ISSN: 2178-7727
DOI: 10.17648/acta.scientiae.6205

Specialised Knowledge Network Activated in Teacher Education to Answer to a Mathematical Why on Fraction Division

Jeferson Gomes Moriel Junior [a]

[a] Instituto Federal de Educação, Ciência e Tecnologia de Mato Grosso (IFMT), Cuiabá, MT, Brazil.

Received for publication 18 Oct. 2020. Accepted after review 25 Mar. 2021
Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Background: One way to deal with the complexity inherent in teaching is to advance in understanding the relationships between the different types of knowledge used in teaching practice. **Objective**: To characterise specialised teaching knowledge and its connections established in a formative context of elaborating an answer to students on why to invert-and-multiply to divide fractions. **Design**: Qualitative analytical-descriptive study, associating the mathematics teachers' specialised knowledge (MTSK) and a teacher education course anchored in a teaching practice situation. **Setting and participants**: A teacher education workshop for in-service and preservice mathematics teachers, from which we carried out an in-depth analysis of the data collected from two participants (a preservice teacher and a teacher). **Data collection and analysis**: We obtained data during the workshop and an interview with the subjects and used content analysis, the MTSK analysis instrument, the MTSK cluster and the MTSK network diagram enabled to describe the connections between types of knowledge. **Results**: The activated network contains knowledge from all MTSK subdomains, exercising different levels of importance and roles (from the action itself to its support). The route of construction of the answer started from a mathematical layer towards didactic elements, reflecting the level of development (including gaps) of the subjects on justifications of algorithms and didactic aspects. **Conclusion**: A seemingly simple teaching action (answering a student question) mobilised all the MTSK subdomains and activated an intricate procurement network, requiring specialised, reasoned, and intentional teaching preparation.

Keywords: MTSK; specialised knowledge network; mathematics education; why mathematics; division of fractions.

Corresponding author: Jeferson Gomes Moriel Junior. Email: jeferson.moriel@cba.ifmt.edu.br

Rede de Conhecimentos Especializados Ativados em Formação Docente para Responder a um Porquê Matemático sobre Divisão de Frações

RESUMO

Contexto: Uma das formas de tentar dar conta da complexidade inerente à docência é avançar na compreensão das relações entre os diversos tipos conhecimentos utilizados em situações de prática docente. **Objetivo**: Caracterizar conhecimentos docentes especializados e suas conexões estabelecidas num contexto formativo de elaboração de uma resposta para estudantes sobre o porquê inverter-e-multiplicar para dividir frações. **Design**: Estudo qualitativo analítico-descritivo, associando o *mathematics teachers' specialised knowledge* (MTSK) e uma formação ancorada em situação de prática docente. **Ambiente e Participantes**: Uma oficina formativa para professores de ensino básico e licenciandos em matemática, na qual aprofundamos uma análise dos dados coletados de dois participantes (um licenciando e uma professora). **Coleta e análise de dados**: Obtivemos dados durante a oficina e a entrevista com os sujeitos e utilizamos na análise de conteúdo, o instrumento de análise iMTSK, o MTSK Cluster e a Rede de MTSK ativado para descrever as conexões entre conhecimentos. **Resultados**: A rede ativada contém conhecimentos de todos os subdomínios MTSK, exercendo diferentes níveis de protagonismo e papéis (desde a ação em si até seu apoio/sustentação). A rota de construção da resposta partiu de uma camada matemática em direção a elementos didáticos, refletindo o nível de desenvolvimento (incluindo lacunas) dos sujeitos sobre justificativas de algoritmos e aspectos didáticos. **Conclusão**: Uma ação docente aparentemente simples (responder uma dúvida discente na escola) mobilizou todos os subdomínios MTSK e ativou uma intrincada rede de conexões, indicando a necessidade de preparação docente ser especializada, fundamentada e intencional.

Palavras-chave: MTSK; rede de conhecimentos especializados; formação docente; porquês matemáticos; divisão de frações.

INTRODUCTION

This article is part of a broad study that investigates the set of teachers' specialised knowledge that is needed to teach and make learn some mathematics contents, i.e., how to build it in initial or continuing teacher education based on scientific advances. In the excerpt presented here, we highlight the division of fractions content, considering that among operations with fractions, the division has been considered the most mechanical task and the one the teachers and students understand the least (Lopes, 2008; Newton, 2008; Özel, 2013).

Previous studies indicate that teachers and undergraduates have performed better in proposing didactic solutions to problems arising from practice and expand their procedural and conceptual knowledge about the division of fractions when involved in group (including short-term) formative contexts (Tirosh, 2000; Green et al., 2008), based on research, including materials, problems, and representations (Sharon & Swarthout, 2014). One of those problems inherent to practice is to give answers to different students' questions (Nicol, 1998; Doerr, 2006; Leikin et al., 2017), including "the mathematical whys1," as the focus of this article is: *Why invert-and multiply to divide fractions*? In these situations, the teacher should not only know the mathematical justification itself but also know how to teach it respecting the student's level (Peterson, 1972; Lorenzato, 1993; Nobre, 1996; Moriel Junior & Wielewski, 2013; Leikin et al., 2017). However, there are indications that there are usually gaps in knowledge and teacher education (Lorenzato, 1993; Fiorentini, 2005; Santos, 2005; Angelo et al., 2009), and often the teachers have the same difficulties as the students (Bayoud, 2011; Özel, 2013; Slattery & Fitzmaurice, 2014).

Literature mappings indicate that teaching knowledge related to the division of fractions has been characterised from various theoretical frameworks and in various situations, such as during class planning or undergraduate courses (Petit et al., 2010; Fávero & Pina Neves, 2012; Moriel Junior et al., 2019). One way to try to cope with the complexity inherent in teaching is to advance the understanding of the connections between the various types of content and didactic knowledge (Aguilar, 2016; Moriel Junior & Moral, 2017; Zakaryan & Ribeiro, 2017). Therefore, in this article, the objective is to characterise specialised teaching knowledge and its connections established in a formative context of elaborating an answer to students on why they should invert-and-multiply to divide fractions.

To this aim, we conducted qualitative research and the theoretical framework of the *mathematics teachers' specialised knowledge* – MTSK to analyse the knowledge of a preservice teacher and a mathematics teacher in a teacher education workshop. In the next section, we present this theoretical framework.

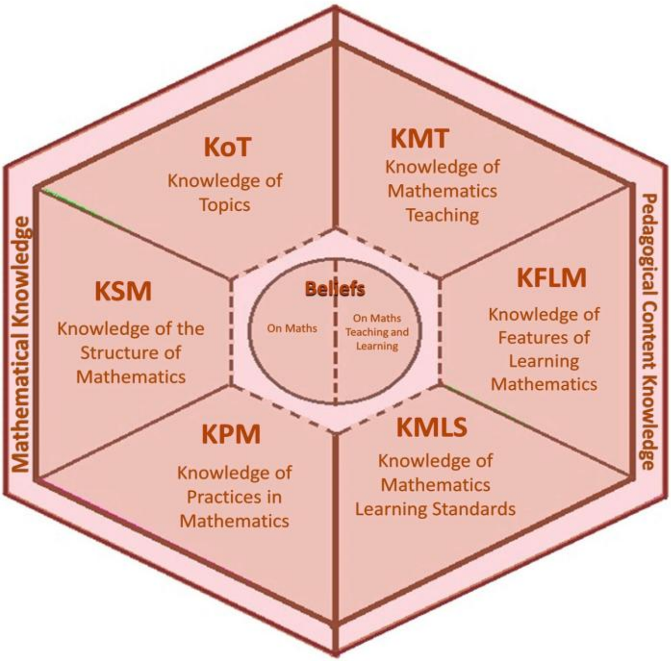
[1] Understood here as the questions (in the form of whys) and respective answers (because) that teachers need to know to justify mathematical procedures and their results (Lorenzato, 1993; Moriel Junior & Wielewski, 2013).

THEORETICAL FRAMEWORK

MTSK is a theoretical model that describes the specific and specialised professional knowledge that a teacher has (or must have) to teach mathematics (Carrillo et al., 2014; Carrillo-Yañez et al., 2018). Considering the main characterisations, typologies, and models done by researchers in the area and advancing regarding the limits detected in them (Escudero et al., 2012; Montes et al., 2013; Kilpatrick & Spangler, 2015; Scheiner et al., 2017), this model was constituted of two domains: Mathematical knowledge and Pedagogical content knowledge. Each of them is divided into three subdomains, presented in Figure 1 with the original acronyms of the English Language, proposed in Carrillo et al. (2014). At the centre of the model are the beliefs about mathematics, its teaching and learning, which permeate the subdomains.

Figure 1

MTSK domains and subdomains (Carrillo-Yañez et al., 2018, p. 6)



The first subdomain of mathematical knowledge is the knowledge of topics (KoT). It encompasses contents to teach, their deep conceptual foundation (Ma, 1999), and their different aspects, such as definitions, interpretations, and properties of concepts, one or more demonstrations of a topic, justifications of algorithmic procedures, examples and counterexamples, realistic models, application situations, and extramathematical uses. It includes knowing the different algorithms and alternative procedures to divide fractions and their justifications (Moriel Junior et al., 2019), the concepts of relationship and first degree function separately, as well as knowing that the definition of the latter is determined by the former, which is configured as an intra-conceptual connection (Vasco et al., 2017).

In the knowledge of the structure of mathematics (KSM) are the interconceptual connections between - prior and future - advanced and elementary topics of different mathematical areas, except those of foundation provided for in KoT, which allow recognising structures of mathematics, as well as seeing it as a system of integrated elements (Carrillo et al., 2018). The connections between two concepts can promote increased complexity, simplification of one with another, use of one in the solution of another, or have cross-sectional aspects in common (Vasco et al., 2017). An example of this type of connection is knowing that the concept of limit of functions can be used to justify the indetermination of division 0/0 (Lima, 1982).

Knowledge of practices in mathematics (KPM) includes the ways to define and demonstrate in mathematics. These are the processes of creating or producing in the area (syntactic knowledge), aspects of mathematical communication, reasoning and proof, elements that structure a demonstration, ways of selecting representations, arguing, generalising, and exploring patterns and regularities. The teacher can use this knowledge, for example, to deal with the solutions created by students (Cano & Flores, 2019) when they are involved in activities of seeking standards and regularities to solve problems and elaborate mathematical constructs, definitions, or proofs.

Among the subdomains of the pedagogical content knowledge, there is the knowledge of mathematics teaching (KMT). It concerns materials, resources, ways of presenting a content and its characteristics (limitations and potential existing in themselves) that allow the teacher to opt for a strategy to teach some content (including organising a series of examples or creating analogies and metaphors). For example, knowing the strategy of teaching fractions using a (circular or rectangular) geometric figure or a model (such as pizzas or chocolates) and knowing that this is (more) suitable to develop the part-whole interpretation (Moreira & Ferreira, 2008). It includes formal or personal knowledge of theoretical elements about mathematics teaching, types of instructional explanations (Charalambous et al., 2011), or problem-solving or modelling-based approaches (Krulik & Reys, 1998; Biembengut & Hein, 2007).

The knowledge of features of learning mathematics (KFLM) includes how students learn mathematical contents (formal or personal models and theories), the characteristics of this process of understanding, common errors and their likely sources, difficulties, obstacles, and the language usually used by learners when dealing with each concept. Examples are the APOS theory, Van Hiele's theory of geometric thinking, Dienes' six-stage theory of learning mathematics, students' common errors when dealing with fractions (Ashlock, 2006; Bayoud, 2011) or even students' interests and expectations (Kaur, 2008).

The knowledge of mathematics learning standards (KMLS) refers to curricular specifications involving what is foreseen at each school stage in terms of content and skills (conceptual, procedural, attitudinal and mathematical reasoning at the various educational moments), minimum standards and forms of evaluation that enable progression from one grade to another, objectives and performance scale in a country (Lacerda et al., 2020).

The construction of specialised knowledge comes both from scientific sources – such as content and didactic books, journals and scientific articles, meta-analyses, legislation, policies, curricula, among others (Courant & Robbins, 1996; Petit et al., 2010; Becker, 2019; Moriel Junior et al., 2019; Silva & Fonseca, 2019; Valente et al., 2020) -, and from professional sources from school culture, derived, for example, from experience and dialogue with other teachers.

The MTSK subdomains, as explained in this section, are efficient in describing the specialised knowledge of a mathematics teacher, even if compared to other typologies. Also, the clarity in the definition of its elements and the unambiguity between them make us use them as categories in our data analysis. Therefore, the theoretical perspective of the MTSK is also adopted as a methodological tool, as detailed below.

METHODOLOGY

The study adopts a qualitative approach, with an analytical-interpretative approach (Bogdan & Biklen, 1991) on the mobilisation of specialised teaching knowledge, with the following phases:

- Data collection during a teacher education workshop;
- Analysis of evidence and identification of evidence of knowledge;

- Data collection through individual interviews on the evidence;
- Knowledge analysis exploring the conversion (or not) of evidence into evidence.

Evidence of knowledge is the verbal, written or attitudinal elements manifested by the subject that suggests to the researcher that some knowledge may have been mobilised, but without providing sufficient and explicit information to ensure it.

The complete research path, including the exploration of evidence, gave rise to opportunities to inquire whether the subject has that knowledge or not and could provide a significant gain in the breadth, depth, and reliability of the results.

The context of this research is one of the grounded teacher education workshops (Kilpatrick et al., 2001; Barbosa, 2011; Olanoff, 2011; Moriel Junior & Wielewski, 2013) conducted by the author for in-service and preservice mathematics teachers participating in the "Observatório da Educação/Observatory of Education" Project (OBEDUC UFMT, Cuiabá funded by CAPES-INEP-SECADI) on the following practice situation: What would you answer to a student who asked you about why to invert-and-multiply to divide fractions? Among the participants of the workshops, four were selected as subjects of the broader research (Moriel Junior, 2014). This article discusses some data from that research that includes two subjects, one preservice teacher and one teacher[2]. The preservice teacher was in the final stage of the course and had two and a half years of teaching experience in schools as a substitute teacher, having worked with the content of fractions and operations with fractions at elementary and high school levels. The teacher has a degree in mathematics, with more than ten years of experience in basic education and a *latu sensu* postgraduate degree in the area.

Regarding data collection, the instruments used during the Workshop (Phase 1) were participant observation, audiovisual recording, and photos of the registers the subjects wrote on the blackboard. Subsequently, in Phase 3, we

[2] Participants of this research signed an Informed Consent Form (ICF) and the project collected data before setting up an ethics committee in the institution in compliance with CNS Resolution No. 466/2012. The author assumes and exempts the journal Acta Scientiae from any consequences arising, including full assistance and possible compensation for damages resulting from any of the research participants, in accordance with Resolution No. 510, of April 7, 2016, of the National Health Council of Brazil.

used audio recordings and photos of the subjects' manuscripts during semi-structured interviews, in which specific questions were asked to explore the evidence detected in the previous phase (Phase 2 - analysis of evidence). We adopted the steps and procedures of reflective interview (Szymanski, Almeida, & Pradini, 2011), including supplementary questions (for clarification, focused or in-depth), such as: What does this mean for you?; Here you mentioned that…; Talk more about...; Why do you think that... (Isiksal & Cakiroglu, 2011).

In the data analysis, we used the technique of content analysis (Krippendorff, 1990) of the transcripts of the workshop and the interview to obtain the units of information (excerpts from the subjects' manifestations), and we systematically compared them with the definitions of the subdomains of the MTSK model. According to MTSK categories with the MTSK - iMTSK analysis instrument (Table 1), we classified the units and analytically explored the knowledge and described the relationships between them.

Table 1

MTSK analysis instrument - iMTSK

Data	**Researcher analysis**		
Manifestation of the subject	**The subject manifested knowledge...**	**associated with...**	**consisting of...**
Excerpt from the episode (Source, line or page)	[subdomain]	[category]	[synthesis of knowledge][a]
[Example] The problem solving class ends when I systematise the concept of Fundamental Counting Principle from students' solutions on matching pants and shirts. (Teacher, 3-5)	*of mathematics teaching (KMT)*	*Teaching theories*	*one of the steps of the 'problem solving' methodology to teach the 'Fundamental Counting Principle': systematisation of the concept 'from students' solutions on [the problem of] matching pants and shirts'*

Note: a. It starts with a definite or indefinite article or a numeral (indicating the amount of knowledge), followed by the central element of the knowledge identified (which is not an action), validating it with data citation. Each excerpt may contain one or more types of knowledge from one or more subdomains and categories, indicating their connections.

In this article, we selected an episode extracted from the Workshop (on how to answer to the reason for the division of fractions) from which we sought evidence of knowledge (Moriel Junior & Carrillo, 2014). An episode corresponds to a fragment with a recognisable beginning and end, with a sequence of actions that configures it and has a complete sense in itself (Carrillo et al., 2013). The interview data were useful to understand whether and how the signs became evidence.

The type of knowledge was coded with the letter 'c,' the numbering and the subdomain MTSK to which they belong, for example: (c1, KSM) and (c6, KMT) merely illustrative.

To describe the connections between each type of knowledge, we adopted the *MTSK cluster* and the *activated MTSK network* diagram, developed exclusively for this work. The first is based on conceptual maps and the knowledge is represented in a hexagonal format, arranged side by side, with a brief description of the evidence, indicating the direction in which it was connected, as well as the level of importance of its use (on a scale ranging from action to support in action). The second is inspired by the idea of neural networks to represent the connections between the two subjects analysed considering the MTSK subdomains mobilised. Both allow an integrated view of what and how a teaching group mobilised specialised knowledge.

RESULTS AND DISCUSSION

We present the results below, starting with the analysis of evidence and signs of knowledge and then the analysis of connections between the type of knowledge mobilised together during the teacher education activity.

During the Workshop, the preservice teacher (PST) went to the blackboard to present the way he had just thought about how to justify to a 6th-grade elementary school student the reason for inverting-and-multiplying in the division of fractions, as shown in Figure 2.

Figure 2

The first part of the episode of justification of invert-and-multiply to divide fractions

PST	It would generate another question, another why, but... Let's see, a fraction [$\frac{3}{5}$] divided by another fraction [$\frac{1}{3}$] is equal to *x*. In this division [image], this number of the denominator [$\frac{3}{5}$] that is dividing will pass [to the other side] multiplying. Then, it will generate the question: why does it [pass to the other side] multiplying? [Writes as he speaks:] Three-fifths equals one-third of *x*. This one is dividing, and it is going to [pass to the other side] multiplying. So three is equal to five times one-third of *x*. So, it is going to be: 3 equals five-thirds of *x*. This one is going to be three times three. Nine equals five *x*. The five passes dividing again. And *x* equals nine over five. It is that method of what is dividing goes multiplying, and what is multiplying goes dividing. So this one [$\frac{1}{3}$ from the initial division] is dividing, right? So I passed it [to the other side] multiplying the *x*, it was $\frac{1}{3}$ of *x*. So, the 5 is dividing, so I passed it multiplying, and it became 5 times $\frac{1}{3}$ of *x*. That became $\frac{5}{3}$ of *x*. Then, I need to isolate the *x*. So the 3 comes back here, it is going to be 9 equals 5*x*. The 5 is multiplying, it goes dividing. It is going to be *x* equals 9 over 5.
Teacher	Actually, the bottom 5 ended up multiplying the top 1. And the top 3 multiplies the bottom 3.
PST	That is the process of cross-multiplication [referring to $\frac{3}{5} \div \frac{1}{3} = \frac{3 \cdot 3}{5 \cdot 1}$].
Researcher	And then, could you explain the invert-and-multiply algorithm?
PST	Yes. And it generated another question, but...

The first knowledge that stands out is that of a representation (symbolic) for the division of fractions: $\frac{\frac{3}{5}}{\frac{1}{3}}$ as division indicated between two fractions, i.e., with a greater trait between dividend and divider (c1, KoT). It was used to create an equation by which the result of the division would be obtained: ‘a fraction divided by another fraction is equal to x’ (4th sentence of the episode). Thus, the student showed to know an interconceptual connection (c2, KSM), in which one mathematical topic (equation) assists in obtaining the result of another (division of fractions). Such knowledge was the starting point of the argument that allowed the group to justify the invert-and-multiply algorithm and the mnemonic rules to divide the fractions: cross-multiplication and the sandwich - or squeeze- rule ($\left[\frac{\frac{3}{5}}{\frac{1}{3}}\right] = \frac{9}{5}$). Therefore, we have that he knows a justification of procedures to divide fractions, which is based on solving an equation (c3, KoT) using the mnemonic rule 'pass it to the other side' (c4, KoT).

Considering that such justification had just been thought by the subject, what is in the episode is the construction of a mathematical argument in which underlies the knowledge of a pre-formal numerical proof scheme (c5, KPM), although the subject cannot call it this way. It is also the instructional approach adopted by him to deal with the practice situation (on how to answer to a student who asked him why he should invert-and-multiply to divide fractions). We thus identify a knowledge of mathematics teaching (c6, KMT) that consists of an instructional explanation (Charalambous et al., 2011) based on the argument that the result of the division $\frac{3}{5} \div \frac{1}{3}$, represented by $\frac{\frac{3}{5}}{\frac{1}{3}}$, corresponds to an *x* value that satisfies $\frac{3}{5} = x \cdot \frac{1}{3}$ and implies $x = \frac{3 \cdot 3}{5 \cdot 1}$ through the application of mnemonic rules (pass multiplying or dividing). Although he ponders that such rules may generate doubt in the student (why do we pass it there?), he concludes the systematisation of cross-multiplication to divide fractions ($\frac{3}{5} \div \frac{1}{3} = \frac{3\times 3}{5\times 1} = \frac{9}{5}$).

To deepen the understanding of this instructional explanation, we explored in the interview some signs of associated knowledge, because we saw that in the first and last speech of the episode, the undergraduate expresses concern about the fact that his approach may generate doubt about a mnemonic rule: why to ‘pass [to the other side]’? This means the knowledge of a characteristic of said instructional approach (c7, KMT). He seems to foresee a didactic consequence, something that could be associated with knowledge of learning difficulties (KFLM). Therefore, we investigated this sign in the interview, asking whether he considered the 'pass to the other side’ a generator of difficulty, obstacle or barrier for the student to learn.

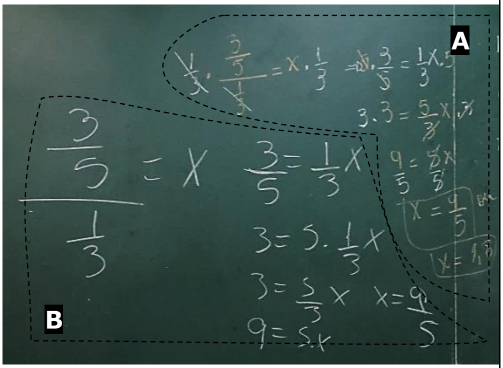
The student's answer indicated that this rule itself is not a problem, but if the student does not have a deeper understanding (the justification for the procedure), he may make the mistake of not using the reverse operation to ‘move to the other side’ of the equation (c8, KFLM). He exemplifies this by showing the error ‘$\frac{3}{2} = x + 3 \leftrightarrow 3 = x + 3 - 2$’and explaining that the student ‘thinks the inverse [operation] of the division will be the subtraction.’ For him, it is a confusion of students, whose probable source of error is a lack of adequate conceptual knowledge that should include understanding ‘what is being done’ (i.e., the justification of the rule) and knowing the reverse operations (c9, KFLM). He adds that to help the student overcome this error, a teacher ‘must reinforce with the details, which are the properties, show what is an inverse, which are inverse operations’ (c10, KMT).

Finally, the student used the episode example $\frac{\frac{3}{5}}{\frac{1}{3}} = x$ to show that ‘instead of you doing [the multiplication by $\frac{1}{3}$ on both sides], $\frac{1}{3} \cdot \frac{\frac{3}{5}}{\frac{1}{3}} = x \cdot \frac{1}{3} \leftrightarrow \frac{3}{5} = \frac{1}{3}x$you can [pass the $\frac{1}{3}$ multiplying:] $\frac{3}{5} = \frac{1}{3}x$. This slows down the process. It saves ink, it saves time. It makes it easier.’ With this, he shows that he knows an arithmetic justification of the ‘pass-to-the-other-side’ rule (c11, KoT) based on the multiplicative principle of equality (c12, KoT), that is, on a mathematical property (important as an object of teaching) that assists in simplifying and solving equations by establishing that multiplying or dividing by the same number (other than zero) the two members of an equality results in a new sentence that is still an equality, and can be synthesised as: datum $a \neq 0 \rightarrow (a \times b = a \times c \leftrightarrow b = c)$ (Medeiros, 2009) .

In the second part of the episode, the teacher starts from the argument the preservice teacher wrote on the blackboard and deepens the justification of the mnemonic ‘pass-to-the-other-side’ rule used (Figure 3).

Figure 3

The second part of the episode of justification of invert-and-multiply to divide fractions

Teacher	Can I talk about this [student's explanation - part B]? [On the blackboard, she writes part A and say:] I'm teaching equations and I don't want them to stay in this ‘pass-to-the-other-side,’ ‘pass-to-this-side,’ but let them know why. If here I divided by a third [$\frac{\frac{3}{5}}{\frac{1}{3}} = x$], then I'm going to multiply by a third [on both sides of equality]. 
PST	Ah, the scale! The scale is perfect!
Teacher	So, if I have the multiplication of a third and a third... It's the scale! So what's left? $\frac{3}{5}$ and here [$\frac{1}{3}x$]. I always say that the number comes first and then the letter. I have to arrange it again. Here I have 5 [in the denominator], if here I divide by 5, then I have to multiply by 5, also, everything I do on one side I do on the other. [...] I'm going to eliminate this here, because it is going to turn 1. So, 3 equals... here is going to be 5 (1 times 5), because he already knows that the numerator should multiply the numerator... There is still a 3 here [in the denominator]. If there's a 3 here, I'll multiply by 3 [on both sides]. It gets longer, but he knows what I am using. So it was 3 times 3, nine. This one with this one are equal, so I'm going to eliminate [$\frac{5}{\cancel{3}}.x.\cancel{3}$], 5*x*. And now, there is a 5 multiplying here, I want to eliminate it, I want the value of *x*. “Ah, teacher, then you'll have to divide." You divide it by 5, and I'll balance it on the other side by five. So, 5 by 5 is 1. I don't need to write the 1. It's going to be *x* and 9 divided by 5. I can say it is either $\frac{9}{5}$ or 1.8. Then over time they will know they have to balance it mentally.
Teacher	I am taking longer in this part for them to know, and not stay in this business of passing here, passing there etc. I went through this: why did the teacher say, ‘pass here’, ‘pass there’? And I was too

	shy to ask. In sixth grade [7th year], after I found out: it is the multiplicative principle.
Researcher	With the multiplicative principle, do you justify passing there?
Teacher	Yeah, it is the reason why it passes. If I do it on one side, I have to do it on the other.

From a mathematical point of view, the teacher performed the same justification for the reverse-and-multiply as the student (c3, KoT), but more rigorously since the mnemonic rule was replaced by the respective conceptual foundation, the multiplicative principle property (according to c12, KoT). His intention was to explain the resolution of the first-degree equation presented by the preservice teacher, but this time justifying the 'pass-to-the-other-side' rule (according to c11, KoT). Symbolic representations were mobilised for division of fractions ($\frac{\frac{3}{5}}{\frac{1}{3}}$ as division indicated between two fractions) (equal to c1, KoT) and, additionally, of a rational number (division indicated or decimal, "$\frac{9}{5}$ or 1.8") (c13, KoT). From the didactic point of view, its manifestation indicates consonance with the student's instructional approach (c6, KMT) based on mathematical argumentation to deal with the issue of practice (how to answer the student's question mentioned).

The teacher deepens the role of multiplication with fractions. At the first moment this occurs, she said that the student already knows that numerator multiplies numerator' (2nd speech), demonstrating knowledge of the procedural development level expected at that instructional moment (c14, KMLS). It also uses this operation as an auxiliary concept to solve the 1st degree equation, which, in turn, allows finding the answer to the division of fractions. This characterises a deepening of the auxiliary connection mobilised by the student (c2, KSM), expanding from two to three interconnected mathematical concepts, namely, multiplication of fractions, equation and division of fractions, the first two being auxiliaries in solving the latter.

Reflecting on the whole process, the teacher argues that 'it gets longer, but he [the student] knows what I am using.' This demonstrates awareness that using the property to solve the equation extends the number of steps compared to using the mnemonic rule. It is the knowledge of a feature of this justification: the process gets longer if we use the multiplicative principle instead of the 'pass to the other side' (c15, KoT). On the other hand, the last part of the sentence ('[...] but he [the student] knows what I am using') evidences the mobilisation of didactic knowledge, especially that related to a potential as an instructional explanation (c16, KMT): it allows to clearly express to the student *what* is being used, that is, the fundamentals involved including concepts, properties, mathematical operations, etc. This is a potential for the teacher, because she expresses at this and other moments of the argument that values the students' conceptual understanding – 'that they know why' (1st speech); 'for them to know and not stay in this business of pass here...' (3rd speech). With this, she intentionally makes the option of teaching in this way (using the multiplicative principle), instead of another (using the 'pass-to-the-other-side' rule).

Finally, from the didactic point of view, the teacher implies that the teaching of equations and their properties (such as the multiplicative principle) occurs in the 7th year of basic education, as is actually provided for in the current curriculum parameters (Brasil, 2019). At the end of the Workshop, the group is led to discuss how appropriate the constructed instructional explanation can be for 6th or 7th-grade students, and the teacher says that it depends on the 'prior knowledge and cognitive development' of the class. Nevertheless, the results are important, considering that it is a content to be dealt with throughout elementary school (Kilpatrick et al., 2001; Moreira & David, 2005; Lopes, 2008) and show that all the MTSK subdomains were mobilised by the subjects involved in the task of responding to the student's whys (Table 2).

Table 2

Summary of the knowledge that was identified in the episode

Knowledge of...	**MTSK Subdomain**
1. **a symbolic representation for division of fractions: $\frac{\frac{3}{5}}{\frac{1}{3}}$ as division indicated between two fractions, using a longer underline between dividend and divider**	KoT
2. **a connection between division of fractions and equation: the latter assists in solving the operation**	KSM
3. **a justification of the invert-and-multiply algorithm and mnemonic rules (cross-multiplication and sandwich rule $\left[\frac{\frac{3}{5}}{\frac{1}{3}}\right] = \frac{9}{5}$) to divide fractions: based on solving an equation**	KoT
4. **the mnemonic rule concerning the resolution of the 1st degree equation: 'pass to the other side'**	KoT

5.	**a pre-formal numerical proof scheme (underlying mathematical argument)**	KPM
6.	**an instructional explanation of why invert-and-multiply to divide fractions: argument that the result of the division $\frac{3}{5} \div \frac{1}{3}$, represented by $\frac{\frac{3}{5}}{\frac{1}{3}}$, corresponds to a value x that satisfies $\frac{3}{5} = x \cdot \frac{1}{3}$ and implies $x = \frac{3 \cdot 3}{5 \cdot 1}$, obtained by applying mnemonic rules (pass multiplying or dividing), thus justifying the algorithm of cross-multiplication to divide fractions ($\frac{3}{5} \div \frac{1}{3} = \frac{3}{5} \cdot \frac{3}{1}$)**	KMT
7.	**characteristic of the instructional approach (c6): may generate doubt about why 'pass to the other side'**	KMT
8.	**a common error of students when applying the 'pass to the other side': not using the appropriate inverse operation, for example $\frac{3}{2} = \mathrm{x} + 3 \leftrightarrow 3 = \mathrm{x} + 3 - 2$**	KFLM
9.	**a probable source of error c8: lack of conceptual knowledge (including justification of the rule and inverse operations)**	KFLM
10.	**an instructional approach to help the student overcome error c8: reinforce the details that are the properties and inverse operations**	KMT
11.	**an arithmetic justification of the 'pass to the other side' rule: based on the multiplicative principle of equality**	KoT
12.	**property multiplicative principle of equality**	KoT
13.	**two (symbolic) representations of a rational number: '$\frac{9}{5}$ or 1.8' (division indicated or decimal)**	KoT
14.	**a procedural development level expected for the multiplication of fractions at this instructional moment: 'he/she knows that numerator multiplies numerator'**	KMLS
15.	**a feature of c3: the process 'gets longer' if we use the multiplicative principle instead of the 'pass to the other side.'**	KoT
16.	**a potential of c6 as an instructional explanation: it allows to clearly express to the student 'what' is being used (concepts, properties, and mathematical operations)**	KMT

In summary, the results so far show that among the various possible approaches to be used to deal with the practice situation of responding to a student why invert-and-multiply to divide fractions - such as material manipulation or search for patterns from examples or realistic problems, among others (Guerra & Silva, 2008; Li, 2008; Moriel Junior & Wielewski, 2013; Moriel Junior et al., 2019) -, the one constructed by those involved was based on mathematical argumentation. This is an instructional explanation based on the premise that the result of the division $\frac{3}{5} \div \frac{1}{3}$, represented by $\frac{\frac{3}{5}}{\frac{1}{3}}$, corresponds to an x value that satisfies $\frac{3}{5} = x \cdot \frac{1}{3}$ and that after applying rules (by the student) or properties (by the teacher) implies $x = \frac{3 \cdot 3}{5 \cdot 1}$, so that one arrives at the systematisation of the invert-and-multiply algorithm ($\frac{3}{5} \div \frac{1}{3} = \frac{3 \times 3}{5 \times 1} = \frac{9}{5}$).

The explanation has the following characteristics: (i) its construction occurred from a mathematical basis (c1, c2, c3, and c4), unveiled didactic layers (c7, c8, c9, c10, c14, and c16) and then mathematics layers again (c11, c12, c13, and c15); (ii) its structure holds similarities to other justifications, such as $\boldsymbol{\frac{a}{b}} \div \boldsymbol{\frac{c}{d}} = \frac{\boldsymbol{\frac{a}{b}}}{\boldsymbol{\frac{c}{d}}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{1} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$ and as the based on operator $\frac{a}{b} \div \frac{c}{d} = K \Rightarrow K \times \frac{c}{d} = \frac{a}{b} \Rightarrow \left(K \times \frac{c}{d}\right) \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c} \Rightarrow K \times \left(\frac{c}{d} \times \frac{d}{c}\right) = \frac{a}{b} \times \frac{d}{c} \Rightarrow K \times 1 = \frac{a}{b} \times \frac{d}{c} \Rightarrow K = \frac{a}{b} \times \frac{d}{c} \Rightarrow K = \frac{ad}{bc}$ (Li, 2008; Lopes, 2008; Özel, 2013; Moriel Junior et al., 2019); (iii) uses a pre-formal numerical test scheme, considered more accessible to students and more familiar to teachers than formal justifications (Tirosh, 2000; Barbosa, 2011); (iv) requires students to have prior knowledge of the 6th year of basic education - multiplication of fractions, elementary algebraic thinking, and property of the multiplicative principle of equality -, as well as of the 7th year, associated with algebraic language, unknown, and equation (Brazil, 2019).

Given the whole scenario presented, it was possible to reach the following three conclusions.

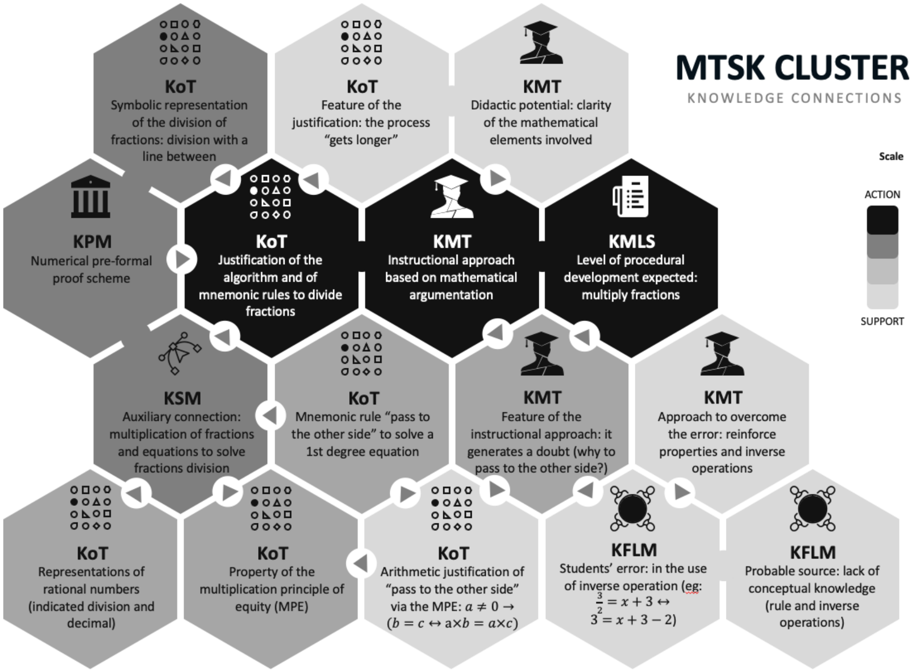
The first is that the network of specialised knowledge activated jointly by a preservice teacher and a teacher from the practice situation (to tell a student why he/she should invert-and-multiply in the division of fractions) connects elements of all MTSK subdomains, with different roles and levels of protagonism.

In the MTSK *cluster connection* map (Figure 4) we show how the knowledge elements mobilised vary on a scale ranging from greater protagonism in *action* (in black) to greater *support* to others (light grey). In it, there are layers of elements (indicated with equal colours) that interact with each other and support the other layers more causally linked to professional performance in the face of the practice situation (tell a student why the division of fractions). The layer that plays the greatest leading role in *action* has three central elements – the instructional approach based on mathematical argumentation (KMT), the justification of the invert-and-multiply algorithm (KoT), and the level of procedural development that the student must have to understand it (KMLS) - for which there are other (clearer) layers of *support*, including both didactic and mathematical subdomains.

These results are in line with the literature in the sense that the knowledge of mathematics teaching (KMT) is related to or supported by the other five subdomains MTSK, the content (KoT, KPM, and KSM) and the didactic domain (KFLM and KMLS), reinforcing the integrated and interdependent nature of the specialised knowledge (Aguilar, 2016; Moriel Junior & Moral, 2017; Zakaryan & Ribeiro, 2017).

Figure 4

Map of connections of specialised knowledge in the situation of answering to a student why he/she should invert-and-multiply to divide fractions



The second conclusion is that the route of the knowledge manifested jointly by a preservice teacher and a teacher in the construction of the answer to why the division of fractions started from a basically mathematical layer (KoT, KPM, and KSM) towards didactic elements (KMT, KMLS, and KFLM). When faced with the practice situation, the subjects first sought to answer it from a mathematical point of view. It was secondary and not systematic the reflection on how appropriate the instructional explanation could be for the interested student audience, considering that it is a content of the 6th year of elementary school (Brasil, 2019).

Therefore, our results explain a greater focus on the production of the answer than on its legitimation, unlike other cases in which teachers considered "mainly aspects inherent to their students' reading, to assess the possibility of this justification establishing or not a productive interaction in the classroom environment" (Barbosa, 2011, p. 1).

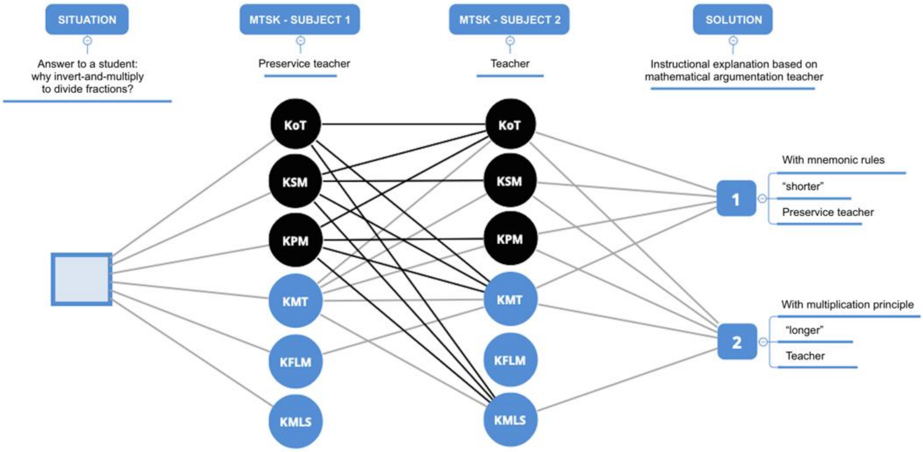
On the other hand, it is understandable that the situation was more challenging from a mathematical point of view, considering that the literature indicates gaps in teaching knowledge about the division of fractions (Bayoud, 2011; Özel, 2013; Slattery & Fitzmaurice, 2014). Moreover, historically the contents of basic education (and its justifications) have lacked a systematic approach during initial training, with articulations between theory and practice, between school and academic mathematics, productively for teaching demands (Santos, 2005; Fürkotter & Morelatti, 2007; Nacarato & Passos, 2007; Moriel Junior & Cyrino, 2009; Fiorentini et al., 2016; Crecci et al., 2017). These two situations occurred in the previous education of the research subjects, as they reported in the Workshop.

Such indications are reflected in the jointly activated MTSK network (Figure 5), whose route begins in the mobilisation of the knowledge of the mathematical dimension – representations of the division of fractions (KoT), a pre-formal numerical proof scheme (KPM), and a connection between equation, multiplication, and division of fractions (KSM) - going towards didactics (but with a scarcity of aspects of learning, KFLM and KMLS).

The network identified culminates in two instructional explanations (KMT) based on mathematical argumentation, a 'shorter' one (undergraduate used the mnemonic rule 'pass to the other side') and a 'longer' one (the teacher's, with the multiplicative principle of equality).

Figure 5

MTSK network activated jointly by subjects from the practice situation



The third conclusion is that specialised teacher education plays a fundamental role in the construction of the knowledge that allow the mathematics teacher to adopt the "best strategies" to deal with situations of teaching practice. By mapping the knowledge connections using the MTSK cluster and the activated MTSK network, we reinforce in this article the complexity and specificity of what a teacher can or should know to teach and learn mathematics (Aguilar, 2016; Moriel Junior & Moral, 2017; Zakaryan & Ribeiro, 2017). We identified that the level of development (including the gaps) of the subjects as to knowledge *of justifications of basic mathematics algorithms*, belonging to the category *procedures* of the subdomain of the topics (KoT) of the mathematical domain (MK) was reflected in the configuration of connections and the type of route taken during the construction of the solution to the practice problem: an instructional explanation in the form of mathematical argumentation.

If we consider that there are other ways to answer to the students' mathematical why (Guerra & Silva, 2008; Li, 2008; Moriel Junior & Wielewski, 2013; Moriel Junior et al., 2019), some more suitable than others for some circumstances or school stages, it is plausible to state that each of them can generate (or have as a background) other networks and routes of knowledge activated. Thus, the teaching action of building didactic solutions and deciding between one and the other does not depend on a gift, vocation, or some type of a "notorious expertise" of content, but rather on elements of specialised knowledge that allow it to intentionally establish complex and well-grounded networks. This requires the formation of (prospective) teachers with adequate treatment of the contents of basic education (mathematical domain) integrated with a didactic preparation in terms of knowledge about teaching (KMT), characteristics (KFLM), and mathematical learning parameters (KMLS) that can be used productively in the classroom. For example, covering, in the formation, formal elements about *good instructional explanations* (Charalambous et al., 2011), including justifications of procedures with productive interaction in the classroom (Barbosa, 2011), promotes an advance in specialised teacher professionalisation to answer to the students' whys (and other questions), including perspectives that lead the teacher to seek a more intelligible instructional approach based on information from the students' *background*, considering types of materials, techniques, activities, examples, and level of concreteness/abstraction more appropriate to the student's school context or stage. In this case, it seems reasonable to state that the activated MTSK network (Figure 4) and the connection map (Figure 5) would have a configuration with more emphasis on the didactic subdomains than we identified from the mathematical argumentation-based perspective.

FINAL CONSIDERATIONS

In this article, we characterise specialised knowledge with emphasis on how they were mobilised and connected by a preservice teacher and a teacher during the elaboration of an answer to the students on why to invert-and-multiply to divide fractions, during a teacher education workshop. By identifying such knowledge and mapping the connections, it was evident that the knowledge of mathematics teaching (KMT) was related to or sustained by the other five subdomains MTSK, the content (KoT, KPM, and KSM) and the didactic domain (KFLM and KMLS), reinforcing the integrated and interdependent nature of the specialised teaching knowledge. In summary, we were able to advance the understanding that:

- The network of knowledge activated from the practice situation (answer to students the mathematical reason for the division of fractions) mobilised and connected elements of all MTSK subdomains of specialised knowledge, exercising different roles and levels of protagonism, being possibly more linked to action itself or to support;

- The route of the types of knowledge manifested in the construction of the answer to why the division of fractions started from a basically mathematical layer (KoT, KPM, and KSM) towards didactic elements (KMT, KMLS, and KFLM) reflects strongly the level of development (including gaps) that the subjects had regarding the knowledge about *algorithmic justifications* (KoT), among other didactic aspects;
- Specialised teacher education plays a fundamental role in the construction of a complex, grounded, and intentional network of knowledge that allows the mathematics teacher to adopt the "best strategies" to deal with situations of teaching practice (including answering to students).

The results present several elements on the complexity and diversity of specialised teaching knowledge, on how they may be related to others (the MTSK network activated), on its possible reflections in practice and implications for teacher education. Several studies have shown the importance of discussing mathematical whys in teacher education, however, our work goes ahead by analysing the situation from the point of view of specialised knowledge focusing on the connections involved.

The action of answering a student's question, a seemingly very simple action, requires an elaborate network of professional knowledge in the field of education. We found that one can explore and develop all subdomains of specialised knowledge of mathematics teachers from the elaboration of answers focused on student understanding. Therefore, teacher education plays an extremely important role in this process, because the level of development (and gaps) of knowledge of the (prospective) teacher is the fine line that separates giving answers that foster or not a productive or an unproductive interaction with students.

It seems reasonable that this work may interest teacher educators, teachers, and undergraduates who seek tools to reflect on their practice and make it more science-based, from the MTSK categories, the elements of knowledge found here and their connections on how to answer to the mathematical why for the division of fractions. The methodological tools that were first published in this article (the *MTSK cluster* and the *activated MTSK network*) may be of interest to researchers investigating the connection between specialised knowledge.

Our results may have applications in teacher preparation by bringing elements from all MTSK subdomains useful to guide a specialised (self-

)education or (self-)reflection focusing on didactic situations of answering students' questions, including the whys. Such elements can be materialised in guide questions such as: 1. What knowledge or skills is the student expected to have already developed at that level of education (KMLS)? 2. What kind of argumentation/proof (KPM) or (counter)examples and justifications do I know (KoT)? 3. What resources, materials, metaphors, instructional explanations, strategies, and didactic approaches do I know and have available for this situation (KMT)? 4. What phenomena and other mathematical elements (such as properties, definitions, etc.) do I need to mobilise (KoT) and what connections between them are needed (KSM)? 5. What is the level of adequacy of all that I know and have available to favour the student learning process (KFLM)?

About the limitation of the work, we understand that it was possible to map the connections of knowledge exhaustively, but not exclusively, because we are clear that the configuration and routes found here are not the only ones possible, being limited to the subjects' data and their knowledge available/constructed until that moment in their lives.

The answers to why mathematics were constructed in that context are closer to a “traditional” perspective of teaching and, therefore, further studies are necessary to understand the answers from other perspectives. In this sense, the researchers are open to the questions: What knowledge and connections are needed for teachers to use manipulative materials, games or technology to teach division of fractions, their procedures, and their justifications? Would they be the same ones found in this article, but connected differently? Would student doubt arise about the *invert-and-multiply* algorithm if students were involved in a learning context based on fraction division problems, or mathematical modelling, or in the search for patterns from examples, numbers, or shapes? What can all this entail for teacher education and improvement of education in the country?

We will continue to seek answers in this sense with investigations on sets of teachers’ specialised knowledge (and its connections) needed to teach and make learn mathematics contents and also how to build it in initial or continuing teacher education based on scientific advances.

ACKNOWLEDGEMENTS.

This work was funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, FAPEMAT

(Universal Notice 42/2016) and support from IFMT (Resolution 10/2015/CONSUP). In memory of our friend Prof. Dr. José Carrillo Yañéz, leader of the group that created the MTSK theoretical framework.

AUTHORSHIP CONTRIBUTION STATEMENT

JGMJ conceived this work, collected the data, developed the methodology for analysis, analysed the data, and wrote this article.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, JGMJ, upon reasonable request.

REFERENCES

Aguilar, A. (2016). *El conocimiento especializado de una maestra sobre la clasificación de las figuras planas. Un estudio de caso* (220 f.). Tese de doutorado, Educação Matemática, Universidade de Huelva, Huelva.

Angelo, C. L., dos Santos, J. R. V., & Melão, W. S. (2009). Licenciandos em Matemática e Situações da Matemática Escolar: um Estudo Exploratório sobre a Formação Inicial de Professores de Matemática. *Revista Brasileira de Ensino de Ciência e Tecnologia*, *2*(3), 41-59. http://revistas.utfpr.edu.br/pg/index.php/rbect/article/view/552

Ashlock, R. B. (2006). *Error patterns in computation: Using error patterns to improve instruction*. Pearson Education.

Barbosa, E. P. (2011). Os Por Quês Matemáticos dos Alunos na Formação dos Professores. In: *Anais da XIII Conferência Interamericana de Educação Matemática*, Recife (pp. 1-12).

Bayoud, J. M. (2011). *A comparison of preservice and experienced elementary teachers' pedagogical content knowledge (PCK) of common error patterns in fractions* (180 f.). Tese de Doutorado, American University of Beirut, Beirut. https://scholarworks.aub.edu.lb/bitstream/handle/10938/8658/t-5470.pdf?sequence=1

Becker, F. (2019). Construção do Conhecimento Matemático: natureza, transmissão e gênese. *Bolema: Boletim de Educação Matemática, 33*(65), 963-987. https://doi.org/10.1590/1980-4415v33n65a01

Biembengut, M. S., & Hein, N. (2007). *A resolução de problemas na matemáticas escolar* (4 ed.). Contexto.

Bogdan, R., & Biklen, S. (1991). *Investigação qualitativa em educação: uma introdução à teoria e aos métodos*. Porto Editora.

Brasil. (2019). *Base Nacional Comum Curricular*. MEC. 600 p. http://basenacionalcomum.mec.gov.br/images/BNCC_EI_EF_110518_versaofinal_site.pdf

Cano, M. C., & Flores, E. (2019). Prácticas matemáticas: un avance en su caracterización. In: *Actas del IV Congreso Iberoamericano sobre Conocimiento Especializado del Profesor de Matemáticas*, Huelva (pp. 87-94).

Carrillo, J., Climent, N., Contreras, L. C., Montes, M. Á., Escudero, D., & Medrano, E. F. (2014). *Un marco teórico para el Conocimiento especializado del Profesor de Matemáticas*. Universidad de Huelva Publicaciones. https://doi.org/10.13140/2.1.3107.4246

Carrillo, J., Rojas, N., & Flores, P. (2013). Caracterización del conocimiento matemático para la enseñanza de los números racionales. *Avances de Investigación en Educación Matemática, 1*(4), 47-64. http://www.aiem.es/index.php/aiem/article/viewFile/74/28

Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, Á., Ribeiro, M., & Muñoz-Catalán, M. C. (2018). The mathematics teachers' specialised knowledge (MTSK) model. *Research in Mathematics Education, 20*(3), 236-253. https://doi.org/10.1080/14794802.2018.1479981

Carvalho, A. M. P. d. (1996). O uso do vídeo na tomada de dados: pesquisando o desenvolvimento do ensino em sala de aula. *Pro-Posições*, 7(19), 5-13. http://www.proposicoes.fe.unicamp.br/~proposicoes/textos/19_artigo_carvalhoamp.pdf

Charalambous, C. Y., Hill, H. C., & Ball, D. L. (2011). Prospective teachers' learning to provide instructional explanations: how does it look and what might it take? *Journal of Mathematics Teacher Education, 14*, 441-463.

Courant, R. & Robbins, H. (1996). *What is mathematics?: an elementery approach to ideias and methods* (2 ed.). Oxford University Press.

Crecci, V. M., Nacarato, A. M., & Fiorentini, D. (2017, 04/30). Estudos do estado da arte da pesquisa sobre o professor que ensina matemática. *Zetetiké*, *25*(1), 1-6. https://doi.org/10.20396/zet.v25i1.8649175

Doerr, H. M. (2006, 2006/06/01). Teachers' ways of listening and responding to students' emerging mathematical models. *ZDM*, *38*(3), 255-268. https://doi.org/10.1007/BF02652809

Escudero, D. I., Flores, E., & Carrillo, J. (2012). El Conocimiento Especializado del Profesor de Matemáticas. In: *XV Escuela de Invierno en Matemática Educativa*, México (pp. 35-42).

Fávero, M. H. & Pina Neves, R. d. S. (2012). A divisão e os racionais: revisão bibliográfica e análise. *Zetetiké*, *20*(1), 39-78. http://www.fae.unicamp.br/zetetike/viewarticle.php?id=553

Fiorentini, D. (2005). A formação Matemática e didático-pedagógica nas disciplinas da Licenciatura em Matemática. *Revista de educação*(18), 107-115.

Fiorentini, D., Passos, C. L. B., & Lima, R. C. R. (2016). *Mapeamento da pesquisa acadêmica brasileira sobre o professor que ensina Matemática: Período 2001 a 2012*. FE-Unicamp. https://www.fe.unicamp.br/pf-fe/pagina_basica/58/e-book-mapeamento-pesquisa-pem.pdf

Fürkotter, M. & Morelatti, M. R. M. (2007). A articulação entre teoria e prática na formação inicial de professores de matemática. *Educação Matemática Pesquisa*, *9*(2), 319-334. http://revistas.pucsp.br/index.php/emp/article/viewArticle/906

Green, M., Piel, J. A., & Flowers, C. (2008). Reversing Education Majors' Arithmetic Misconceptions with Short-Term Instruction Using Manipulatives. *Journal of Educational Research, 101*(4), 234-242. https://doi.org/http://dx.doi.org/10.3200/JOER.101.4.234-242

Guerra, R. B. & Silva, F. H. S. d. (2008). As Operações com Frações e o Princípio da Contagem. *Bolema: Boletim de Educação Matemática*, *21*(31), 41-54. http://www.redalyc.org/pdf/2912/291221883004.pdf

Kaur, B. (2008/12/01). Teaching and learning of mathematics: what really matters to teachers and students? *ZDM*, *40*(6), 951-962. https://doi.org/10.1007/s11858-008-0128-6

Kilpatrick, J., & Spangler, D. A. (2015). Educating Future Mathematics Education Professors. In: L. D. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (3. ed., pp. 297-310). Routledge. https://doi.org/10.4324/9780203448946

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press. http://www.sjsd.k12.mo.us/cms/lib3/MO01001773/Centricity/Domain/872/Adding%20it%20Up.pdf

Krippendorff, K. (1990). *Metodología del análisis de contenido*. Paidós Ibérica.

Krulik, S. & Reys, R. E. (1998). *A resolução de problemas na matemáticas escolar* (1 ed.). Atual.

Lacerda, H. D. G., Cabanha, D. S. C., & Maltempi, M. V. (2020). *Formação inicial de professores de matemática em diversos países*. Livraria da Física.

Leikin, R., Koichu, B., Berman, A., & Dinur, S. (2017, 2017/03/01). How are questions that students ask in high level mathematics classes linked to general giftedness? *ZDM*, *49*(1), 65-80. https://doi.org/10.1007/s11858-016-0815-7

Li, Y. (2008). What Do Students Need to Learn about Division of Fractions? *Mathematics teaching in the middle school*, *13*(9), 546-552. https://doi.org/http://www.jstor.org/stable/41182613

Lima, E. L. (1982). Conceitos e controvérsias: alguns porquês. *Revista do Professor de Matemática*, *1*(1), 4-7.

Lopes, A. J. (2008). O que nossos alunos podem estar deixando de aprender sobre frações, quando tentamos lhes ensinar frações. *Bolema: Boletim de Educação Matemática*, *21*(31), 1-22. https://doi.org/https://www.redalyc.org/pdf/2912/291221883002.pdf

Lorenzato, S. (1993). Os “Por quês” matemáticos dos alunos e as respostas dos professores. *Pro-Posições*, *4*(1), 73-77.

Ma, L. (1999). *Knowing and teaching elementary mathematics*: Teachers' understanding of fundamental mathematics in China and the United States: Lawrence Erlbaum Associates Mahwah, NJ.

Medeiros, V. Z. (2009). *Pré-cálculo* (2 ed ed.). Cengage Learning.

Montes, M. A., Contreras, L. C., & Carrillo, J. (2013). Conocimiento del profesor de matemáticas: Enfoques del MKT y del MTSK. In: A. Berciano, G. Gutiérrez, A. Estepa, & N. Climent (Eds.), *Investigación en Educación Matemática XVII* (pp. 403-410). SEIEM.

Moreira, P. C. & David, M. M. M. S. (2005). O conhecimento matemático do professor: formação e prática docente na escola básica. *Revista Brasileira de Educação, 11*(28), 50-62. http://www.scielo.br/pdf/rbedu/n28/a05n28.pdf

Moreira, P. C. & Ferreira, M. C. C. (2008). A Teoria dos Subconstrutos e o Número Racional como Operador: das estruturas algébricas às cognitivas. *Bolema: Boletim de Educação Matemática, 21*(31), 103-127. http://www.redalyc.org/pdf/2912/291221883007.pdf

Moriel Junior, J. G. (2014). *Conhecimento especializado para ensinar divisão de frações* (162 f.). Tese de doutorado, Educação Matemática, Universidade Federal de Mato Grosso, Cuiabá. http://www.ufmt.br/ufmt/unidade/userfiles/publicacoes/79e3fe1d66c40ff5d174dc92c84fc777.pdf

Moriel Junior, J. G. & Carrillo, J. (2014). Explorando indícios de conhecimento especializado para ensinar matemática com o modelo MTSK. In: *Seminário de Investigación en Educación Matemática XVIII*, Salamanca, Espanha (pp. 1-10).

Moriel Junior, J. G. & Cyrino, M. C. C. T. (2009). Propostas de articulação entre teoria e prática em cursos de licenciatura em matemática. *Educação Matemática Pesquisa*, *11*(3), 535-557. http://revistas.pucsp.br/index.php/emp/article/viewArticle/2831

Moriel Junior, J. G. & Moral, G. C. Y. (2017). Conhecimentos especializados para ensinar adição de frações e como se relacionam: um caso sobre erros comuns de estudantes, suas fontes e modos de superá-los. In: *Anais do Congresso Internacional de Ensino de Matemática*, Canoas (pp. 1-12).

Moriel Junior, J. G., & Wielewski, G. D. (2013). Seis possibilidades de resposta para o por quê matemático sobre divisão de frações. In: *Seminário Educação - SEMIEDU*, Cuiabá (pp. 1 - 15).

Moriel Junior, J. G., & Wielewski, G. D. (2017). Base de conhecimento de professores de matemática: do genérico ao especializado. *Revista de Ensino, Educação e Ciências Humanas*, *18*(2), 126-133. https://doi.org/10.17921/2447-8733.2017v18n2p126-133

Moriel Junior, J. G., Wielewski, G. D., & Carrillo, J. (2019). Meta-análise sobre conhecimento para ensinar divisão de frações. *Bolema: Boletim de Educação Matemática*, *33*(65), 988-1026. http://dx.doi.org/10.1590/1980-4415v33n65a02

Nacarato, A. M., & Passos, C. L. B. (2007). As licenciaturas em matemática no estado de São Paulo. *Horizontes*, *25*(2), 169-179. http://www.saofrancisco.edu.br/edusf/publicacoes/RevistaHorizontes/uploadAddress/Horizontes_25_n2%5B8498%5D.pdf#page=39

Newton, K. J. (2008). An Extensive Analysis of Preservice Elementary Teachers' Knowledge of Fractions. *American Educational Research Journal*, 45(4), 1080-1110. https://doi.org/10.3102/0002831208320851

Nicol, C. (1998, 1998/10/01). Learning to Teach Mathematics: Questioning, Listening, and Responding. *Educational Studies in Mathematics*, *37*(1), 45-66. https://doi.org/10.1023/A:1003451423219

Nobre, S. (1996). Alguns "porquês" na História da Matemática e suas contribuições para a Educação Matemática. *Caderno CEDES - História e Educação Matemática*, *40(1)*, 29-35.

Olanoff, D. E. (2011). *Mathematical Knowledge for Teaching Teachers: The Case of Multiplication and Division of Fractions* (236 f.). Tese de doutorado, Syracuse University, New York. http://surface.syr.edu/mat_etd/64/

Özel, S. (2013). An Analysis of In-service Teachers' Pedagogical Content Knowledge of Division of Fractions [Article]. *Anthropologist*, *16*(1-2), 1-5. https://doi.org/10.1080/09720073.2013.11891330

Peterson, J. C. (1972). Fourteen different strategies for multiplication of integers or why (- 1)(- 1)=+ 1. *The Arithmetic Teacher, 19*(5), 396-403.

Petit, M. M., Laird, R. E., & Marsden, E. (2010). *A focus on fractions: bringing research to the classroom*. Routledge.

Santos, R. C. (2005). *Conteúdos matemáticos da educação básica e sua abordagem em cursos de licenciatura em matemática* (200 f.). Tese de doutorado, Pontifícia Universidade Católica de São Paulo - PUC, São Paulo.

Scheiner, T., Montes, M. A., Godino, J. D., Carrillo, J., & Pino-Fan, L. R. (2019). What Makes Mathematics Teacher Knowledge Specialized? Offering Alternative Views. *International Journal of Science and Mathematics Education*, 17(1), 153–172. http://doi.org/10.1007/s10763-017-9859-6

Sharon, V. V., & Swarthout, M. B. (2014). Evaluating instruction for developing conceptual understanding of fraction division. In G. T. Matney & S. M. Che (Eds.), *Proceedings of the 41th Annual Meeting of the Research Council on Mathematics Learning* (pp. 97-104). RCML. http://web.unlv.edu/RCML/RCML%20Proceedings(2014)%20Final.pdf#page=104

Silva, L. P., & Fonseca, L. S. (2019). Analysis of textbooks of Mathematics in relation to the Disposition of Tasks in the light of Attention as a Neurocognitive Mechanism. *Acta Scientiae*, *21*(4), 160-173. https://doi.org/10.17648/acta.scientiae.v21iss4id3887

Slattery, J., & Fitzmaurice, O. (2014). ‘Ours is not to reason why, just invert and multiply’: an insight into Irish prospective secondary teachers' conceptual understanding of the division of fractions. *Irish Educational Studies, 33*(4), 467-488. https://doi.org/10.1080/03323315.2014.982346

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*(1), 5-25. https://eric.ed.gov/?id=EJ602442

Valente, W. R., Almeida, A. F. d., & Silva, M. C. (2020). Knowledge in (Trans)Training and the Role of Experts: Curricula, Mathematics

Education and Teacher Training, 1920-2020. *Acta Scientiae, 22*(5), 65-83. https://doi.org/10.17648/acta.scientiae.6004

Vasco, D., Moriel Junior, J. G., & Contreras, L. C. (2017). Subdominios KoT y KSM del Mathematics Teachers' Specialised Knowledge (MTSK): definición, categorías y ejemplos. In: *Actas de las III Jornadas de Investigación en Didáctica de las Matemáticas*, Huelva (pp. 29-37).

Zakaryan, D., & Ribeiro, M. (2017). Conocimiento de la enseñanza de números racionales: una ejemplificación de relaciones. *Zetetiké, 24*(3), 301-321.