

Using Dynamical Geometry Softwares in the study of Plane Geometry: potentialities and limitations

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ABSTRACT

Background: Some inconsistencies that appeared when teaching Geometry - using Dynamical Geometry Softwares - to Mathematics undergraduate students, inspired this work. **Objectives:** To stress the potentialities and, specially, the limitations of Dynamical Geometry Softwares in order of using it correctly by teachers, pre-service teachers and students for learning and teaching situations and investigations in Geometry. **Design:** Critical analysis study of situations, with examples produced using Dynamical Geometry Softwares, in order to reveal some inconsistencies with respect to the theory. In particular, Geogebra was the software used. **Setting and participants:** Some of the examples presented here, elaborated in GeoGebra, were briefly discussed in classes of pre-service teachers of mathematics aiming at an awareness of the inconsistencies that may appear using a Dynamical Geometry Software. The authors are the unique participants of the elaboration of those examples. **Data collection and analysis:** There was not data collection, but only elaboration of examples in order to provide some arguments for future discussions. **Results:** Examples production shows some limitations of Dynamical Geometry Softwares and that those limitations are insurmountable due to epistemological reasons. **Conclusions:** Awareness of Dynamical Geometry Software limitations is fundamental for its correct use. Those limitations do not invalidate the software potential. On the contrary, being conscious of both potentialities and limitations of a hardware is a necessary condition to a fruitful use of it.

Key words: Dynamical Geometry Softwares; Potentialities; Limitations; Geometry..

O uso de Softwares de Geometria Dinâmica no estudo de Geometria Plana: potencialidades e limitações

RESUMO

Contexto: Inconsistências surgidas com o uso de Softwares de Geometria Dinâmica ao lecionar a disciplina de Geometria para estudantes de um curso superior de Licenciatura em Matemática motivam este trabalho. **Objetivos:** Esse estudo tem por objetivo destacar as

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potencialidades e, em especial, as limitações dos Softwares de Geometria Dinâmica de forma a serem utilizados corretamente por professores, professores em formação e estudantes no ensino-aprendizagem e em investigações em Geometria. **Design:** Esse é um estudo de análise crítica com a produção de situações, com exemplos construídos em Softwares de Geometria Dinâmica, revelando inconsistências com a teoria. Em particular, o software utilizado neste trabalho foi o GeoGebra. **Ambiente e participantes:** Alguns dos exemplos aqui apresentados, construídos por meio do GeoGebra, foram brevemente discutidos com estudantes de licenciatura de um curso de licenciatura em matemática, no intuito de conscientizá-los das inconsistências que podem surgir com o uso de um Software de Geometria Dinâmica. Os autores são os únicos elaboradores desses exemplos. **Coleta de dados e análise:** Não houve coleta de dados, mas sim a produção de exemplos para fornecer elementos para futuras discussões. **Resultados:** A produção dos exemplos no GeoGebra mostra as limitações dos Softwares de Geometria Dinâmica e que essas limitações são intransponíveis por razões epistemológicas. **Conclusões:** A conscientização das limitações dos Softwares de Geometria Dinâmica é fundamental para a sua utilização correta. Essas limitações não invalidam o potencial desses softwares. Pelo contrário, o conhecimento das potencialidades e das limitações de uma ferramenta é condição necessária para saber utilizá-la com proveito.

Palavras-chave: Softwares de Geometria Dinâmica; Potencialidades; Limitações; Geometria.

INTRODUCTION

Dynamic Geometry Software (DGS) has been used for nearly three decades as an auxiliary tool in the teaching and learning of geometry in elementary education (Gravina, 2015; Laborde, 2000, 2001) and in studies and investigations in this area with a focus on both solving and posing problems (Leikin, 2015; Mariotti & Baccaliggni-Frank, 2010). Posing problems as an object of study has been approached with increasing interest in recent years in the field of Mathematics Education worldwide, as several articles reveal (Gontijo, 2007; Gontijo, Carvalho, Fonseca & Farias, 2019; Silver, 1997; Sriraman, 2004). These articles indicate that the students must have experience posing/formulating new problems to participate more actively in the process of teaching and learning mathematics.

In this work, which is part of a thesis on creativity in mathematics, under development by one of the authors, some critical aspects of the DGSs will be addressed, without describing experiments carried out aiming at creating problems in Euclidean Geometry. These experiments/tasks, carried out with a DGS in classes of the mathematics degree course at the Federal University of Santa Catarina, raised some confrontational situations between practice and theory that the future elementary school teachers should be able to understand and explain.

This work is divided into three parts. The first part brings the analysis of the characteristics of a DGS, its potential, and its limitations. The study also highlights the advantages of GeoGebra compared to other DGSs, during the research on geometry problems. The second part will present examples in three situations the authors elaborated carefully in GeoGebra, in which the limitations and contradictions that may arise with the use of any DGS will be evident. It is necessary to emphasize that knowing how to

work correctly with a tool consists of first knowing its qualities and limitations. In the case of the DGS, the limitations are inevitable due to the reasons that will be exposed here. The third part will discuss the calculation errors inherent to the precision limitation of measurements.

DYNAMIC GEOMETRY SOFTWARES: POTENTIALITIES AND LIMITATIONS

The potential of a Dynamic Geometry Software (DGS) as an auxiliary tool for geometry teaching and learning in elementary schools has been widely discussed and disseminated (Gravina, 2015; Laborde, 2000, 2001; Leikin, 2015). A DGS can be described primarily as “a computational ruler and compass” that should relieve students from difficulties and inaccuracies they find when handling a ruler, a compass, and a pencil. In fact, it does. The use of traditional tools often “does not work.” For example, take the construction of a regular hexagon or pentagon inscribed in a given circle with those tools. Although the construction is accurate in terms of its theoretical justification, in practice, the tools are inaccurate, and the polygon usually “does not close.” A DGS is more *accurate* than those three tools, but it is not *exact*, and we will see, during the development of our discussion, the reason for that. However, it is necessary to give the difference between *precision* and *accuracy*. Precision refers to the measurement with a calculable error. In practice, all measurements are subject to an error. Accuracy is the Platonic idealization of a measurement. Therefore, a measurement that is not accurate implies some kind of error, which, if not controlled, can affect results.

The highest potential of a DGS, however, is not its accuracy. Another even more important potentiality is characterized by the adjective “dynamics,” which is the possibility of modifying a figure while maintaining specific invariant attributes related to that dynamic figure. For example, when we plot the heights of a triangle and later modify that triangle, the segments plotted as heights also change to new heights of the modified triangle, which opens up an entirely new world in terms of teaching-learning and research in geometry. In the example mentioned, we can also observe the property of the intersection of the three heights’ support lines at a point called the orthocenter of the triangle. In GeoGebra, this dynamic feature is in the “Move” tool. This movement of the figures seems to be continuous, but only seemingly. An interesting parallel that could help us understand this aspect of the perception of the continuity of a movement would be the projection of a film in analog mode (as in the past, through celluloid material): the rapid succession of a (homothetic) projection on a screen of many pictures recorded on a celluloid roll that our eyes capture as something continuous. A large but *finite*, therefore *discrete*, number of frames, generates the film we are watching. The understanding of this aspect of a DGS helps us to understand its limitations. If they are well understood, which can generate some inconsistencies with the theory, we can verify that they are typical of any DGS. One must be aware that highlighting and trying to understand the limitations of a tool does not lessen its usefulness. Understanding these limitations allows us to work better

and correctly with them and even take advantage of them. These software programs are very similar in their properties, but some allow us to perform specific actions more advantageously in terms of their potential. On the other hand, this work will discuss why some limitations they share are insurmountable.

GeoGebra, the DGS that will be used in the situations in the next section, presents some beneficial aspects if compared to other DGS programs. First, GeoGebra is a free software, therefore, widely accessible. Second, GeoGebra introduced the idea of jointly presenting the algebraic (analytical) and figural (synthetic) representations of a geometric object, respectively, through its “Algebra” and “Graphics” windows. The “Algebra” window informs relevant quantitative data (area and perimeters of figures, lengths of segments, coordinates of points) that can be used in qualitative (synthetic) investigations. This window also makes it possible to “hide” or “display” some of the elements in the figure (meaning that these elements have not been erased and can be displayed back) more easily. Third, GeoGebra, with the possibility of automatically labelling a geometric object allows us, in some cases, to immediately perceive specific inconsistencies (such as the appearance of two points where there should only be one), a consequence of its limitations, something that can go unnoticed with other DGS programs. Little attention has been paid to this quality of GeoGebra, perhaps because little attention is paid to the limitations of the DGS programs in general. Fourth, in the “Rounding” option, GeoGebra brings a numerical precision of up to 15 decimal places, which also allows a better understanding of the *precision/accuracy* duality in measurements. Fifth, the possibility of zooming in or out a figure in the “Graphics” window allows us to analyze a figure in detail and, most importantly, perform a drag test more precisely. A drag test occurs when the “Move” tool is used to check, for example, a specific geometric property, as mentioned above, in the case of the orthocenter of a triangle. This test can also be carried out for quantitative checks, such as when determining maximum or minimum of areas, perimeters, or measures of segments and angles. There is a big difference, in terms of checking in a drag test, between moving a point on a straight line, a ray or a segment, and moving a point on the plane. In the first (one-dimensional) case, the point moves with one degree of freedom, and in the second (two-dimensional) case, the point moves with two degrees of freedom. This dimensional problem addressed by Duval (2012) influences our ability to perceive geometric properties of figures. An example of the first case is the investigation, via a DGS program, of Heron’s minimization problem and, for the second case, Fermat’s minimization problem (Pasquali, 2004, p. 20 and 61). There are other essential potentialities of GeoGebra, common to other DGS programs that will not be addressed here. Next, this study will present an analysis of the reasons for the limitations of a DGS.

Two reasons explain why inconsistencies caused by the limitations of a DGS can occur: one, of an epistemological order, and another, of a physical (material/technological) order. The epistemological explanation has its origins about two and a half thousand years ago, when the Greek mathematicians (geometers) discovered the existence of pairs of incommensurable segments (Fritz, 1944). This was perhaps one of the most dramatic moments in the history of mathematics: Platonic accuracy gave way to precision or

approximation of measures. Rightly so, mathematicians only came to understand this fact consistently at the end of the 19th century. The irrational numbers added inaccuracy to our machines, however modern they may be. The impossibility of obtaining a measure with accuracy remains in our world essentially of rational numbers, and the DGS programs do not escape this. The second explanation is physical: the computer screen is not (also as a consequence of the epistemological reason mentioned above), nor can it be a “*continuum*,” and not even a dense set (in the topological sense) of points on a plane. It is only a finite set, therefore discrete, of points (pixels) that generate the images. For this reason, the measurement process in a DGS can be undermined.

EXAMPLES OF SITUATIONS THAT HIGHLIGHT THE LIMITATIONS OF A DGS, CAUSING THEORETICAL INCONSISTENCIES

This section will bring three situations that show limitations that can generate inconsistencies with the use of GeoGebra (or any other DGS). The examples described in them were used in activities with students of the mathematics course (both for teaching and researching, Licenciatura and Bacharelado, respectively) at UFSC, to make them aware of the DGS limitations. As students of a mathematics higher education course, they already had a basic understanding of mathematics (for example, in-depth knowledge of real numbers). The activities carried out with the students (their answers) will not be reported in this work. These examples deal with possible inconsistencies in the tangencies of a straight line with a circumference and a circumference with another circumference, the identification of lines due to the sensitivity of the measurement, and the impossibility to obtain intersection with geometric loci or even in the inaccuracy of geometric loci. As will be seen here, these limitations can confuse the students, not contributing to investigations aimed at posing new problems and conjectures in geometry. Possibly, as the definitions of the images on the computer screens (screen resolution) improve, some of the inconsistencies may not occur again; however, the measurement processes will never be exact.

The relationship between the knowledge of the potential and limitations of a tool such as a DGS, as well as all the concepts that emerge when each student uses the tool to solve geometric problems - and the theoretical meaning of this knowledge- is what characterizes the so-called *semiotic potential* of the tool (Bussi & Mariotti, 2008; Mariotti, 2013; Stormowski, Gravina & Lima, 2013). The following is one aspect of GeoGebra’s semiotic potential.

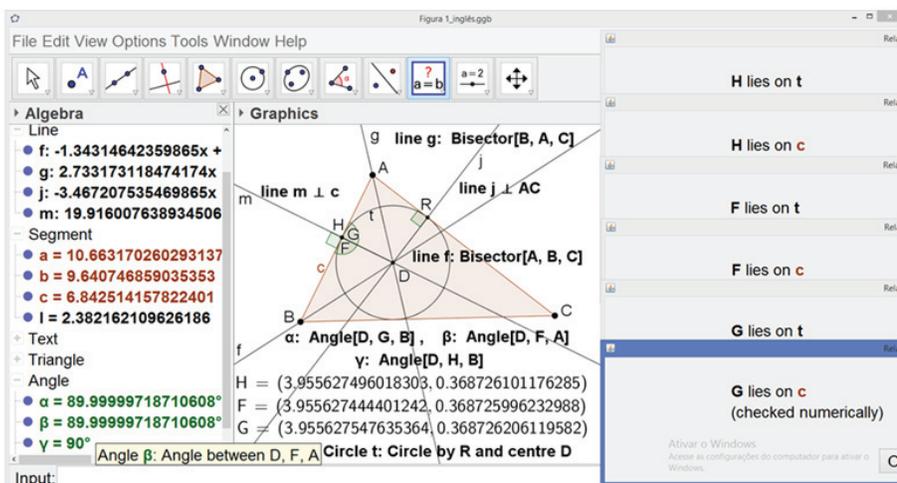
Inconsistencies in tangency

This type of inconsistency can occur when the “Intersect Two Objects” tool is used to construct (correctly, from a theoretical point of view) a tangent line to a circle, and two contact points appear at its intersection. The two points are not visible (they overlap),

but their automatic labels reveal them (if we keep this option in “Labelling”). More specifically, students were asked to construct any triangle and trace the circumference inscribed in it. They found an inconsistency in the tangency. For a more precise and thorough explanation, the authors’ construction will be presented.

Figure 1 is a copy of the print screen of a construction in GeoGebra, in which we tried to put as much information as possible, either in the “Algebra” or in the “Graphic” window using labels, texts, angle labelling, and point redefinition text, to convincingly expose the inconsistency in the tangency. The details in the figure are explained below. Figure 1 shows all values and objects mentioned in the explanation.

Figure 1
Inconsistency in the tangency of the circumference inscribed in a triangle



In a given triangle ABC , the bisectors f and g of the angles of vertices B and A , respectively, were plotted, finding the incenter D of the triangle. These bisectors were drawn according to the ruler-and-compass (theory) construction method, but the same result was obtained by using the “Bisector” tool. Then, a perpendicular j was drawn from D to the side AC of the triangle, finding point R where j intersects that side. Thus, DR is the radius of the circle t inscribed in the triangle, i.e., it must be tangent to the three sides of the triangle. Using the tool “Intersect Two Objects,” we tried to obtain the point of tangency of the inscribed circle t with the side c (AB) of the triangle. This point should be the intersection point of the circle inscribed with c . Surprisingly, two labels were exposed, F and G . Next, the angles α (DGB) and β (DFA) were measured, obtaining measurements smaller than 90° , but very “close” to this value (see the “Algebra” window). Then, a perpendicular m was drawn through center D to the c side (AB) of the triangle, obtaining the point H at the intersection of m with that side. Thus, a right angle γ was obtained, with the vertex at that point. With the option of rounding of 15 decimal places

(confirmed by the values in the (“Algebra” window), we could observe that the three points **F**, **G**, and **H** were all distinct (those coordinates were transported to the lower part of the “Graphics” window). On the other hand, using the “Relation” of points (see the right part of the “Graphics” window), we could see that these three points are collinear (all are on the **c** side of the triangle) and are on the inscribed circumference **t**, which, theoretically, is impossible!

The two points of ‘tangency’ may not appear initially, though; but a small disturbance of the triangle (moving one of its vertices) can make these points appear. Throughout this work, we used the GeoGebra Classic 5.0.562.0-d version. GeoGebra Classic 6 and other known software program were also tested¹, with the same results. This inconsistency with the theory may limit an investigative study. Suppose that a circumference obtained through some properties is, apparently, tangent to a given line in the figure. The test of the intersection of the two objects may indicate that there is no tangency but concurrency, which may be wrong. Of course, it is always possible to move some points, using the software dynamics to check whether the tangency is true or false.

Measurement sensitivity

Figure 2 shows a copy of a print screen of a construction of the experiment described below.

In the “Graphics” window, the axes were displayed, the rounding option was established to 15 decimal places and three points were labelled: the origin **O** = (0,0), the point **P** = (0,4) on the *y* axis and point **Q** = (4,0) on the *x* axis. Then, through its coordinates, a set of points on the positive *x* axis “very close” to the origin, and the respective lines passing through **P** and each of these points were constructed. The respective angles, with vertices at these points and passing through **P** and **Q** were measured:

point **A** = (10^{-15} ,0), line **g** (**PA**) and we measured $\alpha = \mathbf{QAP} = 90^\circ$,

point **B** = (10^{-14} ,0), line **h** (**PB**) and we measured $\beta = \mathbf{QBP} = 90^\circ$,

point **C** = (10^{-13} ,0), line **i** (**PC**) and we measured $\gamma = \mathbf{QCP} = 90^\circ$,

point **D** = (10^{-12} ,0), line **j** (**PD**) and we measured $\delta = \mathbf{QDP} = 90^\circ$,

point **E** = (10^{-11} ,0), line **k** (**PE**) and we measured $\varepsilon = \mathbf{QEP} = 90^\circ$,

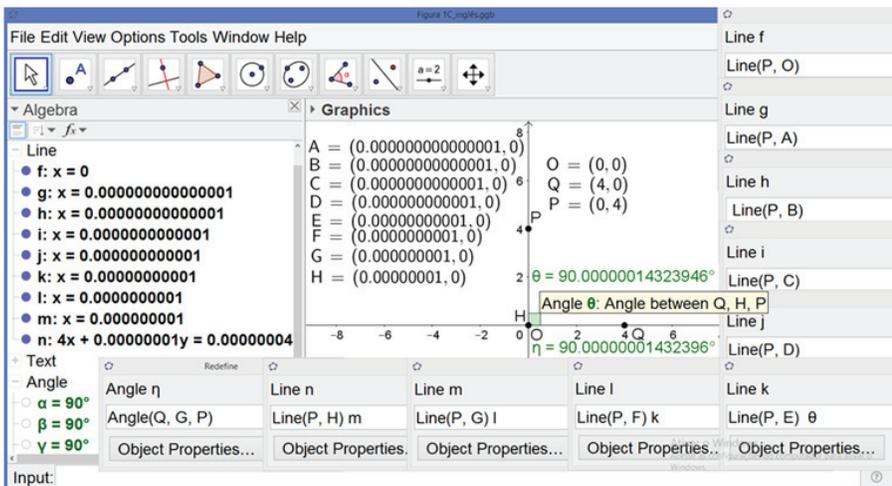
point **F** = (10^{-10} ,0), line **l** (**PF**) and we measured $\zeta = \mathbf{QFP} = 90^\circ$,

point **G** = (10^{-9} ,0), line **m** (**PG**) and we measured $\eta = \mathbf{QGP} = 90.00000001432396^\circ$,

point **H** = (10^{-8} ,0), line **n** (**PH**) and we measured $\theta = \mathbf{QHP} = 90.00000014323946^\circ$.

¹ The other software was Cabri® II Plus, version 1.4.3. In this case, apparently, a single point appeared, but this is because Cabri does not automatically label its points. When we placed the cursor over the point, the software asked: “Which object?” indicating a second point there, which was not detected immediately.

Figure 2
Measure inconsistency - distinct and simultaneously coincident lines



The line **f** through **P** and **O** was also plotted (coincident with the y -axis). The coordinates of points **O**, **P** and **Q** and points **A** to **H** have been transcribed to the “Graphics” window to make the equations of the lines **f** through **n** visible in the “Algebra” window. This window displayed the measurements of three of the angles listed above, α , β , and γ . The labels for all angles were hidden, except for the last two, η and θ , which were transcribed to the “Graphics” window and the window displayed in yellow the information on the definition of the angle θ . Finally, through the “Redefine” tool, it is possible to display, in this window, which points define the lines from **f** to **n**, and which points define the angle η (this information covering a lower part of the “Algebra” window).

What are the inconsistencies shown in Figure 2? Lines from **g** to **m** pass through **P** and, therefore, cross the y -axis, but their equations in the “Algebra” window reveal that these lines are parallel to that axis. The angles measured with vertices at points **A** to **F** and sides passing through **P** and **Q** (angles α , β , γ , δ , ϵ and ζ) measured 90° (not all shown in the figure) and, from the angle η , at the vertex at point **G**, this measure becomes coherently a “little” greater than 90° , although the corresponding line **m** is still, according to its equation in the “Algebra” window, perpendicular to the x -axis. From point **H** onwards, the window displays, then, a straight line (**n**) with a negative angular slope and the corresponding angle θ with a measure “slightly” greater than 90° , which is also mathematically consistent. Therefore, we can say that the measurement sensitivity (allowing us to distinguish the geometric objects) is of the order of 10^{-8} units.

Impossibility of obtaining intersection with geometric loci

Let us suppose we want to establish the geometric locus of a point with specific properties when studying a given construction in a figure. The “Locus” tool makes it possible to find this set better than enabling “Trace on” the point. This geometric locus - usually a curve - appears in the “Graphics” window and is indicated in the “Algebra” window as **lg...= Geometric Locus...** It is not possible to know the equation of such a curve or to which class of curves it belongs. However, if there is a suspicion (intuition) of which curve it might be, then we can try to characterize it through some geometric (not necessarily analytical) property that is satisfied by that curve. For example, if the curve plotted as a geometric locus looks like a circumference, we can plot three segments with ends on that curve (chords), where any two are not parallel, and check if the perpendicular bisectors of the three segments are concurrent. This can be done, because it is possible to mark a point in a geometric locus using the “Point on Object” tool. In the case of a conic, there is a property that says that the midpoints of parallel “chords” (segments with ends at the conic) are collinear, with the straight lines containing the midpoints of each family of parallel strings concurrent at the point of intersection of the conical axes, if it is an ellipse or a hyperbola, and are parallel to the conical axis if it is a parabola (Yefimov, 1964, p. 109). However, for the conic as a geometric locus obtained with the software, the property described above is impossible to verify, while for the circumference, it may be found inconsistent with the theory.

Figure 3 is a copy of a print screen of a given construction through the “Locus” tool in Geogebra, in which the geometric locus of the centers of tangent circles simultaneously to two given circles of different radii (hidden in the figure) was obtained. This geometric locus is known to be a hyperbola. In the figure, there is only a branch of the hyperbola built in two parts, which appear as **lg1** and **lg2**. To try to apply the property mentioned in the previous paragraph, point **I** was marked in **lg1** and point **J** marked in **lg2**, the chord **IJ** was drawn, and its midpoint **L** was marked. Then, a point **K** was marked in **lg1** and the straight line **j** was drawn parallel to segment **IJ**. Finally, we tried to mark the intersection of **j** with **lg2** (using the “Intersect Two Objects” tool), but the software did not respond, probably because the geometric locus is obtained numerically. Thus, we could not verify a possible conjecture² if it were possible to intuit that the curve might be a conic.

² Cabri © II Plus labels this intersection, but the property did not verify, indicating problems with the software. It is actually more confusing than that. An intersection point is marked, but the software does not confirm that the point is over the object.

the geometric locus of the centroid of triangles of fixed base **AB** and inscribed on a fixed circumference *c*. As the distance from the centroid to the midpoint **M** of **AB** is equal to one-third of the length of the median **CM**, then this locus is a circle *c'* homothetic to circle *c* with a centre of homothety **M** and a ratio of one-third.

Figure 4

*Point P, the intersection of the perpendicular bisectors **m** and **n** of the **DE** and **EF** chords, respectively, is not in the perpendicular bisector **p** of the **FH** string*

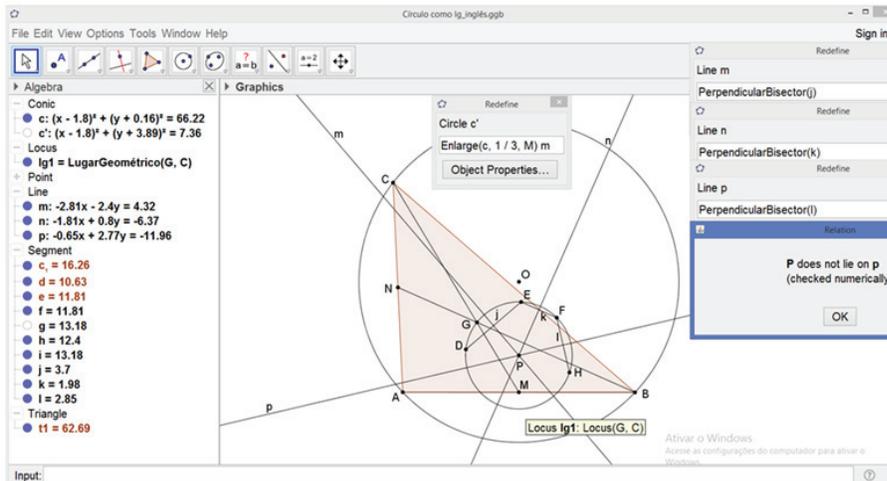


Figure 4 shows locus **lg1** in the “Graphics” window. The circumference *c'*, homothetic to the circumference *c*, does not appear in this window, but is highlighted in a “Relation” box at upper part of the window, and is indicated in “Algebra” window. Points **D**, **E**, **F** and **H** were taken in **lg1**. We can see that the point of intersection **P** of the perpendicular bisectors **m** and **n** of the chords **DE** and **EF**, respectively, is not at the perpendicular bisector **p** of the chord **FH**, as it is indicated in the table “Relation,” therefore, **lg1** would not be a circumference. The problem here could also be circumvented by choosing three different positions for the vertex **C** in the circumference *c*, then marking the three centroids corresponding to the **ABC** triangles, and tracing the circumference that passes through these three centroids. After that, it would be enough to check whether any other centroid of the **ABC** triangle, with **C** going through *c*, would be on that circumference.

These inconsistencies do not jeopardize an optimal investigative work, but the software user (teachers and students) must be aware that they can occur and be aware of the (epistemological and physical) reasons that cause them.

The next section will bring the analysis of the types of errors that induce calculations that may lead to some inaccuracies, and present a simulation of the plane to shed light on

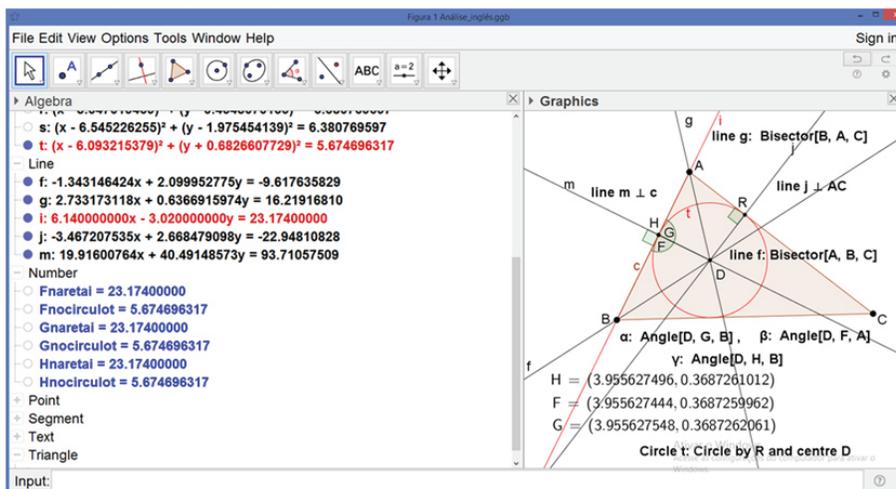
how three distinct and known collinear points also appear as belonging to a circumference, as in the example shown in Figure 1.

PRECISION AND INACCURACY: CALCULATION ERRORS AND THE LIMITED PHYSICAL REPRESENTATION OF THE PLANE AS A SET OF DISCRETE POINTS

How is it possible that the inconsistency verified in the example in Figure 1 occurred? How does GeoGebra perform its calculations? Figure 5 is a copy of a print screen of the same situation, but with the calculations made by GeoGebra to verify whether the three collinear points **F**, **G** and **H** can be on the same circumference, by replacing their respective coordinates in the equations of the straight line **i** (which contains the **c** side of the triangle) and the circumference **t**.

The coordinates of the different points **F**, **G**, and **H** (transported to the bottom of the “Graphics” window) have been replaced in the equations of line **i** and the circumference **t** (both in red in the “Algebra” window). The results obtained (numbers in blue in the “Algebra” window) with rounding of 10 significant digits by replacing the coordinates of the points **F**, **G**, and **H** in the equation of line **i** are identified as “Fnaretai,” “Gnaretai,” and “Hnaretai,” respectively, while the values obtained by replacing the coordinates of points **F**, **G**, and **H** in the equation of the circumference **t** are identified as “Fnocirculot,” “Gnocirculot,” and “Hnocirculot,” respectively. Note that these values are identical, which is consistent with the relevance of those points being on the line and the circumference.

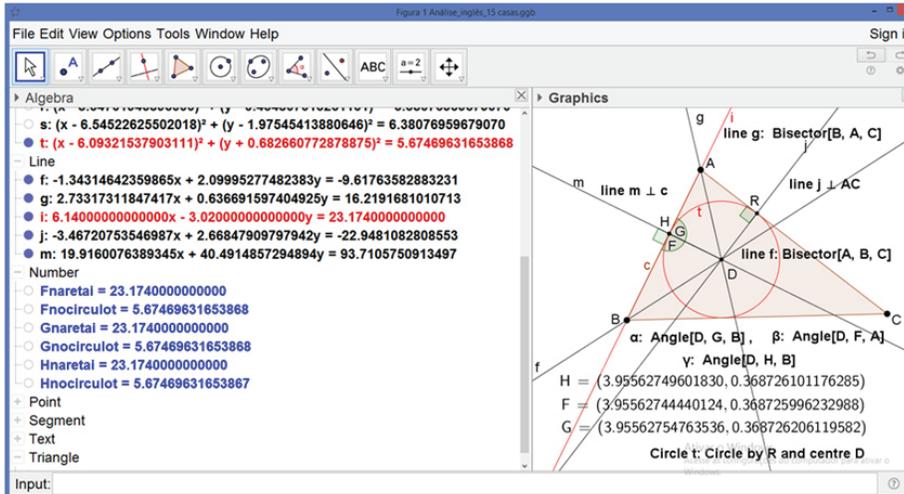
Figure 5
GeoGebra calculations for the coordinates of points **F**, **G**, and **H** in the equations of line **i** and the circumference **t** with rounding of 10 significant digits



However, if the same construction is done with rounding of 15 significant figures, then the result for “Hnocirculot” becomes (minimally) different from the other two, which was theoretically expected. Figure 6 shows this.

Figure 6

GeoGebra calculations for the coordinates of points *F*, *G*, and *H* in the equations of line *i* and the circumference *t* with rounding of 15 significant digits



These results show that GeoGebra identifies its objects in the “Graphics” window, according to calculations that indicate *rounding errors*.

Rounding errors are associated with the number of significant digits with which a given system, or software, works. On the other hand, the so-called *truncation errors* occur in processes with an infinite number of operations or with a large number of digits (whole or decimal) and which, for practical reasons, are truncated (Barroso et al., 1987, p. 12). Rounding errors can spread in successive operations, and the results can also depend on the order in which the operations are performed. We will reproduce below an interesting example found in Barroso et al. (1987, p. 14).

Consider the following system of two linear equations

$$\begin{aligned} 0.0030 x_1 + 30.0000 x_2 &= 5.0010 \\ 1.0000 x_1 + 4.0000 x_2 &= 1.0000 \end{aligned}$$

The exact solution of this system is $x_1 = 1/3$ and $x_2 = 1/6$.

GeoGebra solves the system correctly with rounding, as shown in Figure 7, that is a copy of a print screen of the system resolution by matrix reduced echelon form (matrices **m1** and the corresponding echelon matrix **m2**). A potential problem is if there is an early

rounding, in which $5001/3$, which arises from multiplying the first equation by $1/0.003 = 1666.66\dots$, is rounded up to 1667 , then the system, after a few more operations between lines, becomes

$$x_1 + 10000 x_2 = 1667$$

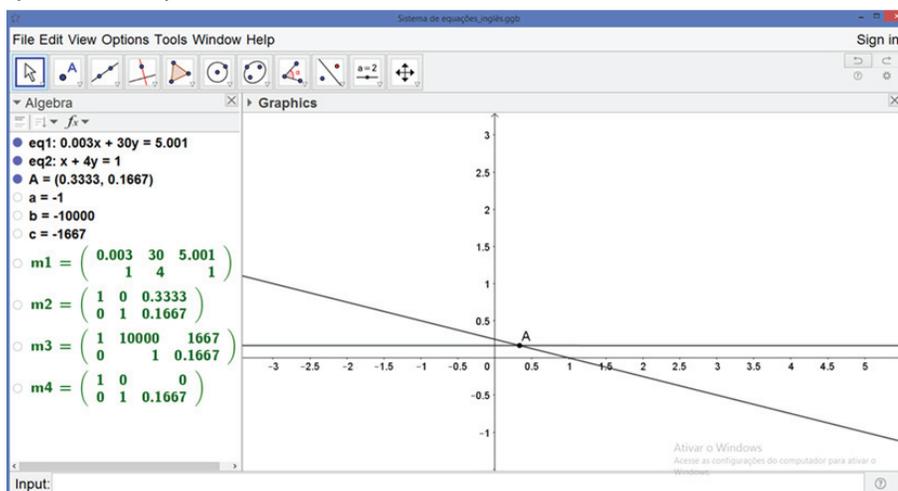
$$x_2 = 0.1667$$

This system would give solutions $x_1 = 0$ and $x_2 = 0.1667$, with x_1 being wrong for the original system. See in Figure 7, the matrices **m3** and its echelon **m4**.

In Figure 7, the figural representations of the two equations in the “Graphics” window show two lines that intersect at point **A**, whose coordinates give the solution of the system.

In this example, there is, in fact, no problem with GeoGebra calculations, but these calculations can be changed by rounding, as seen in Figures 5 and 6.

Figure 7
System resolution by matrix reduced echelon form in GeoGebra



Resuming the example in Figure 1, it would be interesting to approach it from the figural point of view, considering that the computer screen plane is not a *continuum*. Doing a simulation, let us assume that the pixels on the computer screen are represented by the intersection points of a 0.001 distance mesh, as in Figure 8. Here, line **r** (in red) has an irrational angular slope, and therefore, it would not pass through any of the mesh’s vertices. Then, in this mesh, the line would be given by (isolated) discrete points, some marked in black in the figure, plus the three points **F**, **G**, and **H** (in red). Also, the circumference **c** that passes through these three points is plotted. Both line **r** and circumference **c** are represented in the figure for reference only, and their equations are represented there.

Figure 8

Line and circumference given by discrete points in a simulation of the pixel mesh

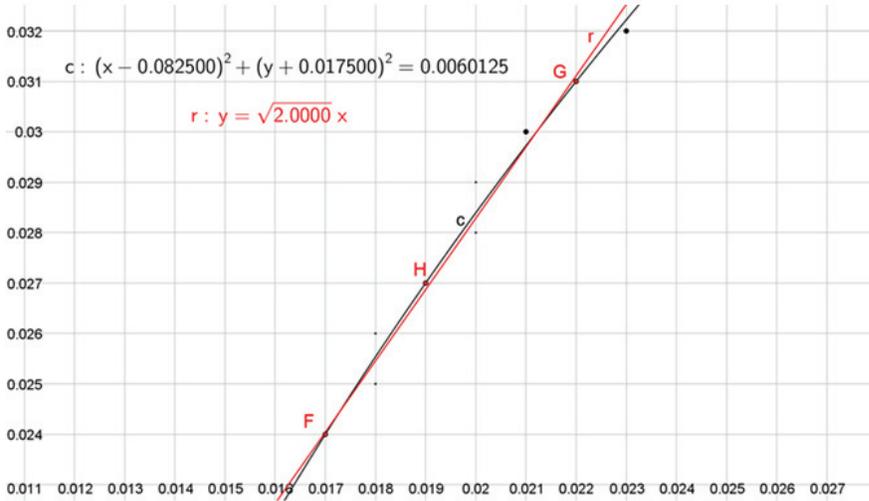
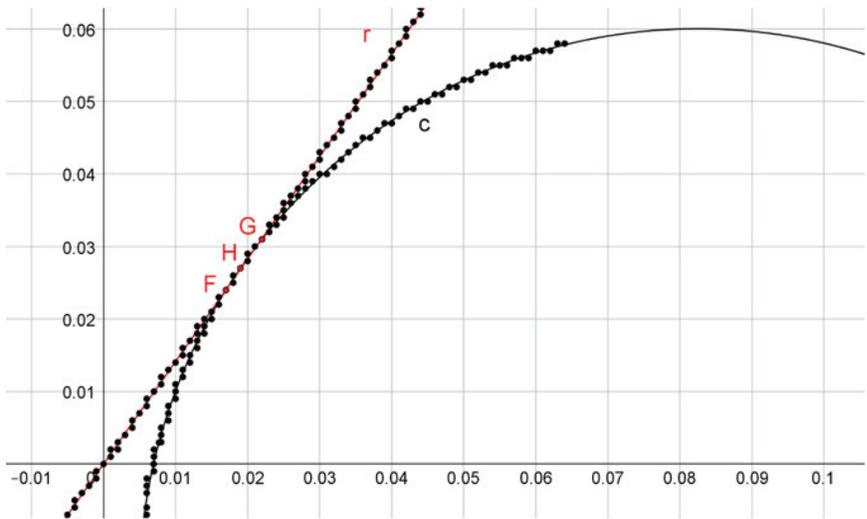


Figure 9

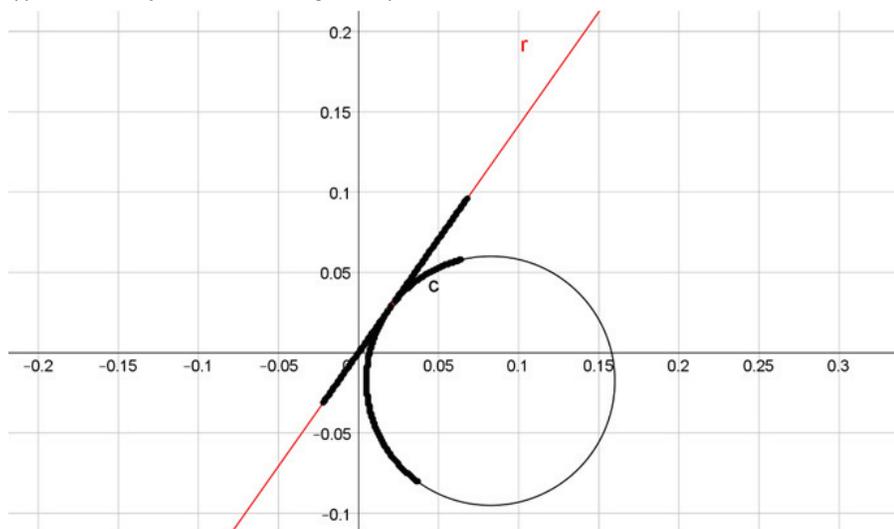
Line and circumference given by discrete point, in a simulation of the pixel mesh and zooming out



The zooming on the figure allows us to perceive the mesh points that make up the line and the circumference (Figure 9). Observe the points of the mesh accumulating in the two curves.

Figure 10

Line and circumference given by discrete points in a simulation of the pixel mesh and zooming out. Note the apparent continuity of the curves through these points



With even greater zooming out, we can already see the accumulated mesh points forming apparent continuous curves (Figure 10).

The simulation in Figures 8, 9, and 10 shows how GeoGebra could recognize three distinct and collinear points **F**, **G**, and **H** that are in the same circumference. Figure 10 gives the illusion that the line and the circumference are tangent.

CONCLUDING REMARKS

Dynamic Geometry Software programs (DGS) are powerful tools for teaching and learning and investigations in geometry. This work aimed to analyze the characteristics that reveal the potential and limitations of those software programs, particularly the GeoGebra. The main characteristic of a DGS is dynamism, which allows the user to move objects in a figure without changing previously established geometric properties. This characteristic, carried out by dragging objects (by using the 'Move' tool, in GeoGebra), allows us to discover and observe invariants in the figure. Besides, it enables, for investigation purposes, quantitative analysis on the variation of measures of areas, perimeters, lengths, and angles. These measures depend on the accuracy of the software, and this is where its limitations become evident.

The limitations of a DGS have been analyzed here through three situations that show meticulous examples. The reasons that generate the limitations are a consequence of a

fact (the existence of pairs of incommensurable segments), proven by the geometers of ancient Greece, that implies the impossibility of carrying out an accurate measurement. Also, computers are physically limited, as they display a discontinuous plane, formed by discrete points. Why does this matter, if it is possible to have reasonable precision in measurements? This is not the right question. Such limitations generate inconsistencies with the theory, and these inconsistencies are revealed here in the examples elaborated. The first example presented a situation of tangency of circumference and line, in which the software reveals three distinct points as belonging to both the line and the circumference. The second example shows lines that pass through a point on the y -axis, and through a point on the x -axis, both distinct from the origin. However, such lines appear in the “Algebra” window as parallel to the y -axis. Besides, there can also be some impossibilities generated by the limited accuracy. For example, the analysis of the geometric loci that the software generated can become an unfeasible task, as was shown in the example of the third situation in this work.

This work also discussed rounding errors resulting from the software’s precision limitations that can lead to erroneous results in solving a system of equations. Still, following this line of discussion on measurement accuracy, a simulation of the discrete “plane” was presented, understood by the software to justify the inconsistency about tangency that arose in the first example

We must emphasize that these limitations do not invalidate the software. On the contrary, we must know all the potentials and limitations of a tool to work with it. Teachers must be aware of these potentialities and limitations to explore the tool’s semiotic potential that will emerge from the experiences resulting from the tasks assigned to students. The inconsistencies that may arise can cause students to be confused and create obstacles (as they will no longer believe in theory) and prevent teachers from using the software safely in their classes.

AUTHORS’ CONTRIBUTION STATEMENT

JLRP wrote this work as part of an ongoing thesis under the supervision of MTM. Both authors discussed the ideas presented in this work and reviewed the final version.

DATA AVAILABILITY STATEMENT

The data and figures presented in this work will be available by the corresponding author MTM, upon reasonable request.

REFERENCES

- Barroso, L. C. et al. (1987). *Cálculo numérico (com aplicações)* (2ª ed.). São Paulo: Harbra.
- Bussi, M. G. B., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a vygotskian perspective. In: M. A. Clements et al (Eds.), *Handbook of international research in mathematics education* (Chap. 28, pp. 746-783). Routledge.
- Duval, R. (2012). Abordagem cognitiva de problemas de geometria em termos de congruência (M.T. Moretti, Trad.). *Revemat*, 7(1), 118-138.
- Fritz, K. V. (1944). The discovery of incommensurability by Hippasus of Metapontum. *Annals of Mathematics*, 46(2), 242-264.
- Gontijo, C. H. (2007). *Relações entre Criatividade, Criatividade em Matemática e Motivação em Matemática de Alunos do Ensino Médio* (Tese de Doutorado em Psicologia). Universidade de Brasília, Brasília, DF, Brasil.
- Gontijo, C. H., Carvalho, A. T., Fonseca, M. G., & Farias, M. P. (2019). *Criatividade em Matemática – conceitos, metodologias e avaliação*. Editora Universidade de Brasília.
- Gravina, M. A. (2015). O potencial semiótico do GeoGebra na aprendizagem da geometria: uma experiência ilustrativa [Versão Eletrônica], *Vidya*, 35(2), 237-253.
- Laborde, C. (2000). Dynamic geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics*, 44, 151-161.
- Laborde, C. (2001). Integration of technology in the design of geometry tasks with cabri-geometry. *International Journal of Computers for Mathematical Learning*, 6, 283-317.
- Leikin, R. (2015). Problem posing for and through Investigations in a Dynamic Geometry Environment. In: F. M. Singer, F. M., Ellerton & J. Cai, (Eds.) *Problem Posing: From Research to Effective Practice* (pp. 373-391). Springer.
- Mariotti, M. A. (2013). Introducing students to geometric theorems: how the teacher can exploit the semiotic potential of a DGS. *ZDM Mathematics Education*, 45, 441-452.
- Mariotti, M. A. & Baccaglioni-Frank, A. (2010). Generating Conjectures in Dynamic Geometry: The Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning*, 15(3), 225-253.
- Pasquali, K. C. (2004). Máximos e Mínimos em Geometria Euclidiana Plana (Trabalho de conclusão de curso). Universidade Federal de Santa Catarina, Florianópolis, SC, Brasil.
- Silver, E. A. (1997). Fostering Creativity through Instruction Rich in Mathematical Problem Solving and Problem Posing. *ZDM – The International Journal on Mathematics Education*, 29(3), 75-80.
- Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematical Educator*, 14(1), 19-34.
- Stormowski, V., Gravina, M. A., & Lima, J. V. (2013). Tecnologia na aula de matemática: a importância do potencial semiótico. *Novas Tecnologias na Educação*, 11(3), 1-10.
- Yefimov, N. (1964). *A brief course in analytic geometry*. Moscou: Peace Publishers.