? _<i>Twelve.</i> _How do you know that the result is twelve? _ (the student thinks and redoes the activity) <i>Ten, six plus four, gives ten</i> (without leaning the tables against each other).</p>
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Participant [5] first answers twelve, probably making an estimate. After manipulating the tables, but without coupling them, she counts and answers ten, without paying attention to the fact that the tables should be together.

Another category of procedures that has been verified is what we call *Mix-Chairs*. Students who presented procedures such as Mix-Chairs understand that there is a relationship between the number of tables and chairs, however they find it difficult to follow the pattern of one chair per place at the table. *Table 3* below shows the use of this type of procedure.

Table 3  
*Mix-Chairs Category.*

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<b>Mix-Chairs</b>	<p>[3] <i>_If I put another table, will there be three more seats? _Yes. _I asked this question in another school and another student said the result is 10, do you think he is right or that he is wrong? _If you put two, look, one, it is four, and if you couple two tables together, it is six, and if you add one more table, it will be nine, I can put one more. _And at the head of the table, can't we put another chair? _Yes, then there was going to be one here and another one here (placing two chairs in the same place at the table).</i></p> <p>[6] <i>_If he put another table, how many chairs could he put? With three tables? _Twelve. How did you get to it? _Because I counted, there are six chairs here, then I counted (pointing at the tables, and counting places where there could be no chairs), seven, eight, nine, ten, eleven, twelve.</i></p> <p>[12] <i>_If he puts three tables, how many places will he have? _Sixteen. _How do you know it's sixteen? _You just think with your head. _But how did you think with your head? _We have this here, then you put three more, it is sixteen. _Can you show here with the pieces? (the student takes the chairs and starts to distribute without a definite order or count) _It is twelve.</i></p>
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Student [6], for example, introduced chairs at the division of the tables, without following the initial pattern of a chair for each place on the table. So this student counted several places. Student [3] inserts two chairs at the same place at the table, which jeopardizes her appreciation of a numerical regularity in the increase in the number of places.

One of the students initially replied that three more chairs should be placed at the table. However, after the researcher insisted a little that he tried again, he realizes that he only needs two more chairs. We highlight this isolated response as another category, called *Right-Chairs*. This category gets its name because the correct number of additional places arose after the student tried to manipulate the chairs and realized that he did not need to add the chair at the head of the table. *Table 4* illustrates this experimentation.

Table 4  
*Right-Chairs Category.*

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<b>Right-Chairs</b>	<p>[4] <i>_If he wants to put another table here (there were already two), how many seats will he have? _Nine. _There are other chairs here, if you want to think. _(after trying again). Eight. _Another colleague said it is different. Do you think he's right or wrong? _I think that he is wrong. _And what could he have done wrong? _The same thing I did, I didn't know I had three more chairs to put.</i></p>
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Student [4] tries to formulate an answer imagining that the three places that are not at the junction of the tables were empty. However, when he decides to try, he finds that there is no need to remove the place at the head of the table. Thus, empirical abstraction plays an important role in this change in answer. The researcher's counter-argument emphasizes that the child is sure about the last answer, but this certainty is due to experimentation with the material made available.

We call *Rule-Plus-One* the category of procedures adopted by some students to add a place for each table introduced. Students who opted for this type of procedure did not get at the right answer. Although they tried to formulate a relationship between the number of tables and chairs, the lack of experimentation hindered the analysis of these participants, as shown in *Table 5*.

Table 5  
*Rule-Plus-One Category.*

<b>Rule-Plus-One</b>	
	[2] <i>_With two tables you have six, right? If he put another table there, how many places would he have? _Seven. _How did you get to it? _I imagined, I was looking at the chairs to see.</i>
	[7] <i>_If he puts another table, how many chairs do you think you can use? _Seven. _How do you know it's seven? _Because six plus one is seven. _So, if he puts another table, will he have another place? _ (after new manipulation attempts and some time for reflection) No, it will be eight. _How do you know it will be eight? _No, it is seven. _What is your final answer? _Seven.</i>
	[11] <i>_On each place of the table there can only be one chair, so if you put one more table, how many places will you have? _Six. _So, how many chairs do you have here? _Seven. _Then, six or seven? _Seven.</i>
	[13] <i>_If he puts another table (there were already two tables), how many places do you think there will be? _Seven. _Why seven? _Because he put one more.</i>

As can be seen in *Table 5*, some students who opted for this type of procedure were in doubt when choosing their final answer. Student [13] makes it clear how he is building his rule. In the line “because he added one more”, he attributes the unitary increase in places to the unitary increase in tables. We can see that there is a belief that there is a two-way relationship between objects.

Despite the constitutive difficulties of the problem presented, including regarding the fact that the expression “the most” is in the questions addressing students, some

managed to get to the right answer, using appropriate means to infer the rule that lists tables and places. The most successful category of procedures in activity was what we call the Rule-Plus-Two, which consists of adding two seats for each table introduced. This characterizes the algebraic pattern present in the situation presented to students. The answers are shown in *Table 6*, below.

Table 6  
Rule-Plus-Two Category.

<b>Rule-Plus-Two</b>	<p>[8] _And with four tables? _Now, it got difficult, hold on. _You can get the little dolls here, if you want. _Give me ten. _If I wanted to put another table, what do you think it would be? _Ih, now, it is difficult. Eleven, no, wait, twelve. _Is there any mathematical relationship there, every time you put a table, is there anything with the numbers? _It is difficult. _Let's remember, with one table, four places, with two tables, six, with three tables, eight, with four tables, ten, five tables twelve. _Yes. _What is going on every time I put a table, how many places more do I have to put? _Three. Let's think about the first case: how many places did I have when I placed two tables? _Six. _Then, how many more did I have? _Ih, you gave us a complicated account. Let's think again: I had four places, so I decided to put a new table, how many more places now? _Six. _Então how many more are there, besides the ones we had? _Oito, there were eight. _So, is this your answer? _Yes.</p> <p>[9] _Just doing the math: with one table, four places, with two tables, six places, with three tables, eight places. With four tables, what would be the number of places? _Ten. _How do you know it's ten? _Because it takes two more. What if I put another table? _Twelve. And another table? _Fourteen.</p> <p>[10] _Then it increases by...? _Two. _Each table that you increased, increased two more chairs? _Yup.</p>
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Student [9] infers the rule that lists tables and the number of places from some examples given by the researcher. The confirmation of the recognition of an algebraic pattern is in his statement “because it takes two more”, indicating that there is an understanding of the relationship “for each table introduced, two chairs must be added”. It is noteworthy that, although some students were able to obtain the correct answer to the problem, some suggestions given by the researcher were necessary. Students [8] and [10], for example, only arrived at the answer after several clarifications.

Most of the procedures adopted in activity, which analyzed the students' ability to recognize algebraic patterns, was to add three chairs for each table introduced. We call this category of Rule-Plus-Three *procedures*, illustrated in *Table 7*.

Table 7

## Rule-Plus-Three Category.

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<b>Rule-Plus-Three</b>	<p>[14] _If he puts another table here, how many more chairs will he have to put? <i>_Three.</i> (handling the chairs, the participant counts twice the same place, as she considers the chair that was at the end as one of the side chairs in the next configuration, with one more table, and, adds two more chairs, thus creating the rule of adding three more chairs each time a table is introduced).</p> <p>[16] _And if I wanted to put another table there, how many more chairs would I have to put? <i>_Four.</i> <i>_Four more? _Yes, one chair here, another here, no, three.</i> <i>_Explain how did you think that (pointing at the chairs and tables available for the experiment). <i>_As I gathered here, there is no way to put another chair here, there is one here and two here, it will be three</i> (does not consider the chair that was already at the table, counting one more).</i></p> <p>[18] _And if he wanted to put three tables, how many more chairs would he have to put? <i>_Three.</i> <i>_Can you explain why you thought of three? (the student uses the material made available, but counts the chair at the head of the table twice, which was already there when there were two tables). <i>_With three tables, you put a chair here, another here, and another here.</i></i></p> <p>[20] _If he places another table (there were already two), how many seats will he have? <i>_Nine.</i> <i>_Why do you think it's nine? <i>_Because three plus three is six, plus three is nine.</i></i> <i>_Why did you add three? <i>_I don't know.</i></i></p> <p>[23] _How many more chairs would he need to put here if he wanted to put another table here, in addition to the ones that are here? <i>_I think he would put six more.</i> <i>_Let's suppose he wanted to put another table, if he wanted to put another table together. How many chairs would he have to place to complete the tables? <i>_Three, like this one, one, two, three</i> (pointing at three places at the table, but considering the place of the head of the table in the two counts). <i>Plus this one, four</i> (and counts again the place at the other end). <i>But, since these two remain</i> (referring to the chairs at the two heads of the table), <i>then he would only need to get two more.</i> <i>_And if he wanted to put another table? <i>_Ouch, then, it would be, he would take four more chairs, which altogether would form twelve.</i></i> <i>_Ok, with two tables, he would need six chairs, with three tables, how many chairs does he need? <i>_Nine</i> (repeating places in his count, stating that three more chairs would be needed).</i></i></p> <p>[24] _And if he wanted to join three tables, how many places would he have? <i>_One, two, ..., it is six.</i> <i>_Six, too? _Yes.</i> <i>If he puts another table? <i>_No, eight.</i></i> <i>_How did you get to it? <i>_In this way</i> (and moves the chair away from the head, showing that there will be two more places available). <i>_So, will he have to put more? <i>_One more table.</i></i> <i>_Ok, another table, and how many more chairs? <i>_Uhm, two.</i></i> <i>_And if he still wants, after that, to put another table, how many more chairs will he have to put? <i>_Four more.</i></i> <i>_Ok, you say two here, and two more here? <i>_Yes.</i></i> <i>_Do you think there is a certain mathematical relationship? For each table he puts, does he need to place a certain number of chairs? Do you think there is a specific number? <i>_Yes.</i></i> <i>_How many? <i>_How many tables?</i></i> <i>_If he puts one more table here, how many more chairs will he have to bring (there were already two tables)? <i>_Eight.</i></i> <i>_If he puts another one? <i>_Eleven.</i></i> <i>_Why would he have to put three more chairs yet? <i>_Yes.</i></i> <i>_Each table he puts, how many more chairs does he have to put then? <i>_Ok, if it is separated, he will have to put four, if it is all together, he will have to put three.</i></i></i></p>
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Students in the Rule-Plus-Three category do not realize that the end chair does not need to be removed before another one can be placed. This is related to the lack of conservation in the participants' thinking. Student [18], for example, indicates the places where chairs can be put, saying "with three tables, you put a chair here, another here, and another here", and does not realize that the chair at the end does not need to be removed. It was interesting to note that the student [24] considered the possibility of not putting the tables together, correctly observing that if the tables were not put together, he would

need four more places, but incorrectly inferring that he would need three more places, in case it was necessary to couple the tables, that is, without considering that the end position would be preserved.

Two students disregarded the abutted tables but managed to formulate a rule based on the number of “extra” places that would be made available. These participants showed to perceive the role of the expression “extra” in the problem, however, as they did not realize the tables were abutted, the rule they constructed was ineffective to solve the situation. This category was called *Four-More*, because, by disregarding the tables abutted, all seats should be filled with new chairs, in the view of these students. These procedures are illustrated in *Table 8*, below.

Table 8  
*Rule-Four-More Category.*

<b>Four-More</b>	<p>[15] _Look: before there were four, right, then he put one more table, how many more places did he have to put here? _Four. _And if he puts another table, how many more places will he have to put? _Four. _And if he puts one more later? _Four. _Will it always be the same thing? _Yes.</p> <p>[22] _How many more would he have to put, if he were to put a third table here? _Four. _Four more? _Yup. _So, each table that he puts, he needs to put four more chairs? _Yes.</p>
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We observed that students [15] and [22] perceive an algebraic pattern in the situation presented. If it were not specified in the problem that the tables should be together, these procedures would have been effective. But for the situation presented, this type of procedure has become ineffective.

### **Discussion of Categories in the Light of Strategies**

In the work by Blanton et al. (2015), the strategies used by the subjects of that study for the situation of the tables, regarding the functional thinking of the participants, are divided into three types: drawing, use of recursiveness and use of the functional rule.

The *drawing* strategy consists of reproducing the tables in written form, as many as are requested. For example, when the child is asked how many places it will take for ten tables to be introduced, he/she needs to draw the ten tables to visualize the concrete situation of having the physical layout of the tables and places.

The *use of recursiveness* is a strategy that consists of comparing the evolution of the two quantities. The child does not need an actual representation, but needs to compare the numerical evolution of the number of places, which varies according to the number of tables that are placed. For example, when we ask the child how many chairs will be needed for ten tables, he counts in pairs: two - six, three - eight, four - ten, five - twelve, six - fourteen, seven - sixteen, eight - eighteen, nine - twenty, ten - twenty-two.

The *use of a functional rule* was the most sophisticated strategy found by Blanton et al. (2015). The child capable of formulating an explanation based on a functional rule is able, for example, in the problem of ten tables, to realize that he/she must multiply the number of tables by two and add two more chairs, as this is the rule of association between the two magnitudes, which is built from experimentation with the materials made available for the activity to be carried.

From the categories of procedures that arise from this research, we can identify relationships with the strategy of designing and using recursiveness. None of the students used the strategy of the functional rule.

The categories of No-Rationale, No-Putting-Tables-Together, and Four-More procedures are not associated with any of the types of strategies present in the work of Blanton et al. (2015), because in the case of No-Rationale causality is not present in the students' thinking, and in the case of No-Putting-Tables-Together and Four-More procedures, the basic premise that the tables were together was not fulfilled, thus the construction of the rule is not consistent with the reality observed by the actual situation of the problem. Therefore, we consider such categories to be *preoperative* procedures.

We can affirm that the categories of Mix-Chairs, Right-Chairs, One-More-Rule, and Three-More-Rule are associated with the drawing strategy since such procedures reflect the need for concrete representation of the proposed situation. The mistakes of students who used such types of procedures refer to the absence of reversibility schemes and a deformity of the empirical abstraction, but the understanding of the need for table junction and the notion of a relationship between the number of tables and number of chairs reveal a great advance in the construction of algebraic patterns and the presence of causal relationships in the schemes these students produced. As drawing is a concrete way of operating, we can say that the procedures that we associate with the drawing strategies produced *concrete and functional* strategies in our research.

There is a strong similarity between the procedures presented by students who used procedures of the Two-More-Rule type and the strategy of using recursiveness in the work of Blanton et al. (2015). In both, students can relate the quantities involved directly and can predict results by observing the relationship between the number of tables and chairs. We can say that, in our research, *functional-recursive* strategies were used.

### **Level of Answers and Operative Invariants**

Level I is characterized by the little use of causal schemes in the strategies used. It is subdivided into three sub-levels, according to the types of procedures the

participants used. Level II is characterized by the understanding of the need to put the tables together. This is what makes the difference between Level I and Level II. In the first level, students are not concerned whether the tables are together, which demonstrates an advance in the representation of the situation presented, although mistakes in the distribution of chairs or the absence of reversibility schemes are evident in the strategies students used.

The strategies in Level III overcome the difficulties of the previous levels since students who used Level III strategies realize the importance of putting the tables together and operate the places effectively, which allows for a correct construction of the functional rule that relates the number of tables and the number of chairs. From the procedures and strategies presented in this section, it was possible to construct *Table 9*, showing the levels of the answers to the situation of algebraic patterns treated from activity.

Table 9  
*Levels of the Answers for Activity.*

<b>Category</b>	<b>Description</b>
Level IA	There is little or no causal relationship in the strategy used. The child estimates the number of places with no rationale, not relating it to the number of tables.
Level IB	There is little or no causal relationship in the strategy used. The child does not realize the need to put the tables together. He/she randomly distributes the seats at tables.
Level IC	There is little or no causal relationship in the strategy used. The child does not realize the need to put the tables together. He/she distributes four seats to each table that is introduced.
Level IIA	The child does not realize the need to put the tables together. He/she does not understand that each place on the table should only house one chair, as in the model presented. The child places more than one chair in some places.
Level IIB	The child realizes the need to put the tables together. The child only gets the number of chairs right after experimentation, by needing to put the chairs at the places, and still being unable to mentally formulate the rule that relates the number of chairs as a function of the number of tables.
Level IIC	The child realizes the need to put the tables together. The child has a two-way relationship between chairs and tables.
Level IID	The child realizes the need to put the tables together. The absence of reversibility causes the head of the table to be counted twice when a table is added, distorting the functional rule that relates tables and chairs.
Level III	The child uses recursiveness to predict how many seats are needed for any number of tables added.

The relationship between students' strategies, procedures, and levels of answers regarding Activity referring to the idea of algebraic patterns, is summarized in *Table 10*, below.

Table 10  
*Procedures, Strategies, and Levels of Algebraic Pattern.*

<u>Procedure</u>		<u>Strategy</u>		<u>Level</u>
No-Rationale	→	Preoperative	→	IA
No-Putting-Tables-Together	→	Preoperative	→	IB
Mix-Chairs	→	Preoperative	→	IC
Right-Chairs	→	Functional-Concrete	→	IIA
One-More-Rule	→	Functional-Concrete	→	IIB
Three-More-Rule	→	Functional-Concrete	→	IIC
Four-More-Rule	→	Functional-Concrete	→	IID
Two-More-Rule	→	Functional-Recursive	→	III

The need to “put tables together” can be considered as a concept-in-action, which mobilizes the theorem-in-action of “counting places each time a table is introduced”. This count is present in the functional-concrete strategies, in which the child realizes that there is a relationship between the number of tables and chairs, but is not sure that he/she can build and use a mathematical calculation to estimate the number of places, from the number of tables.

In Level III, the use of the theorem-in-action “add two places each time a table is introduced” is evident, based on the concept-in-action of the “permanence of places at the ends of tables”, because the subjects of Level III realize that the chairs at the end of the tables are never removed, and two seats are always introduced at the sides of the new table.

## CONCLUDING REMARKS

We identified four operative invariants: the theorems-in-action “counting the places each time a table is introduced” and “adding two places each time a table is introduced”, respectively linked with the concepts-in-action “putting tables together” and “permanence of chair at the ends of tables”.

## CONTRIBUTION OF EACH AUTHOR

VCB was responsible for data collection, for organizing the writing of the article and for the general checking of consistency between methodology and results. JAS was

responsible for guiding the theoretical assumptions, the methodological guidelines, and monitoring the writing. Final considerations were discussed and written by both authors.

## DATA AVAILABILITY STATEMENT

The data supporting the results of this study are all made available by the authors, VCB and JAS, in the tables that make up the text of the article.

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