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Didactic Explanation and School Mathematical Discourse: The Case of Variation

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ABSTRACT

Context: An important area of research in educational mathematics is social communicative practices in classroom organization. **Objective:** To locate and analyse the forms of introduction and development of variation in teaching situations. **Design:** Using a qualitative-interpretive approach, specifically an ethnographic study, for the analysis of socially shared practices among teachers, when they use explanation in the classroom and its correlation in the extended classroom. **Environment and participants:** The research participants were three professors (one physicist and two mathematicians) who taught the subject Mathematics I. On average, their groups consisted of 37 students. **Data collection and analysis:** Information was collected through audio-recorded and transcribed classroom observations. A detailed sequential study was carried out on teaching situations to describe the work done in each intervention that precedes or proceeds to yet another situation and thus construct the categories of analysis. **Results:** Due to the interactive nature, the construction of explanations is seen as an object of analysis and this implies that the minimum units are sequences of interactions, since the construction of discursive resources and meanings for variation was addressed. **Conclusions:** During the classes we recorded different types of explanation, models in which the notion of variation is modelled, the teaching representations when explaining the contents through numerical, algebraic, and natural language representation.

***Keywords*:** Explanation, Variation, Prediction, School mathematical discourse.

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RESUMEN

Contexto: Un área importante de investigación en matemática educativa son las prácticas sociales comunicativas en la organización áulica. **Objetivo:** Localizar y analizar las formas de introducción y desarrollo de la variación en situaciones de enseñanza. **Diseño:** Mediante un enfoque cualitativo-interpretativo, específicamente un estudio etnográfico, para el análisis de prácticas socialmente compartidas entre docentes cuando usan a la *explicación* en el aula y su correlato en el aula extendida. **Entorno y participantes:** Los participantes de la investigación fueron tres profesores (un físico y dos matemáticos) que impartían la asignatura Matemáticas Básicas en un Tecnológico. En promedio sus grupos estaban conformados por 37 alumnos. **Recopilación y análisis de datos:** Se recolectó la información mediante observaciones de aula, audiograbadas y transcritas. Se realizó un detallado estudio secuencial sobre situaciones de enseñanza a fin de describir el trabajo hecho en cada intervención que antecede o procede a otra situación más para construir las categorías de análisis. **Resultados:** Debido al carácter interactivo, la construcción de explicaciones es vista como objeto de análisis y esto implica que las unidades mínimas sean secuencias de interacciones, pues se atendió la construcción de recursos discursivos y significados para la variación. **Conclusiones:** Durante las clases, registramos diferentes tipos de explicación, modelos en los que se modela la noción de variación, las representaciones docentes al explicar los contenidos mediante la representación numérica, la algebraica y la del lenguaje natural.

***Palabras clave*:** Explicación, Variación, Predicción, Discurso matemático escolar.

INTRODUCTION

If teaching is conceived as a form of communicative practice of a social nature, then oral, written, gestural or figurative discourses constitute the means for learning that emanate from those classroom practices. In this way, the speeches that some emit and others interpret based on the media and resources to specify them correctly are nourished by different languages. Ideally, the classroom is a space for mutual understanding, for negotiating curricular contents and forming shared meanings: in this sense, the act of teaching is fundamentally constituted thanks to communication (Edwards & Mercer, 1987). While the classic approach suggests that we analyse what we say and especially how we say things in the classroom, nowadays, we focus on an *extended classroom* that favours the transversality of knowledge (Cantoral, 2019).

According to Candela (1999), discourse is considered a means to study *communicative social practices* in the classroom organisation, as an explanation, since every intervention is oriented towards understanding some idea, notion or concept and, in that sense, the notion of explanation allows clarifying the effective forms of communication for learning. Thus, to study the teachers' discourse, we planned to attend class episodes where they used didactic explanations (Sierpinska, 1994) and discursive resources to make the notion of *variation* accessible to their students. Therefore, this analysis focused on the role played by the teachers' explanations in the construction of meanings and how and in which situations they used such resources.

We were also interested in researching how the teachers developed the discursive resources to structure the explanation in the classroom and their role in the construction of knowledge inside and outside the classroom. We consider that one of the objectives of a teacher is to achieve students' understanding of specific mathematical knowledge, or in its deepest dimension, of the knowledge taught. However, in this article, we are interested, above all, in the role of *variation* in shaping their explanations.

The explanation encompasses those discursive resources that favour the understanding of notions, facts, phenomena or objects, which go beyond mere description or characterisation when trying to find the causality or the principles that produce it. Not only does it generate a reflective stance, it is also an explicit means available to educational actors (teachers and students) to exchange mathematical ideas with each other and correlate them with the learners' life, providing motives to make some data, a phenomenon, a finding or a process understandable in the course of the didactic action. In fact, it is a factor that is privileged in the significant didactic action.

The teacher explicitly assumes a responsibility, that of "explaining" to establish with explanations, cognitive, social, and affective links with their students, which in turn have various forms, ranging from simple comments, illustrations, problem solving to the construction of reasoned arguments and school demonstrations (Wittmann, 2021).

Another focus is bringing a sceptical view on a very widespread conception that the discourse in the classroom is authoritarian, and the teachers' communicative formats are characterised by rigidity and scant reflection, without rigorously paying attention to how the information finally reaches the students. This is because the knowledge is considered finished and consistent with curricula. Hence the teacher is assigned the role of the authority or the one who "knows the truth." However, from the teachers' explanations, we see a process of adaptation to the discursive interactions with their students.

Some researchers study aspects of the teachers' traits, such as personality or characteristics. They are conceived as professionals who not only act but reflect on their actions and, therefore, are able to generate knowledge: this perspective is known as *reflective paradigm of the teachers' thinking*. At present, we accept that the mere reflection does not modify the complexity of the classroom reality, so we find works that seek to know the didactic and disciplinary knowledge that are used in mathematics class discourse. This research contributes elements in another direction, it opens the analysis of teaching discourse in contexts framed by explanatory interactions.

One of the ways to have access to information on how the notion of *variation* is introduced and developed (Cantoral, Moreno – Durazo, Caballero, 2018; Johnson, 2015) is to study the teacher's discourse (Sierpinska, 2004), but also the discourse in the social interaction in the classroom (Reséndiz, 2006, 2019). Thus, the research problem was delimited with these questions: Which role does *variation* play in the teacher's *discourse*? What role does the notion play in the *explanation*?

THEORETICAL FRAMEWORK

Studies on *variational thinking and language* show that *variation* and *change* operate on isolated data through narratives of sequenced facts, arranging them with certain rationality, one after the other. In other words, what happens between one datum and another requires an argumentation of a causal nature based on experience, which, when shared, is typified as a *reference practice* (Cantoral, 2020). In the school environment, variation is usually understood as the study of change, and is reduced to a comparison of successive states, whether figurative, numerical or symbolic, thanks to an operability that relates the first datum, say A, with the second, let us call it B. Usually, it is a question of comparing them through A – B subtractions, A / B ratios, figurative or symbolic sequences A, B...

Now, when we talk about change in both mathematics and science, we think of magnitudes whose difference is quantified, providing a numerical reference system. However, an amount is also a conception of the individual and the community with which he/she dialogues of a measurable facet of a given entity. This means that a quantity refers to something more than the measure alone used in the material world or the pairing between unit and number. In this way, magnitude, by coming from practices, will have a numerical sense with its own meaning, that is, information that helps in decision-making and helps to organise the action.

In this dimension of variation, some transversality of knowledge and heuristic and abductive reasoning are needed. To this end, the action of *comparing* precedes and accompanies the activity of *varying* and this, in turn, *predicting*, *estimating,* or *guessing* practices. In short, there will be no conception of variation without a reference system and some gradation of successive variations (orders of variation).

This research shows how the notion of variation appears in teaching and its development in the process of negotiating meanings through *explanation*. In their study, Sierpinska (1994), Mopondi (1995), and Reséndiz (2004, 2006) point out that didactic explanations are those offered by the teachers (or the students) and are aimed at a better understanding with more familiar and frequent bases for teaching. In the classroom, however, there are several alternatives to address them; hence both educational actors try to build their versions based on the academic mathematical discourse (dME) as an objectifiable dimension. There, the participants provide their explanations to understand and guide the social agreements. We can then assess, within each of the classes, the resources used by the teachers and the contributions of their students, or vice versa, and the effect on the construction of shared knowledge.

To study the teachers' discursive elements about variation, we analysed the situations where they explain different topics of study: variation and the notion of function or the concept of derivative; we also paid attention to the students' explanations when they express the need for greater exposure and how this modifies the explanation in the course of an interaction. For this, we focus on the role of the explanations during mathematics class when we intend to teach concepts and mathematical processes linked to variation.

In short, the *explanation* is one of the means that the teacher uses to "make students understand or make sense" of something. It constitutes the object of communication, a debate, a discussion. Likewise, it can appear as the communication of helpful information, or a means that can quickly facilitate transmission or argumentation. It seems to be linked to reasoning (Duval, 1999), and its objective is to find understanding (Sierpinska, 1994, Reséndiz, 2006).

Sierpinska differentiates scientific from didactic explanations. Scientific explanations seek to achieve an understanding of the more conceptual bases - for example, of an abstract mathematical theory - while the didactic explanations are aimed at an understanding with more familiar and frequent bases in teaching (an image, some prior knowledge, or experiences). Since the role of the explanation is to make the meaning of a formal object accessible (whether it is a method, a procedure, a term or a formal propositional statement), it is the means that teachers use for their students to show evidence of their understanding. Its objectives are manifold: teaching, convincing, ordering, or obtaining a formative and informative advantage. Block and Laguna (2020), in their view, assume it from a dual perspective, such as the *ageing of didactic situations* and as scenarios of teacher interaction from sociocultural perspectives.

Thus, we intend to identify the didactic explanations, the discursive elements, and the negotiation of meanings to which the teacher resorts, taking into account that the dME is expressed in the classroom, providing a scenario for the teacher and the students to represent, think, speak, agree or disagree. Here we analyse what and how ideas are expressed in the class, under the premise that it is in this classroom discourse where we reach an organisation for explanatory purposes, since every intervention is oriented towards shared understanding.

RESEARCH METHODOLOGY

In the framework of the general strategy we chose -qualitative research-, it is important to point out the general procedures to gather the information and the logic that underpins our study.

In trying to answer the research problem, we conceived that teaching is teacher-led; however, the students' influence is decisive for many teachers' actions when teaching.

Stage of interest

Classroom activities include dialogues that generate opportunities to learn, as students must verbalise and reconstruct their solutions and resolve conflicts. To initiate such instruction, the teacher and students need to mutually define the expectations and obligations in the classroom; hence, *interactive communication* constitutes the core activity of the teaching and has its goals.

Due to the characteristics of the educational actions in which the teachers of the higher level participate, one type is fundamental for our research: regular and daily classes. In this way, we will consider the daily classroom activity of students in the first semester of an engineering course. For this, we selected teachers who are mainly dedicated to teaching.

Study participants

As an element of utmost importance, we have considered the *teachers* bearers of the knowledge that will be staged in the classroom. The participants were three professors who teach the mathematics subject in the common core of the different engineering careers. They were randomly chosen from among the teachers who offered the course.

We spoke with each teacher, telling them about our interest in observing and registering how they presented the concepts of function and derivative, to which all agreed. It is worth mentioning that this institution is always open for research in mathematics education.

The observations lasted long and only during the classes in which the teachers taught the concepts described, since they are spaces for the study of variation.

Sources of information

The information was collected under usual conditions through observations of classroom activities. We needed to record the classes in audio and elaborate the field notes to triangulate with a data source to:

- Obtain information to illustrate what happens in the classroom under "normal" conditions
- Bring teachers close to the group, but without causing significant changes in their daily forms of work and relationships. This, to make it easier for us to have real registers
- Understanding teachers' ways of acting in educational activities
- Collect information on what happens in social interaction, i.e., in the educational process where teachers and students participate
- Have elements of interpretation of events, *from the perspective* of the subjects under study
- Assimilate what happens in educational processes where teachers participate

We recorded the class observations to reconstruct the undocumented aspects, rescue the routine, the unconscious, a part of the school reality that remains unseen (Candela, 1999), i.e., to design registers that allow reconstructing what was observed in the light of subsequent and more elaborate conceptualisations than those that emerged at the first moment (Rockwell, 1987; Erikson, 1986) of the research. After that, we transcribed the class registers entirely.

Phases of the research

As a whole, our study included these phases:

- Approach and delimitation of the problem
- Fieldwork
- Description
- Interpretation and communication of results

We conceived the phases in a differentiated sequence, since each one emphasises a specific issue, but constantly returned to the previous phases for review.

RESULTS AND ANALYSIS

A first topic we wanted to explore concerned the ideas used by the engineering teachers on the notion of variation. We did not only want to find the definition used, but we were also interested in the entire cognitive and social network with which they mobilise the idea through their *explanations* in the classroom and how these, in turn, are modified through discursive interaction to reach agreements. The discursive sequences were of diverse extension, however, as the classroom interactions are complicated, we chose to separate only the teacher's explanations with examples that suit this research purposes.

Due to its interactive nature, the construction of explanations, seen as an object of analysis, implies that the minimum units are sequences of interaction, not decontextualised phrases or messages (Candela, 1999), since it is necessary to attend to the construction of discursive resources and the meanings about variation. In this way, we chose from the explanations given by the teachers on the notion of variation, and especially how they play different roles. For example, let us quote four sequences:

I. Tabulating the analysis of the numerical variation
II. Graphing the variable and its variation at a given point
III. Explaining based on everyday situations
IV. Using parameters as main variables

Sequence I. Tabulation for the analysis of the numerical variation

A first approach to the notion of variation in the classroom is given through *tabulation*. In the teachers' explanations rests the idea of the relationship between sets, interpolating, progressively approximating, rotating, going up or going down. Let us start this section with examples based on the didactic explanations where they register the discrete numerical variation through tabulation: The teacher explains the rational functions and then suggests that a tabulation be made to treat the linear behaviour. Such a strategy is essential since the mathematical notion of function is being approached.

In the following extracts, we identify the reciprocal explanations by using: **P**, for the teacher, **Am**, for the male student, **Af**, the female student and **As**, for several students speaking at once. We only chose extracts from a larger collection.

Extract 5.3

> P: See whether that expression can happen, look well at it, it is the same thing here, it is exactly the same thing. If I had $(x) = \frac{x^2-9}{x+3}$, I say $\frac{(x+3)(x-3)}{x+3}$, this is equal to $x - 3$ for $x \neq -3$. I'm doing the same thing; it means exactly the same thing. What is this, then?
>
> - Am: A straight line.
> - As: That's a straight line!
>
> P: I mean, this is exactly a straight line, isn't it? We could do a brief tabulation to see roughly its behaviour; for example, I can give you some values to observe, yes? The teacher says that except for -3. So, I can give you any value, like $-4, -2, -1, 0, 1, 2, 3$. Those are some values that I can give you... can you see how curious the graph of this line is? When it says $y = x - 3$, then -4 they would be $-7, -5, -2, -1, -4, -3, -2,$, right? -1.
>
> - Am: 0
> - P: 0 Right? With that, we already have an idea of how the graph is.
>
> A (Gpo-1), p. 33.

The teacher asks the students, after the simplification, for the result "What is this, then?" The students identified a straight line. Then the teacher uses this answer to comment and take them to the tabulation.

With the numerical variation table, you will have an idea about the behaviour of the graph; this expression can be noticed in the last turn of the sequence. The teacher employs the majority's opinion, as in the expression: "That's a straight line!" In this situation, the teacher makes slight modifications to the explanations built during the interaction, promoting complementarity between the students' versions and their own. On the other hand, the teacher's interventions play the dual role of requesting explanations and guiding them towards the answers they consider acceptable.

The notion of variation the teacher used in this episode during his explanation was the numerical variation, and the model used were represented by the table of values. In this order of ideas, Gutiérrez and Reséndiz (2021) report that secondary school students have some knowledge of variation of movement situations (moving, walking from one point to another, i.e., in daily life contexts) and can even describe it in words, but plotting a graph became complicated.

The didactic phenomenon of ageing of teaching situations (Brousseau, 1986) occurs incessantly in class since the teacher constantly interacts with the students placing the use of knowledge. Although we are analysing selected episodes, it does not mean that the phenomenon of ageing only happens there because, as the theory points out, verified by our empirical evidence, this phenomenon is continuous, happening permanently. The analysis selects, describes, studies, and explains it.

In the following excerpt, the teacher explains that to graph a function, a tabulation is required. Let us say the table is an intermediary for the graph. Some particular values are proposed, equidistributed as -3, -2 and -1.

Extract 5.4

> P: ... $f(x) = x$. We want your graph. We'll tabulate it. We're going to give values to x and we find those of y; for example, we start with $-3, -2, -1$. Why in whole numbers? - someone might ask me. Because they're simple to work with; you can also occupy fractions, but why get into trouble, right? The question might be: why don't we evaluate more numbers? That is at your discretion, i.e., if you consider that with those points you

> can visualise the shape of the graph. If the points are not enough for you, you would have to give more, how many? The ones needed for you to visualise the graph. In this case, to make your graph, we find this: −3 with −3. I think that they already work very well: 1 with 1, 2 with 2, 3 with 3. Why do we unit the points? Because we understand that you can also evaluate the intermediate points and find their respective pairs; then you unite them, and you get this figure that represents a line.
>
> C (Gpo-1), pp. 2-3.

This explanation is interesting because the teacher assumes that the students are going to ask him a series of questions that he asks and answers himself. The first question alludes to the selection of whole numbers to tabulate: "why in whole numbers?". His explanation emphasises that it is easier to work with whole numbers than with fractions. The second question, "why don't we evaluate more values?" - he leaves at the student's discretion because he wants them to visualise the shape of the graph.

The last question, "why do we unit the points?" - is particularly relevant because the teacher's explanation revolves around the intermediate points (interpolation) between the proposed ones. Although by uniting them through a graph, the intermediate points are those that the teacher mentioned would be difficult to tabulate (fractions). The numbers with which he elaborates his teaching design consist of three values near the origin. He uses the numerical variation in his explanation (the number organises the points), while his model is the table of values. Thus, we can visualise the function graph by a representation model.

As in the previous explanation, the following also refers to the succession of points, but here, uniting them is difficult for the students. This difficulty can be controlled by the teacher's class management. He assumes control of the discourse, both at the level of the affirmations and of the answers to the questions he raises. This situation will be of interest in the analysis.

Extract 5.5

> P: These kind of graphs, I tell you, are very common. You should not be amazed. You saw it, many were looking for a point over there, another over here and could not form a succession of points. It turns out that

> this is what happens: I simply notice where the axis of the $x's$ cuts. That's why we feel that the curve is crossing the axis of the $x's$ in 3 points: here, it is crossing it in 2. Generally, the degree of polynomial tells them how many times it cuts the axis of the $x's$. If the degree is1, for example, it is a line, cuts the axis of the $X's$ at a point; if it is quadratic, you can cut it at 2; if it is cubic, you can cut it at 3, i.e., cross it at 3 points, and if it is of the fourth degree, you must cut it 4 times...
>
> A (Gpo-1), p. 30.

This explanation appears with the cubic function $f(x) = x^3 - 4x$. The teacher suggests another strategy different from that of his colleague, which consists of locating the point where it cuts the graph to the axis of the $x's$, hence he makes a tabulation with fewer points y , relying, as a visual resource, on the graphical representation of the function.

To conclude the explanation of the behaviour of the function in terms of the degree of the polynomial, the teacher tries to generalise saying that depending on the degree of the polynomial, it is the times that it will cross the axis x. However, if the students take these mathematically imprecise generalisations literally, they may have a problem if they try to apply such a criterion in different exercises. These generalisation attempts will find counterexamples, they will demand changes in future explanations, so to speak, they are the seed of potential changes in the explanations they will produce in the face of the students' demands in new situations. Although the example is under the control of the teacher's explanation, this is not the case with generalisation.

The explanation of the variation is clearly seen in the use of the table of values model (numerical variation) and in the model of the geometric representation. It is interesting to analyse the teachers' different approaches when trying to graph a function through tabulation. In the previous fragments, it is evident that tabulation has been favoured, i.e., a variation where a number orders the points and the tables show the relationship between two quantities. In the teacher's explanations, the notion of *numerical variation* appears in the sense that it is the number that orders the position of the points and vice versa, while in its model or graphic representation, the table of values guides the graphic construction sequence.

In the various teaching interventions, we identify didactic explanations where, in some way, basic notions around the concept of variation are appreciated: the relationship between sets, approximation, increase or decrease, up-down, turns in both directions, changes the inclination, and the behaviour of the intermediate points (Reséndiz, 2019).

Sequence II. When constructing graphs as the variation of a reference point

In the explanations of the three participating teachers, the idea of moving a reference point such as the origin, the vertex, or the asymptote appears. This idea has been of great importance for constructing the exposures on the graph movement (variation of a reference point, reference line, trigonometric functions): in the following sequence, the teacher's explanation revolves around the displacement of the vertex or the *variation of a reference point*.

Extract 5.11

> **P:** That's why I checked there $(-1, 0)$. There's your graph. Notice: when the number affects the basic function, it raises or lowers it, but when it affects the variable of the function, it moves it directly. In this case, to where did you move it?
>
> **As:** To the left!
>
> **P:** To the left. We could think that because it has more $(+)$ it moves it to the right, but it does not.
>
> **Af:** What if it were $f(x) = (x - 1)^2$, would it move it to the right?
>
> **P:** It would move it to the right. I told them that one would think that with... (+) it is to the right and if it is... $(-)$ to the left, but it is not so, it is the other way around. ... if it is $(+)$, it moves to the left and if it is... $(-)$ to the right, that is what we are observing, ... it would be answered through its tabulation, locating the points.
>
> C (Gpo-1), p. 11.

We are working with the quadratic function $f(x) = (x + 1)^2$. The teacher's explanation attends to the displacement of the vertex when the function is affected by a number (*"when the number affects the variable of the function directly moves it"*) and compares it with the basic function $f(x) = x^2$, that, when influenced by a number, causes the vertex to go up or down. The teacher asks to where the graph moved and uses the opinion of the majority to legitimise his explanation, since the students answer in unison: *"To the left!"*

A student takes as a reference the example formulated by the teacher to generalise the explanation of the movement or variation of the vertex by asking *"What if it were* $f(x) = (x - 1)^2$*, would it to move it to the right?"* The teacher says that if it is $(+)$, the graph moves to the left and, if it is $(-)$, it moves to the right. The notion of variation is given in relation to a moving reference point, while the validation falls on the tabulation. The teacher uses familiar expressions to illustrate the movement of the vertex (up, down, right, left) and uses two types of explanation for the notion of variation: the natural language model and the geometric representation model. In this regard, Parada, Conde, and Fiallo (2016) point out that every individual needs to build, interpret and connect various representations of ideas, make observations and conjectures, formulate questions and produce persuasive and convincing arguments, using everyday language to express their ideas.

The didactic relationship implies a collective construction since teachers and students participate by suggesting adjustments and negotiations. Here are two examples where the teacher's explanation focuses on the displacement of the vertex.

Extract 5. 15

> **P:** Well, what is the graph of the function? If we add 1 to that function, for example, and it remains $y = x^2 + 1$
>
> **Am:** The origin is moved in $y = 1$.
>
> **P:** Are you sure we would be doing that? I would say that y it is going to be what it is worth in $x^2$and that we would be doing, adding 1, where is it in x? In 0, put 1, in 0 I put 1, and in 1 when it's worth it 1, now y , how much is it going to be?
>
> **As:** ¡2!
>
> **P:** It will be 2 (...) and then the formula would remain the same. What was the only thing that happened? - that the curve shifted one unit up, and if we wanted to go down, what could we do?
>
> **As:** Subtract!

P: Subtract 1, now, what would be the graph of $y = x^2 - 1$, we can put this $y = x^2$, and if we return1, what is going to happen? When you open vertex $(0, -1)$ where it cuts the x-axis in $1, -1$ and this is the graph of $y = x^2 - 1$, of $y = x^2$. If I subtract, what happens to the curve?

Am: We move it

P: How many units do we move?

B (Gpo-1), p. 99.

He begins by explaining the basic quadratic function $y = x^2$ that, when added a unit ($y = x^2 + 1$), shifts its origin in $y = 1$. The statement *"the vertex is moved in* $y = 1$ *"* was made by a student, although not requested. By subtracting one unit from the basic quadratic function ($y = x^2 - 1$), the vertex is shifted one unit down and, by adding one, it goes up. The teacher uses the term "shifts" that the student said when he asserts: *"the curve shifted one unit up."*

When the teacher requests that the students express their opinion, through questions, he motivates explanatory interventions, and it is of great interest for the students to be able to "move" the vertex from their initial position. The teacher tries to generalise, saying that if they add a quantity to the basic function, the graph moves up, and if they subtract it from the function, it moves down. In this situation, there are two types of explanation where the notion of variation intervenes: the natural language model and the geometric representation model, which serves to visualise the movements.

Among students, mathematical thinking develops as they can take control of their mathematical activities, orchestrated by the teacher (Contreras, González, and Reséndiz, 2020). On the other hand, Cantoral and Montiel (2001) affirm that when using a graphing strategy either to build, interpret, or transform a graphical representation, a particular way of mathematical thinking is opening in the student.

Finally, the theory of situations (Brousseau, 1986) is based on a constructivist approach, which acts on the principle that a notion is built through teaching situations; hence we consider that, in the classroom, a shared discourse is created between teachers and students. Similarly, the socioepistemological theory (Cantoral, 2013) is based on a sociocultural approach to knowledge and analyses the classroom as a scenario of shared construction, where the aulic discourse is regulated by the dME.

Let us see how the vertex of a quadratic function $y = x^2$ shifts when a unit is added to it in x and gets $y = (x + 1)^2$.

Extract 5.16

> **P:** It's a parabola $y = (x + 1)^2$. We can even see it like this, $y = x^2 + 2x + 1$, right? Then, -2 , how much is it? The -2 is 1. Then the curve would be like this −to see if we agree−, while the basic form would be until here, which is $y = x^2$; the form remains the same. We must understand that it is the same curve, and the only thing that the line does is move it towards where?
>
> **Am:** To the left.
>
> **P:** To the left. And if we wanted to move it further to the left, what would we have to do? What must be replaced in the basic form? The x by $x + 2$ ($y = (x + 2)^2$). If I want it to the left, until -10, then where would F be located? if the original function is $f(x) = x^2$ and I want to take it to the vertex that is in -10, what do I do?
>
> **Am:** It would be: $y = (x + 10)^2$
>
> B (Gpo-1), p. 101.

The teacher's explanation refers to the function x^2. It is the same shape as the curve, but now it moves one unit to the left and, if you want to move more to the left, you would have to give any negative number, like the -10.

Here it is highlighted that teacher Bruno makes the binomial squared. We can see this at the beginning of this excerpt; however, there is a whole discussion about the fact that a student develops the binomial squared in teacher Carlos's group. At the end of the explanations on the movement or displacement of the vertex, the teacher summarises the topic: graph $y = f(x) + c$ displaced c units up, graph $y = f(x) - c$ displaced c units down, graph $y = f(x + c)$ displaced c units to the right and graph $y = f(x + c)$ displaced c units to the left.

Two types of explanation involve the notion of variation. One is the natural language model (up, down, left, and right displacements), and the other is the geometric representation model, which allows visualising the displacement of the graph or parabola as a complete identity.

The students’ previous interventions are different versions of the answer to the question, and, in some cases, they contrast with each other and with the teacher’s explanations. The students manage the movement of the origin (they do this in the episode, although, as we know, it is rather the vertex) and its displacement to the left or right to build an explanation for the graph behaviour. Here, the explanation of the notion of variation occupies the geometric representation model, through which the behaviour of the parabola upwards or downwards is visualised. In the previous sequence, Bruner (1988) points out the construction of knowledge is not an isolated individual process but a social process of joint creation in a culture.

Throughout these sequences, we systematically observe that the teacher refers to the relationships between the displacement of a graph with the variation of parameters. The mix of languages used in these examples comes from a natural everyday reference; ideas, expressions, metaphors, generalisations based on what is most familiar to students are used. Moving a glass, a chair, or -now- a graph will not require a greater explanation, a mathematical explanation. Instead, it will require didactic explanations, explanations that are familiar and serve the understanding. Explanations that, mathematically, are not considered insofar as they act on specific objects. In this sense, the space of discursive interactions between students and the teacher when it comes to notions such as variation is considerably expanded, resulting in an atmosphere of agreements and negotiations. In our opinion, if this is consubstantial to mathematics class, it is more substantial and more persistent when it comes to paramathematical notions. We are well aware that this situation would not have occurred if the discussion had laid its foundations on a concept with an explicit definition, such as the integral, the limit, or the derivative. Hence the relevance of this analysis.

In these teachers’ interventions, we could identify the explanations where their different interpretations of the notion of variation can be recognised. The idea of taking a specific reference point, such as the vertex, the asymptote, or the periodic shape, was of great importance for constructing the interventions based on the movement of an object, a graph. Hence the change in the explanation went from the concept to the mathematical object. Notions of movement were attributed to the graphs and main reference points, how it moves, rises or falls, travels, progresses, shifts, etc. The explanations where the variation appears were mainly based on the external representation model (on the graph), perhaps due to the possibility of synthesising information in a single drawing.

Sequence III. Verbal expressions as a reference for everyday situations

During trigonometric functions session, the teacher addresses trigonometric relationships and asks what those relationships or number mean (the opposite leg divided by the hypotenuse, for example). The student replies that *"They are like reason of change,"* and the teacher resumes it because he did not convince the student; he argues that, when we talk about change, we allude to a variation of the quotient, and we also call it ratio, but we do not gain anything, we only change the name (rates of change-variation of quotients), or the ratio between the opposite leg and the hypotenuse is the trigonometric relationship between the sine...

Extract 5.21

> **P:** 0.5, 60º sine, the opposite leg divided by the hypotenuse, and this gives us 0.8660, but we have the same question what does it mean? I know how to calculate this, and if I don't, the calculator does; we just press a few keys. However, I would be interested in answering this question: What do those numbers mean?
>
> **Am:** They are like rates of change
>
> **P:** Yes, when we talk about change, we talk about variation of the quotient, i.e., it is a ratio, but we gain nothing because we understand that it is a quotient, which is also called ratio. We're just changing its name, right? We cannot deny what your colleague says. If it is true that this is a relationship through ratio, as a division -do you agree?-, it is the ratio between these sides, between the legs, pardon, the ratio between the opposite leg and the hypotenuse, and the opposite leg and the hypotenuse is the trigonometric relation sine. The sine is the relationship that the opposite leg has with the hypotenuse; the ratio that it has is like grabbing an orange and saying "I'm going to divide it between two people". What is the ratio of an orange in relation to these two people? It would be taking the orange and dividing it in two: the ratio would be half orange...

C (Gpo-1), p. 53.

To convince the students of the ratio between the opposite leg and the hypotenuse, the teacher formulates an analogy: *"It is like grabbing an orange and saying I am going to divide it between two people. What is the ratio between an orange and these two people?* It *would be to take the orange and divide it into two: the ratio would be half an orange."* He tries to explain an unfamiliar situation by comparing it with a similar one, although little explored by the students so that they can understand and share their explanations.

The teacher's explanation is based on an alleged *variation of the quotient* and exemplifies it or relates it to another idea with which he resorts, a little forcedly, to the use of expressions that he considers familiar to the student. In this didactic relationship typical of the classroom, the teacher seeks to "diminish" or "invalidate" the student's intervention and preserve his role as a guide to the debate. He uses this unconscious argument in the problem since it is not just any quotient, but a quotient of increases, i.e., the rate of the changes. Let us say that he explains without justifying the participation of the student in said interaction. He achieved the appearance of a metaphor like the one previously described, i.e., it is the interactions between student and teacher that regulate the explanation and consequently the ageing of the situation (Brousseau, 1986, Reséndiz, 2006). This happens in the framework of the dME, which as a reason system (Soto & Cantoral, 2014), regulates this process. Without such intervention, the teacher would simply have defined the concept.

Sequence IV. Using parameters as main variables

In the teachers' explanations, the manipulation of parameters allows them to visualise the changes in the graphs of functions. The teacher compares them, the function, in this case$f(x) = x$, and this with 2 units added to it$f(x) = x + 2$.

Extract 5.24

> **P:** This is a basic function, $f(x) = x$i.e., the basic function of $f(x) = x + 2$; this 2 is rotating it. So that you understand what I mean by basic, here the basic of $f(x) = x$ it would be around here... look at how it rotated it, how much did it rotate it? It was not so simple there. Now, what is its inclination? Well, that's what we're going to talk about. Now I think that with this we can go to a translation, let's put it like this and let's add it... what does the sum do to the translation? That's what

we're going to find out. To look at this, again, I have the representation; if we evaluate it at –3, it would be –3+2; if we evaluate it at 3, it would be 3+2, how much?

As: 5

Am: It's +5, right?

P: It's +5. Now I briefly draw the basic, that is $y = x$. What did it do to it?

As: It raises it!

P: It uploaded it, I mean, it moved it, by how many units?

As: 2

P: 2, so this number, what it does is to move it on the axis of the $y's$. We can see it like this, raise it by 2 units. What is it for? to make the graphics a little faster, and that in a moment they can help us. For example, to graph $f(x) = x - 3$...

As: It lowers it!

P: By how many units?

As: 3

C (Gpo-1), p. 5.

The first comment of the teacher about the variation or change of parameter is that it 2 rotates the graph of the function with respect to the origin. Such an idea is not correct since the graph is moved, raised, or lowered. Then he handles the idea of *translation* in that same exercise by adding 2 units to the function. When the line leaves the origin, there is a *translation,* and to verify the movement of the line, he performs a tabulation.

When evaluating the function, the teacher asks, *"what did it do to it?"* the students answer, *"it raises it!"*. They did not use the teacher's explanation, i.e., the idea of translation. However, the teacher mixed the two: *"raised"* and *"translated"*. The parameters work as a whole; the parameter sign determines the behaviour of the function, its translation (go up or down).

Here the teacher's explanation of the notion of variation is based on a graphical representation model that illustrates the entire behaviour of the graph.

Let us see one of the teacher's explanations, where the parameters that multiply $f(x) = sen\, x$ are varied so that the modifications of the graph are perceived.

Extract 5.22

> **P:** When you put a value there, when you have the variable there, it will affect the domain. What we are going to *vary, depending on what changes I make to it, sin x* and when it affects the domain, the image will also change. This indicates that your graph is not going to be the cosine function, but sine of x...
>
> C (Gpo-1), p. 58.

Teacher Carlos generalises what he did in the sequence of the extract and takes it to the quadratic function $f(x) = x^2$, which he names basic. His explanation is carried out through tabulation, trying to acknowledge the effects of 2 when added to the function. Unlike the previous extract, where the teacher refuses to consider the opinion of the majority (*"it raised it!"*), here the term translation is no longer used, but that of *"it raises it!"*

Extract 5.25

> **P:** The basic expression that we are going to consider is the following $f(x) = x^2$ and we are going to consider it as the basic of the quadratic functions. To know it, we are going to tabulate it, and from there, we start to make its graph. We're going to give it some, and then we'll locate the points. Well, we already have its graph: what we observe is that we can evaluate it in any number, a rational, an integer, an irrational, right? We can start by applying the properties that we have already seen, which ones? The ones with the graph handling. For example, let's make graph $f(x) = x^2 + 2$... what does 2 do to it?
>
> **As:** It raises it!
>
> **P:** How much are we going to raise it?
>
> **As:** By 2!
>
> **P:** Very good, two units. I try to make its shape; that it is not the same, it is not important, what we are trying to understand is that what 2 does is to raise the graph by two units.

C (Gpo-1), p. 8.

First, the function is tabulated, and once the students recognise it, the teacher leads them to use what he calls *a graphing technique*. As noted, the vertex is a reference point that varies when a number is added.

In this excerpt, we have the teacher's explanation when solving a task: "Graph the functions when the parameters $f(x) = sen\,x, f(x) = sen2x$, and $f(x) = 2sen\,x$ vary, and find some differences and similarities between them."

Extract 5.26

P: What would the function look like? Let's see, the first function is $f(x) = 2senx$, right? Well, it says here that twice the sine 0 is 0; twice the $\pi/2$ is 2 , and twice the sine in π is 0. If you remember the function that we outlined at the beginning, the sine function, we find that the first was of this style, right, where they only walked between 1 and -1. If you look, it was -1 y 1; that was the oscillation that the range values took. You are going to see what happens to the double: it doubled. In other words, the function doubled; if it were triple, then the function would be three times more; if it is 4 or 5 times, its amplitude increases.

Am: Is it a wave, teacher?

P: It's a central wave.

Am: Teacher.

P: Well, it just increases the amplitude, what comes out here is more, isn't it? It would be this, sorry, it's like this then, right? Because $\pi/2$ is 1, then it is 1: it increases twice. This is what I wanted you to observe. The other is the double sine function, which says: sine of the double angle, i.e., twice the sine of 90, twice the sine of 0, twice the sine of 180, twice the sine of 270 , or twice the sine of 360, what happens now with the function? What about the function $f(x) = sen\,2x$?

Am: Where it was a frequency now are two frequencies.

A (Gpo-1), p. 60-61.

Its position begins with the basic function $f(x) = senx$, which ranges from – 1 to 1, taking it as a reference. Subsequently, it attends to the change it undergoes when it is multiplied by 2: it doubles. A student, seeing the shape of the graph, asks, *"Is it a wave?"* to know if he should go on observing the wave. The teacher replies that it is a central wave.

The last version is $f(x) = sen\ 2x$, sine of the double angle, notes the teacher, and asks the following question: *"What happens to the function* $f(x) = sen\ 2x$*?"* One student answers, *"Where it was a frequency, now they are two frequencies."* Although the teacher knows that such an answer is correct, he does not immediately validate it and proposes more values to evaluate the function and arrive at the desired answer. He tries to generalise his explanation, saying that the graph will have some behaviour, according to the parameters. If the function multiplied a 2, it would double, and if it multiplied a 3, it would triple, and so on, since the value that will multiply the function will modify its domain and range: the parameters work as a whole. The basic function $f(x) = sen\ x$ plays an important role in the graphing of the functions. The teacher's explanation of the variation was built under the graphical representation model by manipulating the parameters, making the graph changes visible. In the teachers' discourse, explanations that show their notions about the variation in terms of parameters (rotated, translated, and raised) are identified. Graphics such as translation, inclination, rotation, displacement, raise or lower were assigned a geometric meaning.

Thus, we have two types of explanation of the variation: the algebraic (constants) and geometric representation models (parameters). These episodes show how the teacher's explanation keeps in the margins of the pedagogical contract (D'Amore, 1999). His speech has a pattern: "I explain about graphs and validate with tables". The teacher's message through his actions is captured by the students through the regularity of his appearance and will appropriate it as a response strategy.

This is how the phenomenon of ageing (Brousseau, 1986) also takes into account the "classroom habit" (Soto & Cantoral, 2014) and not only the effect of the transposition of knowledge. The space of meanings in which teachers and students coexist is built in the school culture. They bring for their dialogue aspects of previous experiences and, above all, the students' and the teacher's experiences in more general areas.

CONCLUSIONS

We identified a diversity of perspectives within a pattern of teachers' explanations (about the notion of function and their ideas about variation, such as that of parameters –rotate, translate- or the assignment of a geometric meaning to functions: translation, inclination, rotation, displacement, raise or lower, increase or decrease). They attributed additional notions of movement to the graphs through their reference points such as vertex, origin, or asymptote (through expressions such as displace, raise or lower, travel, move or scroll).

We consider that the strategy of moving a reference point (the vertex, the origin or the asymptote) was of great importance for teachers to construct their explanations around the movement of the graph and, thus, they emphasised the role of the notion of variation. To elaborate their explanations, they used primitive functions, such as $f(x) = x, f(x) = x^2,\ f(x) = x^3$ and $f(x) = sen\ x, ...$

During the classes, we recorded types of explanation, models, in which the teachers' notion of variation, the representations or models they used to explain content can be seen:

- The numerical
- The geometric representation
- The algebraic
- The natural language

These ways of explaining the notion of variation in the classroom are created under the discourse built by both the teacher and his students, taking into account the specificity of the knowledge at stake but regulated at once by the dME as hegemonic. According to the socioepistemological theory, the learning situation generates a series of interactions that make the communication and the exchange of ideas functional. Thus, the episodes analysed in the classroom are closely linked with the search for a satisfactory explanation for the actors of the didactic interaction. Let us say that the teacher produces a message Mi (Message from teacher *i*), which is altered with the first doubt or reflective expression of the student Ai. Let's say Mi → Ai → Mi', however, the modified message Mi' retains the structure of Mi but changed the communicative discourse. For example, it uses the daily metaphor to "give a communicative sense."

In that network of meanings, what emerges is a normative explanation for both teacher and student. $M \to A \to M' \to A' \to \cdots$ Let us say that the content of the message itself is modified, but also the roles of the interaction. In the analysis carried out alone, we have three teachers, approximately 110 students and two contents (function and derivative), which produce three$\times 110 \times 2 = 660$ episodes. We only chose some of them for this analysis, the variation with specific representations and a selection of the interactions. We chose the most relevant ones to testify facts that theory explains through phenomena, such as the ageing of situations and the hegemony of the dME.

In the situations analysed, it is notorious that they occur based on a set of relationships between teacher, students, and knowledge with an objective in the framework of school work. These relationships are of interest to the study because they are the basis of explanations that bring knowledge. As we said before, the teacher explains to the student based on the *millieu*, or more strongly, based on his hegemonic reason system, by making use of each of the elements available at the given time, as was the case of a "half-orange", "move the wave", "raise the vertex," etc. This fact is a notable feature of the *phenomenon* since it draws on the resources available to the teacher and the students when modifying the explanation, which is changed to the extent that it is used.

The explanations are altered based on the interaction, fostered by a search for complementarity between the students' and the teacher's versions. The teacher's interventions of double function, requesting explanations and trying to guide them, regulate the course of the class. The teaching situation is modified by the explanations provided.

The notion of variation, which is the centre of study, is strongly supported by the numerical variation, and the most rescued model was the table of values, but in all cases based on different forms of knowledge. We insist that these resources are not part of what the mathematical content of the classroom typically considers; instead, they are the resources with which the aulic explanation is amalgamated. The function is neither the table nor the formula, the graph, the daily metaphors, but an arbitrary correspondence relationship known as Dirichlet's definition. Its representation is not the concept itself, but it is how the teacher makes it appear in class and how it is validated in textbooks. However, metaphors to cultural know-how (know-how as knowledge put into use), metaphors to everyday life, are indispensable to amalgamate them, are part of the message. It is the metaphor located where the explanation of the classroom sits.

Regarding the didactic phenomenon, specifically of ageing of the teaching situations, the teachers constantly interact with the students by using the know-how explained, or the didactic explanation. This occurs most intensely when they deal with the notion of variation, which is not the explicit object of teaching, insofar as it is neither introduced into the classroom through a definition nor is it treated in the books explicitly, since it is not a categorical characterisation. It appears together with the daily use language that provides the context; it varies according to physics or engineering and is translated into the game of formal and non-formal discourses. This fact makes ageing faster since the number of eventualities is greater than when categorical concepts are introduced.

We believe that generalisation -or overgeneralisation-, analogy, systematic repetition, reformulation, and crossing of languages are factors that contribute the most to the phenomenon based on the explanations. When the teacher, as we saw in one of the episodes, tries to generalise saying, "depending on the degree of the polynomial, [we will know how many] times it will cross axis x," he is making an incorrect generalisation that will cause problems for the students if they take them to the letter because soon they will find counterexamples; this requires permanent changes in the teacher's explanations.

This study on the role of explanation in mathematics class tries to locate, analyse, and explain how the phenomenon of ageing of teaching situations develops. These passages described above show that the origin of the change in the teacher's explanation can be multiple and very complex. For example, the use of an invalid generalisation, the exhaustive repetition of an argument or the combination of languages induce factors of change in the teacher's discourse and particularly in his explanations. At the theoretical level, there are the roots of ageing.

When the teacher requests that the students express their opinion through questions, he motivates explanatory interventions, and it is of great interest for the students to be able to "move" or "displace" the vertex from its initial position. The teacher tries to generalise, saying that if they add a quantity to the basic function, the graph moves up, and if they subtract it from the function, it moves down. The student must share this reference system. By analysing explanation in class (school or extended), aspects of the classroom discourse provided neither by textbooks nor by the DME are revealed.

AUTHORSHIP CONTRIBUTION STATEMENT

ERB and RC have contributed to the study considerably. ERB was responsible for studying the discourse in the classroom, both the transcription and the identification of the categorical elements. RC made a final review of the contributions and their adaptability to the theoretical approach. Both decided on the structure and content of each section.

DATA AVAILABILITY STATEMENT

The data supporting this research will be made available under reasonable request. ERB is responsible for data custody.

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