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Some Examples of the Phenomenon of Metadidactic Slippage in School Practice

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ABSTRACT

Background: In didactics of mathematics, the problem of metadidactic slippage (*glissement metadidactique*) evidenced by Guy Brousseau has been shown for decades. But the school didactic practice proposes behavioural models (mathematics teaching-learning) from which it is manifest that the subject is completely unknown. **Objectives:** This article intends to present and discuss the metadidactic slippage problem and give some negative examples of its influence, in particular, about the naive interpretation of the so-called Pólya heuristic regarding problem solving in mathematics. **Design:** Theoretical research in didactics of mathematics. **Setting and participants:** focuses on the school didactic practice of problem solving in mathematics. **Data collection and analysis:** Negative examples chosen from among those most diffused in the school world are analysed in the light of modern didactics of mathematics to identify metadidactic slippage in them. **Results:** Thanks to the slippage, the student learns a scheme, or an algorithm, not the desired mathematical topic T, which remains a mystery to the student (and sometimes also to the teacher). **Conclusions:** Before trying to "improve" the teaching-learning of mathematics with temporary and drastic measures, it is better, at least, to study it modestly.

Keywords: Metadidactic slip, Problem solving, Polya heuristics.

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Algunos ejemplos del fenómeno del deslizamiento metadidáctico en la práctica escolar

RESUMEN

Contexto: En Didáctica de la Matemática se ha demostrado durante décadas el problema del deslizamiento metadidáctico (*glissement metadidactique*) evidenciado por Guy Brousseau. Pero la práctica didáctica escolar propone modelos de comportamiento (enseñanza – aprendizaje de la Matemática) desde los cuales se evidencia que el tema es del todo desconocido. **Objetivo:** este artículo tiene la intención de presentar y discutir el problema del deslizamiento metadidáctico y dar algunos ejemplos negativos de su influencia, en particular en lo que respecta a la interpretación ingenua de la llamada heurística de Pólya relativa a la resolución de problemas de Matemática. **Design:** Investigación teórica en Didáctica de la Matemática. **Entorno y participantes**: se centra en la práctica didáctica escolar de la resolución de problemas de Matemática. **Recogida y análisis de datos:** Ejemplos negativos elegidos entre los de mayor difusión en el mundo escolar son analizados a la luz de la moderna Didáctica de la Matemática para poder identificar deslizamientos metadidácticos en ellos. **Resultados:** Gracias al deslizamiento, el alumno aprende un esquema, o un algoritmo, no el tema matemático T deseado, que sigue siendo un misterio para el alumno (y en ocasiones también para el profesor). **Conclusiones**: Antes de pretender "mejorar" la enseñanza – aprendizaje de la Matemática con medidas coyunturales y drásticas, es mejor, como mínimo, estudiarla con humildad.

Palavras-chave: deslizamiento metadidáctico, resolución de problemas, heurística de Polya.

POLYA AND TROUBLESHOOTING

The informative, non-scientific production of the great Hungarian American mathematician George Polya (1887 - 1985) was developed between 1945 and 1967. It consists of two famous books translated into several languages: 1. How to Solve It; 2. Mathematics of Plausible Reasoning Volume I: Induction and Analogy in Mathematics; Mathematics of Plausible Reasoning Volume II: Patterns of Plausible Reasoning. (Polya, 1945/1967, 1954). In these informative books, Polya illustrated and demonstrated to the public his personal way of facing and solving problems, a brilliant technique admired by all those mathematicians who have appreciated his excellent results in probability, number theory, combinatorial calculus and in the study of some particular series. Valuable, for the knowledge of Polya's work, is his extensive and learned posthumous memoir written by the American mathematician Ralph Philip Boas (1912 - 1992), who also co-authored with Polya (Boas, 1990).

This type of analysis is not unique in the world of mathematics, on the contrary, it follows, as it were, a tradition. For example, in 1910, the French psychiatrist and journalist Édouard Toulouse (1865 - 1947) published a famous book in which he narrates the analyses he carried out at the late 19th - early 20th century on Henri Poincaré (1854 - 1912), one of the most brilliant mathematical creators of all history (Toulouse, 1910), after having observed his way of working for a long time and having discussed with him his work habits and his modes of creative thinking. From this book emerges a mathematical Poincaré - human being, which moves away from the stereotype of the mathematician, from multiple points of view (D'Amore & Sbaragli, 2020).

Unlike what has been written about Poincaré and Hadamard, the narration of Polya's methods and the public confession of how he achieved his results became, for some readers of the time, a kind of "general methodology of problem solving," something like a "successful heuristic" that, with superficial considerations, was advertised as a modality to use in the classroom. The internal and personal "rules," which Polya brilliantly and generously enumerates and describes with examples, were naively regarded as a blueprint worthy of being followed in the teaching process, with the conviction that learning would be its logical consequence.

In those days, nobody talked about the didactics of mathematics. As a discipline, it had not been created yet. Guy Brousseau began to conceive it precisely at the end of the 60s, continuing throughout the 70s, and ending with the creation of a true theory of mathematics learning at the end of the 80s.

Now, what Polya wanted to suggest to his readers was and still is very clear today, based on his words: to present himself as a model, since it was a successful model, and propose its stages as examples that anyone could follow.

Today, although the history of those highly effective personal instruments, such as in Polya's case, is considered great historical and psychological interest, no one would dare to believe them scientifically suitable for studying the didactics of mathematics with direct application in the classroom. Perhaps this could come to the mind of those who have not studied at all -or have done so poorly- the didactics of mathematics. Polya's instruments are usually acclaimed or cited favourably by those who do not know what has happened in recent decades, thanks to the didactics of mathematics and the research that has been developed within them. At the cost of repeating ourselves, therefore, we reiterate that an eventual citation of Polya from a historical or perhaps psychological perspective may be interesting, but certainly not from a didactic point of view, as we will show in the following paragraphs.

Before addressing this specific argument, we must present one of the research issues that the didactics of mathematics discipline has faced in recent decades.

THE (NEGATIVE) PHENOMENON OF METADIDACTIC SLIPPAGE

The use in the didactic practice of heuristic systems elevated to models that replace the learning of mathematics with the learning of an analogy, as algorithmic and sequential as possible, is located in the study of a negative and counterproductive phenomenon evidenced by the research properly framed in didactics of mathematics that is included under the name of "metadidactic slippage." However, the teachers themselves sometimes encourage this widespread and dangerous phenomenon.

This phenomenon occurs when one goes from the study of a mathematics topic T, which should constitute a learning object, to the study of instruments that could only serve to the maximum or to illustrate the topic T or to face the resolution of a related problem with that topic T, as a banal scheme and not as true learning (which would imply, as a logical consequence, the correct, appropriate, and general resolution of problems related to topic T). But, if the slippage is successful, the student learns a scheme, or an algorithm, or a generalised example, not the topic T. Some teachers (when they do not know the results of the didactics of mathematics) confuse these two levels, accepting in good faith the situation that appears superficially as positive. Sometimes they even create it themselves and propose it in the classroom, trusting in the suggestions of the "experts," and, therefore, a perfect illusion is created: everyone is satisfied. But the mathematical subject T remains a mystery for the student (and sometimes also for the teacher).

To better understand the situation, we suggest some examples chosen from among those most widely used in the school world.

1. We consider problems of this type, with a great presence in the school world around the world: «3 workers do a certain job in 9 hours. But if there are 6 workers who do the same job, how many hours of work are required to do it? ». This is a proportion with an unknown term. a: b = c: d.

To understand and consequently consciously solve these types of problems, a graphical mechanism known throughout the world as the "rule of 3" was devised a long time ago. This model transforms the arithmetic formulation into a graph, and this seems to make the problem solving more effective. But, as has happened and happens in all countries, after a while, there is no further reference to either the problem or the issue of proportionality, only the graph is mentioned. Learning to use the rule of 3 replaces what was originally the true learning object: knowing and knowing how to use the mathematical object "proportions."

The student learns to handle and use this graph (with arrows that have concordant or discordant meanings). And even if he/she can find the result of that proposed problem, he/she does not learn to solve the problem or similar problems because he/she has not learned the idea of proportionality. He/she only manages to figure out the correct way to place the arrows. If he/she forgets the rule of 3 or if he/she makes a mistake in the placement of the arrows, he/she will not be able to solve these types of problems: the student does not reason, he/she looks for the rule, the algorithm. So much so that if the unknown term is not c but d, the student often does not know what to do, that is, how to place the arrows.

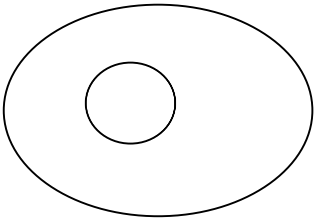
2. Another disastrous example was verified with the arrival in the classrooms of naive set theory in the 70s and 80s as a consequence of an overestimated idea of some mathematicians of a certain prestige, with good intentions, but with little relation to the teaching-learning problems. After a few years, the representation of objects in set theory was introduced into the world of school, and then that is how the use of circles or ellipses was thought to indicate them. After a short time, the set theory study was left aside, and theorising about how to draw and use graphs began, transforming the entire learning process from the naive set theory into the mastery of this purely graphical, low-level activity.

Therefore, if the students learned something, they did not learn logic as the basic language of mathematics, which was the initial objective, they learned to master the drawing of graphs. Another metadidactic slippage. Fortunately, the disorder that followed all this served to eliminate this useless mathematical content from the mathematics study programs, also thanks to the intervention of other mathematicians of equal prestige, such as René Thom (1970/1980, 1973) and Morris Kline (1973) (D'Amore narrates all this in detail, 1999).

We are sure that everyone can understand, by intuition, without having to study a whole… theory of circles, that: if the objects a are only a part of the objects of b (for example, all squares are rhombuses), then it is possible to use as a graphical representation a pair of circles arranged as follows (A represents the set of objects a, B represents the set of objects b) (Figure 1).

Figure 1

A represents the set of objects a, B represents the set of objects b



This ridiculous phenomenon shows that, sometimes, to solve a difficulty, sometimes a little one, a cascade of pseudo-didactic procedures takes place that can lead to useless activities that little by little become an uncontrollable monster.

3. The so-called “tests” of the results of operations, algorithmic mechanisms to verify the correct result of said operations. Everyone knows that these are useless algorithms since they guarantee absolutely nothing. For example, the “proof by nine” of the following multiplication: 137 × 24 = 2271, tells us that the result of the multiplication is correct, against all evidence. The only possible “test” would be to perform (correctly) the division 2271 ÷ 24 and thus verify that the quotient is 137, a modality that allows confirming some learning: multiplication and division operations are one the inverse of the other (Naturally, this “test” can be misleading).

4. The technique of division between fractions. Everyone knows that to perform the division a / b ÷ c / d, the multiplication a / b × d / c (b, c, d ≠ 0) must be performed. But very seldom is the reason for this “rule” explained in the classroom; students and teachers take refuge in the metadidactic slide. In fact, the request is often explicit: “It should be done like this” or “It is enough to do it like this”: this is all there is to know, it is what the teacher expects of his students. In our experience, almost no teacher demonstrated knowing the answer to the spontaneous question: Why?

5. Knowing how to add is a strong point of elementary school, but sometimes it becomes a plurality of algorithms with no other explanation beyond being an instrument and not as knowledge. Not only must the student learn to calculate the result of addition, for example, to be able to answer a problem, but he/she must do so with various algorithmic instrumental

modalities: in the column, “horizontally”, mentally, in the abacus, on the “number line.”

The object of knowledge ceases to be the algorithm of addition, to become the learning of a set of modalities that have little to do with the mathematical meaning of the operation itself. And so, the meaning of calculating the result of an addition in the process of solving a problem is distorted by metadidactic slippage, and the real problem for those trying to solve it is knowing how to perform the addition in so many different ways. The student lose the sense of problem solving and transforms his/her own activity into algorithmic executions.

6. If a number must be multiplied by 10 or by 100, the calculations must not be made, one or two zeros must be added respectively after the figure that occupies the units place, after the last figure. Not only is it not clear why, but it becomes problematic as soon as the multiplicand is not a natural number, but a rational number written both with the comma and as a fraction. All teachers know it. Knowledge becomes a pseudo-algorithmic rule whose applicability is not dominated by everyone. And when it also comes to the analogous rules for division, the negative results of these metadidactic slippages become evident to all teachers.

7. A mathematical object is indicated with a symbol, it can be with a graph (a drawing, a diagram,...); then you stop thinking about the initial abstract mathematical object and everything is relegated to the graph itself. For example, you define a straight angle (plane). (The amplitude of an angle is seldom defined and is presented as intuitive. Usually, it is the angle measure and not the amplitude measure). Instead of clarifying what an angle is, from a mathematical point of view, and that the amplitude is a measure, they limit themselves to drawing an arc a little distant from the vertex, an arc that goes from one side to the other; tendingly, it is made in such a way that it is an arc of circumference (which is why it is called an “arc”) that has the center at the vertex of the angle and any measure of the radius.

This arc sometimes indicates the angle, in others the amplitude. At this point, the angle object is forgotten, and the arch is studied. So much so that there are university students who believe that the angle is the arc and not a part of the plane (referring to the most recurrent definition of angle) (Sbaragli, 2005); and that the length of the arc measures the width of the angle, with the consequence that, depending on where the arc is drawn specifically, the measure of the width of said angle changes. In this case, metadidactic slippage is cognitively dangerous, but most teachers do not realise it.

8. The positional writing of the numerals represents a death trap for the mastery of cognitive aspects, especially due to metadidactic slippage. If Natalia owns 123 marbles, no one doubts that she has 123 units, where each unit is a marble. Therefore, in the numeral 123, there are 123 units. It seems obvious. But if Natalia (for personal reasons) decides to group the marbles into boxes in groups of ten by ten, she would have 12 boxes, and in each one, she would have a dozen marbles, plus 3 loose marbles. Now, Natalia then has 12 dozen marbles. Natalia decides (always for personal reasons) to group the boxes - tens in a larger box, collecting the tens ten by ten; she will be able to collect only 10 tens that she will collect in a box that obviously contains 100 marbles, that is 10 tens, that is one (1) hundred. [Our positional system of writing numbers is called decimal precisely because it is put together ten by ten to move to the top-level grouping: unit → tens → hundreds → thousands (units of a thousand)].

But 2 of these dozen boxes are left outside the large container. Therefore, at this point, Natalia has one (1) hundred marbles, plus 2 dozen marbles, plus 3 loose marbles. No one doubts that she continues to have 123 marbles, that is, 123 units; no one doubts that she has 12 tens plus 3 loose marbles. It should be said that in numeral 123 the figures 3, 2, and 1 represent the values that appear in the unit "places", tens, hundreds of the numeral 123: with greater precision 3 indicates the figure that appears in the units place, 2 indicates the number that appears in the tens place, 1 indicates the number that appears in the hundreds place. It would be all very simple.

But here, the metadidactic slippage is triggered when it is intended to force the student that in numeral 123 "there are": 1 hundred (which coincidentally is correct) 2 tens (which is false because the tens are 12), 3 units (which is also false because the units are 123). The study of the mathematical object "positional writing" is left aside, dealing with this metadidactic slippage, pretending that students learn to tell a falsehood. To be sure that the error occurs in all cases and that it constitutes a heavy burden, colours are sometimes assigned to the writing of the figures in each of the positions: the units are in red, the tens in colour yellow and the hundreds in green (we are inventing this wrong and harmful chromaticism because we do not know whether there is already an agreement to present this nefarious activity).

And so, the correct arithmetic meaning of the mathematical object "positional writing" is no longer explained, and ends up moving to a chromatic writing ... that forces students to use coloured pencils when they must write the numerals, effectively cancelling about 7000 years of history and research. The advantage of positional writing, one of the greatest inventions of humanity in its long history, is precisely the fact that the same number, depending on the POSITION it has within the numeral, acquires a different value; while here all this is annulled shamefully, and a positional writing is not obtained but a CHROMATIC one. Instead of saying “decimal positional system,” it should be called “decimal colour system”.

Sometimes the teacher uses the abacus; but, in the abacus, in the 123 “writing” 123 disc-drives do not appear, in total, only six appear, but their arrangement (1 in the hundreds column, the third from the left; 2 in the tens column ; 3 in the units column), not the colours, is what determines the value. Thus, the abacus, when set up as a model and used correctly, contradicts the results of this metadidactic slippage. If the student is asked: “How many tens are there in 123?”, Many teachers give the wrong answer “2” instead of the correct answer “12.” Even comforted by the fact that the number 2 has been written in yellow. Which explains the negative results found in the responses to international evaluation tests.

We should not consider that examples of metadidactic slippage are present only in elementary and middle schools. We limit ourselves to proposing just one among the many examples that are also found in the early years of high school.

9. The so-called Ruffini rule, famous in the early years of high school in various countries.

The student is studying polynomials and must know how to perform the easy division (2x3-3x2-5x-2) ÷ (x-2), which would lead to the quotient 2x2 + 3x + 1. This topic constitutes an excellent argument for mathematical knowledge. But, in general, he/she are not taught how to do the division, which among other things, is an algorithm that does not offer difficulties. On the contrary, he/she is taught a scheme formed by all the coefficients in play that are placed in a particular table and in a given order.

Learning is no longer the division between polynomials, and it becomes how to organise the coefficients in this table and how to use them. It is this mechanism that replaces learning, the one that textbooks and teachers are waiting for, an evident metadidactic slippage because of which significant knowledge that it would be important to possess is lost.

10. Analogous examples spread throughout the world are constituted by the use of specific tables to perform logarithmic calculations, today eliminated thanks to the widespread use of calculators and computer programs. Calculations were also found to show that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$;

The objective of these equalities (and many similar ones) was related to the fact that the trigonometric tables contained the values of the trigonometric functions of the angles between 0 ° and 90 ° and therefore to calculate, for example, sin110 °, it was necessary to consider sin110 ° as sin (90 ° + 20 °). And so, instead of the study of trigonometry, these infernal and useless algorithmic rules were studied. All this ended thanks to the introduction of the calculator and computer programs. But, we wonder: did these instruments finally lead to the study of true trigonometry as a theory and not as a set of rules?

We stop here, but we could continue with many other examples in each of the domains of mathematics and at all school levels.

CONCLUSIONS

Knowledge and know-how form a pair of metaknowledge with mutual reciprocal influences. Knowledge is the implicit means to activate and manage the know-how. Know-how is the institutional and cultural instrument that allows us to learn knowledge, both our own and those of others. Wanting to treat them in a univocal way, in particular, to think of knowledge as know-how, constitutes a permanent metadidactic slippage. To be effective, each knowledge implicit in a know-how requires new knowledge, which, once established, cannot be considered as such. Errors, misunderstandings, failures that repeat impossible demands and ineffective practices result. From an economic perspective, the knowledge available in the classroom is the capital and the interests are the know-hows acquired; the subtle and uncertain game of living, doubtful and fleeting knowledge with the safe and shared know-how is implicit, the game of the said and the unsaid.

Before trying to "improve" it with short-term and drastic measures, it is better, at the very least, to study it without boasting about it.

AUTHORS' STATEMENTS OF CONTRIBUTION

The authors discussed the theoretical framework to contribute to the production of this article and participated collectively in its construction.

DATA AVAILABILITY STATEMENT

Data supporting the results of this study will be made available by the authors for correspondence, upon reasonable request.

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