ACTA SCIENTIAE
ISSN: 2178-7727
DOI: 10.17648/acta.scientiae.6311

Approach to the Early Childhood Education Teacher's Specialised Knowledge about Length and its Measurement in a Collaborative Context of Professional Development

Mónica Ramírez-García[a,b]
Juan M. Belmonte[b]
Noemí Pizarro[c]
Nuria Joglar-Prieto[b]

[a] Centro Superior de Estudios Universitarios La Salle, Área de Educación, Madrid, España
[b] Universidad Complutense de Madrid, Departamento de Didáctica de las Ciencias Experimentales, Sociales y Matemáticas, Madrid, España
[c] Universidad Metropolitana de Ciencias de la Educación, Departamento de Matemática, Santiago de Chile, Chile.

Received for publication on 2 Dec. 2020. Accepted after review on 26 Jul. 2021.
Designated editor: M. Mar Liñán García

ABSTRACT

Background: To teach mathematics in early childhood education, we consider that solid and rigorous knowledge is needed. Researchers in mathematics education call for the characterisation of this specialised knowledge. **Objectives**: To advance in characterising the knowledge that the early childhood education teacher needs to design and implement activities on length and its measurement. **Design:** Considering the *mathematics teachers' specialised knowledge* analytical model and starting from a review of the literature on length measurement teaching in early childhood education, we approach using a qualitative and interpretative methodology, the specialised knowledge of the early childhood education teacher from two angles: the knowledge that educators-researchers consider that the teacher needs to mobilise to design the activity, and the knowledge mobilised by teachers when they reflect on that activity. **Settings and participants**: The research took place in 2019 in a context of collaborative professional development with four educators and two teachers. **Data collection and analysis**: The collaborative sessions were recorded on video. All analyses were triangulated by the authors of the study. **Results**: Teachers' educators give importance to mathematical knowledge of the contents and teaching-learning theories. Teachers' reflections always start from the students' characteristics and actions, including the emotional aspects. **Conclusions**: Working together enriches both groups and allows to characterise the knowledge by interweaving theoretical and formal

Corresponding author: Nuria Joglar Prieto. Email: njoglar@ucm.es

elements with intuitive and empirical elements, which improves the initial education of the early childhood teacher.

Keywords: Professional development; Mathematics education; Early childhood education; Length measurement; Construction of magnitudes.

Aproximación al conocimiento especializado del maestro de educación infantil sobre la longitud y su medida en un contexto colaborativo de desarrollo profesional

RESUMEN

Antecedentes: para enseñar matemáticas en Educación Infantil consideramos que se necesita un conocimiento sólido y riguroso. Investigadores en Educación Matemática hacen un llamamiento a la caracterización de este conocimiento especializado. **Objetivos**: avanzar en la caracterización del conocimiento que el maestro de Infantil necesita para diseñar e implementar actividades sobre longitud y su medida. **Diseño:** considerando el modelo analítico *mathematics teachers' specialised knowledge* y partiendo de una revisión de la literatura sobre enseñanza de medida longitud en Educación Infantil, nos aproximamos, utilizando una metodología cualitativa e interpretativa, al conocimiento especializado del maestro de Educación Infantil desde dos ángulos: el conocimiento que formadores-investigadores consideran que el maestro necesita movilizar para diseñar la actividad, y el conocimiento movilizado por maestras cuando reflexionan sobre esa actividad. **Contexto y participantes**: la investigación tuvo lugar en 2019 en un contexto de desarrollo profesional colaborativo en el que participaron 4 formadores y 2 maestras. **Recolección de datos y análisis**: Las sesiones colaborativas fueron grabadas en vídeo. Todos los análisis fueron triangulados por los autores del trabajo. **Resultados**: los formadores dan importancia al conocimiento matemático de los contenidos y a teorías de enseñanza-aprendizaje. Las reflexiones de las maestras siempre parten de las características y acciones de los estudiantes, incluyendo aspectos emocionales. **Conclusiones**: el trabajo conjunto enriquece a ambos grupos y permite una caracterización del conocimiento entretejiendo lo teórico y formal con lo intuitivo y empírico, lo cual redundará en la formación inicial del profesorado de infantil.

Palabras clave: Desarrollo profesional; Educación Matemática; Educación Infantil; Medida de longitud, Construcción de magnitudes.

INTRODUCTION

Solid and rigorous knowledge of the bases of the discipline is needed for mathematics teaching in early childhood education to facilitate reflective learning in a playful and functional environment (Ma, 1999). Only teachers with in-depth knowledge of the mathematics they teach will be able to successfully face this challenge (Ball & Bass, 2000). However, during their initial education, teachers are not usually required to take extensive and specific mathematics courses (Oppermann et al., 2016; Alsina, 2020; Nolla et al., 2021). From the research point of view, attention to early childhood education is relatively recent and scarce, which is evidenced in the review carried out by Parks and Wager (2015) of articles published in the previous two decades in the four most relevant journals on this education stage: *Early Childhood Research Quarterly, The Journal of Early Childhood Teacher Education, the Journal of Research in Mathematics Education and The Journal of Mathematics Teacher Education*. The authors mention that these journals pay little attention to research on the early childhood teachers' mathematics education and that the two specific mathematics journals devote minimum attention to early childhood education.

Shulman (1986) raised the need to consider the specificity of the content being taught. To this end, he introduced the distinction between content knowledge and pedagogical knowledge of the subject, which is different from general pedagogical knowledge. In mathematics, Ball et al. (2008) built the MKT (*Mathematical Knowledge for Teaching*) model, which characterises the specialised knowledge a teacher needs to teach it. This does not mean that aspects of the teacher's general pedagogical knowledge do not play a relevant role in their practice, linked, moreover, in very specific ways, with those specialised in the early childhood education stage.

In this proposal, we assume the conceptualisation of teacher's knowledge present in the *Mathematics Teachers' Specialised Knowledge* (MTSK) analytical model (Carrillo et al., 2018), since this has proven to be suited to several recent studies with the aim of understanding and analysing the specialised knowledge of the early childhood education teacher on different mathematical topics, both from a theoretical approach and from the analysis of their practice (Muñoz-Catalán et al., 2017; Muñoz-Catalán et al., 2019; Muñoz-Catalán et al., in press b).

Usually, the early childhood education teacher dispenses with the symbolic and formal apparatus inherent in the mathematical contents of later stages. Specialised knowledge of the subject (*Mathematical Knowledge*, MK) must include elements of the mathematical essence of the contents they address, their fundamental features, and their role in the construction of the mathematical concept itself. However, in some cases, existing categories within the MTSK model seem to be limited according to their current definition to identify and express specific aspects of the early childhood teacher knowledge. This limitation seems to be more recurrent in the subdomains of the MK than in those of the *Pedagogical Content Knowledge* (PCK) (Muñoz-Catalán et al., in press b), so, in this chapter, we decided to start from our reflection about the MK.

It seems acceptable to consider that the mathematical content a teacher knows and how he/she knows it is highly conditioned, on the one hand, by the educational stage in which he/she works and how he/she approaches it in class, and, on the other, by how he/she has accessed that knowledge both in his/her initial professional education and through professional development activities. In other words, professional experience and education are the two fundamental ways through which he/she builds his/her professional knowledge, in particular, the specialised knowledge necessary to teach mathematics.

The mere observation of practice may not evidence the specific aspects of teachers' knowledge in early childhood education. For this reason, in this work, we have opted for collaboration between researchers-teacher educators and two female teachers. The researchers designed and brought to the teachers' classes an activity that they all analysed collaboratively immediately afterwards.

This work aims to advance the characterisation of the specialised knowledge that early childhood education teachers must mobilise to design and bring to the classroom activities to introduce length and its measurement. For this purpose, the researchers designed and implemented an activity to introduce the topic in two teachers' classes (a group of four-year-old students and another of five-year-old students). Thus, the study developed two complementary analyses of the mathematics teachers' specialised knowledge when teaching mathematics in early childhood classes.

On the one hand, and from a theoretical approach, the researchers identified and interpreted elements of the teachers' specialised knowledge they think necessary for the teachers to mobilise to design and bring the activity to their classes. The starting point of this approach was the literature review on the teaching of magnitude measurement, particularly the length, in the childhood stage, to look for common and differentiating elements according to the different authors.

On the other hand, and from a practical approach, the elements of specialised knowledge that the teachers put into play were identified and interpreted when, in a joint reflection session, they analysed episodes on the implementation of the activity in two groups of four and five-year-old students. The collaboration between teachers and researchers-teacher educators is a path for the professional development of both groups and, in particular, for the development of research focused on a specialised knowledge to teach that is evidenced in practice, understanding teaching practice as a continuous cycle of planning, instruction, and reflection (Baumfield & Butterworth, 2007).

BACKGROUND AND THEORETICAL FOUNDATIONS

Due to the nature of this research, the theoretical foundations are articulated around two pillars: the mathematics teachers' specialised knowledge model and the studies on length measurement teaching.

Mathematics teachers' specialised knowledge

Research on teaching practice has been increasing in recent years since the teaching process has been acknowledged as a complex, uncertain, and changing scenario, where interesting interactions occur for us to observe, relate, contrast, question, and reformulate (Gergen, 2001), aiming to understand how the teaching processes are developed and how they could be improved. These investigations have placed teachers as the most important internal factor in learning (Hargreaves & Fullan, 2014).

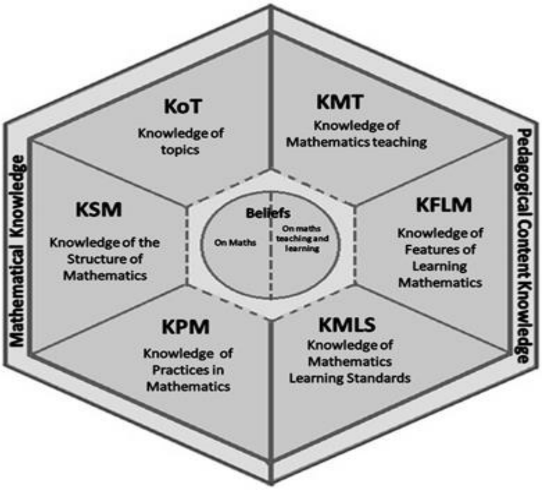
Most teacher knowledge models are based on Shulman's (1986) work. This author believes the specificity of the content being taught must be considered, focusing on the knowledge needed to teach through the vision of the discipline itself. This author introduced the distinction between the *subject matter knowledge* (SMK) and the *pedagogical content knowledge* (PCK), understanding the PCK as a different category from the *general pedagogical knowledge* (GPK). The GPK encompasses general principles of classroom organisation and management and knowledge about general teaching and learning theories. However, thanks to the PCK, the teacher can understand what facilitates or hinders the learning of a given content or how a given topic, problem, or issue can be organised, represented, and adapted to the diversity of students' interests and competencies when presenting that content in the classroom. Therefore, it is a question of thinking of the PCK as the knowledge the teacher needs *to* teach the specific subject and seeing it intertwined with the subject matter knowledge (SMK) the teacher also needs.

We agree with Ball et al. (2008) that the knowledge teachers need to teach mathematics is specialised because it requires a way of knowing mathematics that is specific to the teachers and that must be rooted in the very mathematics, since the abstract nature of the contents necessarily affects their learning and teaching. However, for us, this does not mean that some elements of this knowledge cannot be shared with other professionals (for example, with mathematicians who are not dedicated to teaching), but that the set of those elements is what is considered specialised (Scheiner et al., 2017). This, together with the existence of overlapping subdomains or the adoption of an evaluative approach to teacher knowledge, have turned out to be some of the limitations of the model of Ball et al. (2008), as shown in Carrillo et al. (2018). Therefore, in the proposal presented, we adopt the teacher's conceptualisation of the knowledge present in the *mathematics teachers' specialised knowledge (MTSK)* analytical model (Carrillo et al., 2018), understanding knowledge as the one that the teacher needs and uses, that which is available to him (Schoenfeld, 2000) and, therefore, supports their actions. This model has been helpful to understand the specialised knowledge that the early childhood education teacher mobilises to teach mathematics (Muñoz-Catalán et al., 2019; Muñoz-Catalán et al., 2017; Muñoz-Catalán et al., in press b).

The MTSK model distinguishes six subdomains of knowledge and is organised, following Shulman (1986), between the mathematical knowledge domain (*MK*, in our case) and the pedagogical content knowledge domain (*PCK*, in our case). It also includes the category of beliefs and conceptions as an element that pervades all knowledge (which will not be the subject of study in this work) (figure 1).

Figure 1

Mathematics Teachers' Specialised Knowledge (MTSK) (Carrillo et al., 2018).



Within the mathematical domain, we highlight three subdomains: knowledge of topics (KoT), knowledge of the structure of mathematics (KSM) and knowledge of practices in mathematics (KPM). In turn, there are three subdomains within the PCK: knowledge of mathematics teaching (KMT), knowledge of features of learning mathematics (KFLM) and knowledge of mathematics learning standards (KMLS). Each subdomain is organised into categories or indicators, to which we will refer in the subsequent analysis (Carrillo et al., 2018)

Thus, the MTSK is the analytical model that allows us to locate aspects of the desirable specialised knowledge to teach the concept of length and its measurement in early childhood education from two angles. On the one hand, the features of the specialised knowledge of the teachers that the researchers consider optimal, based on their theoretical sensitivities, for the design of activities that promote the need to measure. On the other hand, the knowledge mobilised by two female teachers with over 25 years of experience when analysing an introductory activity about measuring length, designed and implemented by teacher educators-researchers in their early childhood classrooms, with four and five-year-old students.

Length measurement teaching

Below the most relevant information on the second theoretical pillar of our work is included: length and measuring length in early childhood education.

Not many studies have focused on the specificity of mathematics teaching in early childhood education, and much less on magnitude and its measurement (Alsina, 2006; Belmonte, 2005; Clements, 2010). There is also no consensus on the mathematical contents the teachers might learn at this stage (Alsina, 2020; Nolla et al., 2021). In addition, the curricular proposals often present a fluctuating vocabulary that reveals the confusion between the different aspects involved in the process of measurement.

The teaching of magnitudes and their measurement is included in the curriculum of elementary education, however, concrete proposals in early childhood education have generally been weak, despite its social relevance and the need to establish sound foundations for subsequent educational stages.

It is pointless to approach the teaching of measurement without knowing what characteristics of objects and phenomena are measurable. It is therefore necessary to ensure children's contact with contexts that help them discover the magnitudes, to be able to compare objects and phenomena for their magnitude quantity, and to appropriate references of the units of measurement, among others (Buys & Veltman, 2005; Fernández & Mantecón, 2018).

The length is usually the first magnitude to be taught because it is one of the most elementary and tangible qualities of the objects. Also, it is built in parallel with the natural number because they complement each other in their understanding (Clements & Stephan, 2004). In this way, the length is treated as a discrete magnitude in the first years (although it is a continuous magnitude). Assembly activities can be used to operate.

On the construction of length and its measurement in early childhood education, various authors have given guidelines focused on their main ideas, both general and particularised to length, which we collect in the following table.

Table 1

Concepts associated with the construction of length and its measurement.

Authors	Year	Concepts associated with the construction of magnitudes and their measurement	Concepts associated with the construction of length and its measurement
Fernández & Mantecón	2018	Direct comparison, indirect comparison, conservation principles, and transitivity	Identifying, defining or recognising; relating and operating
Copley	2017	Transitivity, iteration and equitable partition, conservation, origin. Emphasis is placed on the accuracy of language	
Arteaga & Macías	2016	Perceptual comparison, displacement of the object, operability of the transitive property. Stages: Sensory estimation, direct comparison, indirect comparison, choice of unit, irregular measurement system, regular measurement system, decimal metric system.	Direct comparison, indirect comparison, choice of a unit, irregular measurement system, regular measurement system.
Lupiáñez & Castro	2016	Object and attribute selection, choice of the appropriate unit, compare the object with the unit of measurement, express the number of units and the name of the unit	

Clements & Sarama	2014		Understanding the attribute, conservation, transitivity, equipartition, the unit of measurement and the repetition of units, accumulation, origin and relationship between the number and the unit of measurement
Van den Heuvel-Panhuizen & Elia	2011		Qualitative comparison, ordering, numeric value assignment, use of instruments.
Buys & Veltman	2008	Comparison and ordering, measurement with natural and standardised units, use of measuring instruments.	Comparison and ordering, measurement with natural and standardised units, use of measuring instruments.
Belmonte	2005	Consideration and perception of a magnitude, conservation of the magnitude, ordering with respect to the magnitude, correspondence of numbers to quantities of magnitude. Perceptual comparison, displacement of the object, operability of the transitive property. Objectal unit, situational unit, figurative unit.	Preservation of distance, symmetry of distance, inequality of distance. Acquisition of own vocabulary of the length, arrangement of varied objects for the work of the length in classification, comparison, composition, ordering, etc.

According to the previous table, we observe several authors share ideas about the teaching of length and its measurement, which are detailed below, and which are our basis to characterise specific contents of the KoT subdomain:

- Conservation of magnitude: under certain transformations, one must identify which changes in the object leave invariant the characteristic property of the length.
- Direct comparison: no common measure or displacement is used. Children's work naturally uses the ordering of objects, which is intrinsic to the notion of magnitude.
- Indirect comparison: transitivity emerges here. Arguments such as: "if a = b and b = c then a = c" can be constructed. Where element b is the intermediary in the comparison. Naturally, this stage is linked to the conservation of the quantities since they move and, without their conservation, the reasoning does not make sense.
- Use of patterns: at first, this intermediary (the pattern) is larger than the objects to be compared, and then the children use a smaller pattern, convincing themselves that the precision will be greater.

Although it is not part of the process itself, for Belmonte (2005), the notion of unity is developed at the end of it, according to the following mathematical concepts:

- Correspondence of quantities of magnitude to numbers: this correspondence makes us know that one quantity of magnitude is greater than another and how greater it is.
- Objectal unit: the unit is associated with a single object, even in relation to the object to be measured. For example, in the case of length, it is customary to use some constituent part of the object to be measured as a unit.
- Figural unit: the unit loses its relationship with the objects to be measured, although they are still associated with specific figures. The unit continues to identify with some particular form.
- Unit: when the unit is completely freed from the figure, size, and object to be measured, the construction of the true notion of the unit is achieved. It is important to note that a unit is a particular quantity of magnitude but not a specific figure.

METHODOLOGY

The design and development of this research was carried out in the context of a permanent formative seminar, *ARANMATINF*, in which teachers of early childhood education and teacher educators-researchers collaborate. Generally, in the seminars, the teachers bring mathematical problems they had in their classroom, and the educators-researchers propose general ideas and resources so that, together, the group designs a specific activity to address that problem. Then, each teacher or one of the educators brings to the classroom a version of the activity adapted to their contexts and realities. They analyse the sessions together in the subsequent sessions, serving as a starting point for new problems, starting a new cycle (Baumfield & Butterworth, 2007). The sessions of the *ARANMATINF* group are recorded on video. Educators-researchers can refer back to the recordings as many times as necessary to analyse them in detail and from different perspectives -such as microanalysis or in-depth analysis- (Erickson, 2006; Roschelle, 2000).

Due to the nature of the study, we must carry out qualitative research under interpretative paradigms to understand the approach to teaching knowledge since the model suggests that the informants, i.e., the educators-researchers and the teachers, should be observed as documents that reflect their own culture (Pérez Serrano, 2014). In this study, we will observe how participants interpret the activity and what it means to them (Latorre, del Rincón, & Arnal, 2005).

All the researchers in this experience are teacher educators with no formal training or working experience as early childhood teachers. Thus, joint work was necessary, on the one hand, to complement and improve the educators' knowledge and practice, being situated in a context of teaching the measurement of length in a children's class. And, on the other hand, the teachers of this stage could complement their specialised knowledge about the MK domain, which had not been emphasised in their initial training (McIntyre, 2005; Oppermann et al., 2016).

In one of these sessions, the group concluded that they needed to design an activity whose resolution required indirect comparisons with patterns that were repeated during measurement. The fact that the internal logic of the activity established the need to set up comparisons was also emphasised, thus creating the need to measure. Also, it was important to contextualise the activity with meaning for the students at that stage. When designing the activity, the educators-researchers show the specialised knowledge they expect from teachers to design and bring the activity to the classroom. Details of the activity are described in Table 2.

Table 2

Activity designed by the educators-researchers.

Features	Description
Material	A pencil case with three organisers designed to fit some pens. The green organiser (A) has three equal parallel spaces and matching ends, the red one (B) has three equal but not parallel spaces, and the yellow one (C) has three different and not parallel spaces. Fit-in cylinders (2 cm height). Pencils (about 30) of four different lengths (lengths of 10, 9, 8, 7 cylinders).
Class organisation	The empty pencil case is on a table, along with some fit-in cylinders. The pens are mixed in a tray at the other end of the class, where they also have fit-in cylinders.
Students' objective	To get, in a single trip, three pens of the same length as the spaces in the organisers of the pencil case (and insert them).
Instructions for the students	"Here's a pencil case where three pens should go. The pencil case must travel, so we want to fill it with three pens that fit well into the gaps. The pens should be adjusted so that they fill the entire space without leaving any part outside. But there is a condition: the pencils are on that table, you cannot take the pencil case, and you must bring them on a single trip. If you want, you can use these fit-in cylinders, which are next to the case and next to the pencils."
Optimal strategy	Build a bar of the same length as the gaps with the fit-in cylinders. If it is possible to transport the bar, take it next to the pens and find those that are as long as the bar. If the transportation of the bar is not possible, count the cylinders that make up the bar, go to the table where the pens are, build

	another bar there with the same number of cylinders and look for the pens of the same length.
Verifying the strategies	The activity directly sanctions the students' procedures. They must only check whether the chosen pens fit correctly into the spaces. The lengths were chosen so that only one size would fit in there. It was very evident when the size was not right.

Below, we describe the design, the context, and the participants of the activity, and the implementation in the classroom:

Students aged four and five were given a situation. To solve it, they needed to establish an indirect comparison based on a recurring pattern, leading to the idea of a unit. This work is crucial to prepare later aspects of the learning of the notion of magnitude and its measurement, and as we will see, it must be done at the early childhood stage.

One of the aspects the teachers highlighted in the session before the activity was designed concerns the narrative of the situation. As one of the researchers is Chilean, she has a different accent, and that could attract students' attention. This feature was the core of the context: the students should prepare a gift for a Chilean child who likes to draw, so they agreed to send him some typical Spanish pencils. Students must observe the path of the pencil case on a map, understand that the pencils must be fixed in the pencil case so that they do not move during the trip and the gift gets there in optimal conditions.

Figure 2

Pencil cases, pencils, and fit-in cylinders used in the activity.

Two researchers designed a pencil case with some gaps where some pens could be inserted. The pens were far from the case, and the children had to choose exactly the ones that fitted in the gaps between the different sizes. We decided to use fit-in cylinders as a material to induce the iteration of a pattern in the indirect comparison (see Figure 2).

Regarding the implementation of the activity in the classroom, we should note that in the four-year-olds' class, only the green organiser was used (A, parallel spaces), we allowed a single trip, but students were allowed to transport the intermediate bar. It was only intended to mobilise the transitive property that underpins the indirect comparisons.

In the five-year-olds' class, the green (A) and red (B) organisers were used and, after a few initial attempts where they could transfer the intermediate bar, that was forbidden, triggering the emergence of the number as a memory of the quantity, which means an early exercise of the measurement.

The teachers returned the informed consent forms signed to participate in this research about their specialised knowledge.[1]

RESULTS AND DISCUSSION

The analysis of the results is organised from the two angles established in the methodology. Firstly, we specify the categories from the different subdomains of the MTSK that emerge when one designs the activity theoretically and, secondly, from the joint reflection with the teachers on its implementation in the classroom.

Design of the activity

The researchers devised a situation that required finding an object as long as another one given in its absence (KMT; strategies, techniques, tasks, and examples). To induce the iteration of a pattern in the indirect comparison (KoT; procedures, definitions, properties, and foundations), students would be provided with some fit-in material that could be used to construct an object as

1 The authors of the article assume any responsibility and exempt Acta Scientiae from any action that occurs thereof, including full assistance and possible compensation for any damage that results for any of the research participants, in accordance with Resolution 510 of the Council, of April 7, 2016, National Health of Brazil. Given the establishment of the collaborative workshop in the interest of the teachers and researchers themselves, the need to request a report issued by an external ethics committee was not considered.

long as the model (KMT; material and virtual resources, strategies, techniques, tasks, and examples). The transitive property would allow the choice of the pen from the information of the intermediate object (KoT; procedures, definitions, properties, and fundamentals).

The researchers identified difficulties in finding didactic material to perform an activity with the characteristics mentioned before, a pencil case with some established spaces where pens could be inserted was designed especially for this task (KoT; phenomenology, and applications). As seen above, the pencil case designed contains three possible organisers. The researchers sequenced the use of the organisers according to the characteristics of the positions of each space that involve more or less complex procedures (KMT; strategies, techniques, tasks, and examples), (KMLS; sequencing with previous and subsequent topics), (KoT; definitions, properties, and foundations).

The researchers adjusted the four lengths of the pens with a difference of one cylinder between them (KMT; teaching theories). The use of thicker cylinders than the gap to be filled aims to force the children to replace using the objectal unit with the figural one (KFLM; theories about learning, strengths, and difficulties). Both on table with the pencil case and next to the box containing all the pens, some fit-in cylinders were provided, with the idea that children can choose to use them as a pattern (KMT; strategies, techniques, tasks, and examples).

The goal of the task is to get, in a single trip, three pens as long as the spaces in the pencil case organiser (KMT; strategies, techniques, tasks, and examples). The instructions given to the children are carefully thought out so that they receive accurate information about what they need to solve, without giving an indication of how they could do it. In particular, the use of words such as *length, measurement* or *as long as* to cause the spontaneous use of the length measurement for the resolution of the task (KMT; strategies, techniques, tasks, and examples), (KoT; registers of representation) is avoided.

The optimal strategy to solve the activity requires understanding the notion of length measurement and requires using the number as a quantity memory through the discretisation of the length (KMLS; learning expectations), (KMT; strategies, techniques, tasks, and examples), (KoT; procedures, definitions, properties, and foundations). To reach the optimal strategy as a learning objective, children go through intermediate strategies along a learning path (KMLS; sequencing with previous and subsequent topics), (KFLM; learning theories), which is modified through the management of the indicated didactic variables (KMT; strategies, techniques, tasks, and examples). The designed activity allows the verification of the strategies used by the children, as it directly validates the student's procedures (KMT; strategies, techniques, tasks, and examples).

The concretion of the didactic variables (Briand & Chevalier, 1995; Brousseau, 1991) shows the specialised knowledge that researchers intend teachers to acquire, intertwining the mathematical content of the task (KoT; procedures, definitions, properties, and foundations) with didactic aspects of learning expectations (KMLS; learning expectations) and the theories of teaching, strategies, and resources (KMT; theories about teaching, strategies, techniques, tasks, and examples, material and virtual resources), which allow reaching those standards while respecting the development of children's learning about that specific content (KFLM; theories about learning, strengths, and difficulties). Some aspects of the categories indicated above are described below for each didactic variable.

- The number of different sizes of pens: we decided to use four different sizes to avoid using qualitative aspects, such as small, medium, and large, for the visual differentiation of the length quantities.
- The difference between the sizes of the pencils: the lengths of the pens differed just in the height of a cylinder. These are visually different lengths, but that difference does not enable a direct memory of the length.
- The relationship between the pencils' lengths and the cylinders' lengths: the length of each pen could be broken down into an entire number of cylinder heights (7, 8, 9 and 10 cylinders). If you want to encourage the use of the number (measure) to transfer the length, it should be a natural number (work in early childhood).
- The size of the gaps in the organisers: we decided to build two organisers with all equal spaces, so only a quantity of length had to be reproduced. The yellow organiser (C) has spaces of three different sizes, which requires three different reconstruction processes, which the difficulty of the task increases.
- The spatial arrangement of the spaces: the parallel arrangement (organiser A) makes it easier to perceive the equality or not of the lengths.

- The location of pencils and pencil case: to force indirect comparisons, the pencil case and pencils must be separated enough so that no perceptual comparisons can be established.
- The number of trips: limiting the number of trips disables successive approach strategies (trial and error) until the correct size is found.
- The displacement of the intermediary: prohibiting the displacement of the intermediate cylinder bar requires the use of the number of cylinders (patterns) as the only possible information to reconstruct an equal length.

Joint reflection on the development of the activity

We now move on to discuss the teachers' specialised knowledge that emerged during the joint reflection. The discussion session is also part of the collaborative workshops ARANMATINF in which educators-researchers and practicing teachers jointly propose, suggest, design, implement, and reflect on classroom practices with the double objective of professional training and research.

The session began with the educators-researchers presenting the activity. Then, they raised questions for reflection for the teachers to show evidence of their specialised knowledge for mathematics teaching (especially of length and its measurement). In other words, we focused on the teachers' analysis of the development of the activity in class.

It is noteworthy that all teachers' observations tended to be centred on the students. They initially interpreted students' actions from the characteristics of their learning and development, from a general pedagogical and didactic mathematical point of view (KFLM), and verbalised it: "We care about their very thoughts." Some ideas raised during the debate between educators-researchers and teachers about children's behaviours allowed us to identify typical elements of the teachers' knowledge of the KoT and the KMT. This knowledge can be described as artisanal and empirical, as it emerges from classroom experience and is limited in identifying and spelling out mathematical concepts, as shown below.

When the teachers noticed the similarities and differences between the strategies used by the children, they identified the procedures of direct comparison and use of a pattern by saying: "Well, first, they count, directly count how many there are, they know that there is a pattern that is repeated several times and by counting" (KoT; procedures). The teachers also identified in the children's answers the use of the indirect comparison, for example: "And then, the second, the one that took the pen, not the fit-in cylinders, but the pen, with another pen and another pen, because it has to be like this one, they realise that this measure will be kept even if it is not measured with the fit-in cylinders, with the nine times..." In this last comment they sensed the conservation of the length implicit in the indirect comparison (KoT; definitions, properties, and their foundations). The teachers also added: "... this strategy is more abstract, they no longer need the material, they know that with nine they are going to get... I don't know mathematics... but I like that they do it," which evidences that they sensed the transitivity property informally based on the use of the property in the procedure of comparing several amounts of length (KoT; definitions, properties, and their foundations) and that the number represented sufficient information to express the amount of magnitude (KoT; definitions, properties, and their foundations, procedures), (KFLM; learning theories), (KSM; auxiliary connections). Thus, they added that: "... I believe that this would be more evolved... not having to use the reference, he focused on that quality, on that length, those who go no longer need the reference," so it can be deduced that they distinguished indirect comparison as a more evolved strategy than direct comparison (KFLM; learning theories).

During the discussion, the researchers raised questions about possible difficulties in children's strategies, such as the isolation of a magnitude, in this case, the length in a three-dimensional object. The children tried to fit the cylinder bar into the pencil case slots, and it turned out to be wider ("fatter") than the gap. The teachers identified this limitation of the material (KMT; material and virtual resources) but emphasised that it was not a problem for them (KFLM; strengths and difficulties), they saw it as just "one more step" (KMLS; sequencing with previous and subsequent topics):

"... the children had to have the feeling that it was going to work but that it did not fit in there [...] equality... thickness, they serve to make them realise that they are not the same ... they are not the same, one is thicker than the other, but the length is the same. I think it has to be sometimes ... one more step, although they saw that they did not fit, it was a material that allowed them to measure, that was what they wanted, they wanted something that was, that fit, to take into account a..., [...] a dimension, a characteristic".

Again, in response to specific questions, the teachers mobilised an element of the KoT, focusing on the children's answers: the isolation of length

(KoT; definitions, properties, and their foundations), (KFLM; forms of interaction with a mathematical content).

An educator asked how different two strategies of comparison used by the children were, representing them with the pens: with one of the strategies, she held the end of the two pens together on the table, perpendicularly to the table, and with the other, she held them in the air. The teachers commented: "Starting from the base, the length is more likely to reach the height, which is what we are working, maybe the four-year-olds are not aware of the length of the height from a specific base... this must be measured with this and this with this... [she coincides the extremes]", which shows that they identify the need for a common origin in the length comparison procedure (KoT; procedures). The use of the fit-in cylinders is considered appropriate because it is familiar for them, but they understand that not fitting the cylinders properly would imply a poor measurement technique (KoT; procedures).

On the other hand, about the mathematical practices, the teachers realised that the children could anticipate the solution of the activity by ranking the pens they were discarding: "... an adult in that situation, would remove from the box the pencils that were not valid," which helps to solve the task. The teachers' identification of the "trial and error" strategy that some children used to solve the activity can be interpreted as a process associated with problem solving as a way to reproduce mathematics (KPM). It should be noted that the "trial and error" strategy was not considered in the design of the activity, it emerged immediately from the children's actions. Besides, the teachers highlighted as positive that the activity allowed the students to validate their actions since "they do not want to be the ones who say whether they are right or wrong."

Finally, again intuitively from their experience, we highlight the appearance of elements related to the knowledge of mathematics teaching theories when the teachers identified characteristics of the case organisers that affected the students' strategies: "... in the green organiser (A), it is clear that all three are equal, although the children verify it. [...] in the green one (A), you can check visually, but in the red one (B), you cannot check it," so they identified the characteristics of the organisers as a didactic variable (KMT; theories of teaching). In this comment, the teachers also recognised visual perception as a comparison strategy (KoT; procedures).

The teachers commented on the use of the error as a pedagogical ally: "... we do not take advantage of the error, we must take advantage of the error, use the error to move forward, help others, and then also collect..." Thus, they identified the error as something that should be exploited in the teaching-learning process (GPK, general learning theories such as constructivism).

Throughout the debate, the teachers often focused on attending to the emotional behaviours of the children: “...in the learning process, the child should be emotionally balanced, they must feel safe... not only in mathematics, in reading, in creativity... [the teachers] we care about their very thoughts.” At one point, they even remarked that the teacher educator was “cold” when a girl felt blocked: “... there are children that can be demanded to reflect and others who feel more frustrated... sometimes they just need a hug...”

To close the session, the teachers, in a final general reflection, noted that the educators-researchers centred the design of the task on the mathematical content and teaching theories, forgetting, on some occasions, the global narrative that gives meaning to the activity once the didactic objective of the task has been achieved: “When you carry out the activity you were not worried about mixing the three pencils at the end, and that is important for them because it was the motivation of the task...” The teachers, however, focused more on the integral development of the children: “... you [the educators] pay attention to the mathematical aspects of the task that the children are doing, we [the teachers] want the children not to lose interest, not get bored. When they are doing it, we say: turn around so that everyone can see it...” The teachers wanted all the children to understand the task: "... I have seen that they understood the activity...”

CONCLUSIONS

This work aimed to characterise the specialised knowledge early childhood education teachers need to mobilise to design and implement activities in the classroom to introduce length and its measurement, which, in turn, allows the improvement of teachers’ initial and continuous training programmes.

On the one hand, the article describes the specialised knowledge that the educators-researchers understand early childhood education teachers should have when analysing, from the MTSK model, the researchers’ design of an activity addressing the learning of the length measurement. On the other, the researchers analysed the specialised knowledge mobilised by two highly experienced teachers while reflecting together about the implementation of that activity. Thus, our proposal provides an answer to the problem posed by Zeichner (2010): in general, the classroom teacher is not aware of the knowledge currently taught in the university, while the teacher educator is not familiar with the classroom work.

When designing the activity, the educators-researchers prioritised the mathematical knowledge of the contents involved in the task and use teaching and learning theories to build the activity. In particular, they showed the mobilisation of knowledge of the following KoT categories: definitions, properties and their foundations, procedures, phenomenology and applications, and registers of representation; and, in parallel, of the following categories within the KMT subdomain: techniques, tasks, and examples; and teaching theories. As the task progressed, category elements emerged from the other two PCK subdomains (KFLM and KMLS).

However, the starting point for all the teachers' comments was always the students' characteristics, their actions, and the cognitive demand level that sustains them (KFLM). When the teachers explored these ideas in more depth, hints of aspects of knowledge of other subdomains such as KMT or KoT emerged. These could be interpreted as the seeds of some categories that emerged, on many occasions, from general or affective pedagogical events to which they gave great importance. In the joint reflection the educators-researchers expressed their concern about the task having developed very concrete mathematical skills, and the teachers emphasised their interest in the students' well-being, emotional state, and in keeping their attention (GPK).

During discussions within the collaborative group, the teachers openly stated their limitations when explaining mathematical concepts present in the students' activities and actions. And the educators-researchers reinforced their own limitations for not being familiar with the early childhood education classroom culture. With this open debate, the gap between the two approaches to teaching was narrowed.

Based on our work, the importance of the training of early childhood education teachers was ratified from a dual perspective: the theoretical approach, facilitated by the educators-researchers, and the empirical approach, centred in the classroom, contributed by the teachers, to form a model of the specialised knowledge of this professional to teach mathematics, in our case, length and its measurement.

AUTHORSHIP CONTRIBUTION STATEMENT

N. P. and N. J-P. conceived the general idea and objectives of the study presented here. J. M. B., N. P. y M. R-G. were responsible for the theoretical review and the design of the intervention. M. R-G. and N. J-P. coordinated the collaborative sessions in which the information was collected and the methodology was adapted to the context of the study. All authors actively participated in the discussion of the results and in writing the paper. Also, all reviewed and approved the final version of the document.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, Nuria Joglar-Prieto, upon reasonable request.

ACKNOWLEDGEMENTS

This work has been developed in the framework of the project: "Conocimiento especializado del profesorado de matemáticas y formación del profesorado" [Specialised knowledge of mathematics teachers and teacher education] (RTI2018-096547-B-I00, of the Ministry of Science, Innovation and Universities, Spain). It is also linked to the MTSK Network, sponsored by the Ibero-American Postgraduate University Association (Asociación Universitaria Iberoamericana de Posgrado - AUIP).

REFERENCES

Alsina, A. (2006). *Cómo desarrollar el pensamiento matemático de 0 a 6 años*. Octaedro-Eumo.

Alsina, A. (2020) La Matemática y su didáctica en la formación de maestros de Educación Infantil en España: crónica de una ausencia anunciada. *La Gaceta de la RSME, 30*(2), 373-387.

Arteaga, B. & Macías, J. (2016). *Didáctica de las matemáticas en Educación Infantil*. UNIR.

Ball, D.L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler

(Ed.) *Multiple Perspectives on Mathematics of Teaching and Learning*. (pp. 83-104). Ablex.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Baumfield, V., & Butterworth, M. (2007). Creating and translating knowledge about teaching and learning in collaborative school–university research partnerships: An analysis of what is exchanged across the partnerships, by whom and how. *Teachers and Teaching: Theory and practice*, *13*(4), 411-427. https://doi.org/10.1080/13540600701391960

Belmonte, J.M. (2005). La construcción de magnitudes lineales en Educación Infantil. En M.C. Chamorro (coord.), *Didáctica de las Matemáticas para Educación Infantil* (pp. 315-345). Pearson.

Briand, J. y Chevalier, M.C. (1995). *Les enjeux dans la relation didactique*. Hartier.

Brousseau, G. (1991). *Théorie des situations didactiques*. La Pensée Sauvage.

Buys, K. & Veltman, A. (2008). Measurement In Kindergarden 1 and 2. In Van den Heuvel-Panhuizen, M., & Buys, K. (Eds.), *Young children learn measurement and geometry: A learning-teaching trajectory with intermediate attainment targets for the lower grades in primary school*. Brill Sense.

Carrillo, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, A., Ribeiro, M., & Muñoz-Catalán, M.C. (2018). The mathematics teacher's specialised knowledge (MTSK) model. *Research in Mathematics Education, 20*(3), 236-253. https://doi.org/10.1080/14794802.2018.1479981

Chamorro, M. C. & Belmonte, J. M. (1988). *El problema de la medida. Didáctica de las magnitudes lineales*. Síntesis.

Clements, D. H. (2010). Teaching length measurement: Research challenges. *School Science and Mathematics, 99*(1), 5-11. https://doi.org/10.1111/j.1949-8594.1999.tb17440.x

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach*. Routledge.

Clements, D. H., & Stephan, M. (2004). Measurement in pre-K to grade 2 mathematics. In D.H. Clements, J. Sarama & A.M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education*. (pp. 299-317). Lawrence Erlbaum.

Copley, J. V. (2017). *Putting Essential Understanding of Geometry and Measurement Into Practice in Prekindergarten-grade 2*. NCTM, National Council of Teachers of Mathematics.

Erickson, F. (2006) Definition and analysis of data from videotape: some research procedures and their rationales. En J. Green, G. Camili y P. Elmore (Eds.). *Handbook of complementary methods in education research*. (pp. 177-191). American Educational Research Association

Fernández, M. & Mantecón, J. (2018). El número en la construcción de las magnitudes lineales. Magnitudes de Longitud, Peso, Capacidad, Tiempo. En C. Muñoz- Catalán & J. Carrillo (eds.), *Didácticas de las Matemáticas para maestros de Educación infantil*. (pp. 157-159). Paraninfo.

Gergen, K. (2001). Self-narration in social life. In M. Wetherell, S. Taylor y S.J. Yates Eds.) *Discourse theory and practice* .(pp. 247–259). Sage.

Hargreaves, A., & Fullan, M. (2014). *Capital Profesional*. Morata.

Latorre A., del Rincón, D. & Arnal, J. (2005). *Bases metodológicas de la investigación educativa*. Experiencia.

Lupiñez, J & Castro. E. (2017). Medida. En, Castro, E. y Castro, E. (eds.), *Enseñanza y aprendizaje de las matemáticas en Educación infantil*. (pp. 203-204). Pirámide.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the US*. Lawrence Erlbaum.

McIntyre, D. (2005). Bridging the gap between research and practice. *Cambridge Journal of Education*, *35*(3), 357-382. https://doi.org/10.1080/03057640500319065

Muñoz-Catalán, M. C., Liñán-García, M. M., & Ribeiro, M. (2017). El conocimiento especializado para enseñar la operación de resta en Educación Infantil. *Cadernos de Pesquisa*, *24*, 4-19. http://dx.doi.org/10.18764/2178-2229.v24nespecialp4-19

Muñoz-Catalán, M. C., Joglar, N., Ramírez, M., Escudero, A. M., Aguilar, Á. & Ribeiro, C. M. (2019). El conocimiento especializado del profesor de infantil desde el aula de matemáticas. En Badillo, E., Climent, N., Fernández, C., & González, M. T. (eds.). *Investigación sobre el profesor de matemáticas: práctica de aula, conocimiento, competencia y desarrollo profesional* (pp. 63-84). Ediciones Universidad Salamanca.

Muñoz-Catalán, M. C., Joglar-Prieto, N., Ramírez, M., & Codes, M. (in press a) *MTSK desde la perspectiva del profesor de Educación Infantil: foco en el dominio matemático*.

Muñoz-Catalán, M. C., Ramírez-García, M., Joglar-Prieto, N. y Carrillo, J. (2021). Mathematics Teachers' Specialized Knowledge to Promote Algebraic Thinking in Early Childhood Education as from a task of additive decomposition. *Journal for the Study of Education and Development* (RIYA). https://doi.org/10.1080/02103702.2021.1946640

Nolla de Celis, A., Cerisola, A., Fernández, B., y Muñoz, R. (2021). La formación inicial de los maestros en Matemáticas y su Didáctica. *Revista interuniversitaria de formación del profesorado, 96* (35.1). https://doi.org/10.47553/rifop.v96i35.1.85882.

Oppermann, E., Anders, Y. & Hachfeld, A. (2016). The influence of preschool teachers' content knowledge and mathematical ability beliefs on their sensitivity to mathematics in children's play. *Teaching and Teacher Education*, *58*, 174-184. https://doi.org/10.1016/j.tate.2016.05.004

Parks, A. N., & Wager, A. A. (2015). What knowledge is shaping teacher preparation in early childhood mathematics? *Journal of Early Childhood Teacher Education, 36*(2), 124-141. https://doi.org/10.1080/10901027.2015.1030520

Pérez Serrano, G. (2014). *Investigación cualitativa. Retos e interrogantes. I Método*. La Muralla.

Roschelle, J. (2000). Choosing and using video equipment for data collection. En A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 709-729). Lawrence Erlbaum.

Shulman, L. (1986). Those who understand: Knowledge for growth in teaching. *Educational Researcher*, *15*(2), 4-14. https://doi.org/10.3102/0013189X015002004

Scheiner, T., Montes, M. A., Godino, J. D., Carrillo, J., & Pino-Fan, L. R. (2019). What Makes Mathematics Teacher Knowledge Specialized? Offering Alternative Views. *International Journal of Science and Mathematics Education*, *17* (1), 153-172. https://doi.org/10.1007/s10763-017-9859-6

Schoenfeld, A. (2000). Models of the teaching process. *Journal of Mathematical Behavior*, *18*(3), 243-261. https://doi.org/10.1016/S0732-3123(99)00031-0

Van den Heuvel-Panhuizen, M., & Elia, I. (2011). Kindergartners' performance in length measurement and the effect of picture book reading. *ZDM Mathematics Education*, *43*, 621-635. https://doi.org/10.1007/s11858-011-0331-8

Zeichner, K. (2010), Rethinking the connections between campus courses and field experiences in college- and university-based teacher education. *J. Teacher Educ*, *6*, 89-99. https://doi.org/10.1177/0022487109347671