ISSN: 2178-7727
DOI: 10.17648/acta.scientiae.6737

Early-Career Mathematics Teachers' Knowledge in the Multiplicative Conceptual Field

Rafael Neves Almeida[a]
Ruy Cesar Pietropaolo[b]

[a] Universidade Federal de Sergipe, Departamento de Matemática, Centro *Campus* Professor Alberto Carvalho, Itabaiana, SE, Brasil
[b] Universidade Anhanguera de São Paulo, Programa de Pós-graduação em Educação Matemática, São Paulo, SP, Brasil

Received for publication 13 Jul. 2021. Accepted after review 27 Mar. 2022
Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Background: Operations with natural numbers are highlighted in the first five years of elementary school, according to the prescribed and practised curricula. According to the recently prescribed curricula, the mathematics teacher should return to this theme in the 6th grade so that the students consolidate and expand this knowledge. This fact can constitute a great challenge for the teacher who may not have had adequate training to teach it, ignoring the work done in the initial years. **Objectives**: to investigate mathematics teachers' didactic and curriculum knowledge about teaching problems involving multiplication and division - the multiplicative conceptual field, according to the National Curriculum Parameters and Vergnaud. **Design**: the principles of a qualitative study carried out. **Setting and Participants:** five mathematics beginning teachers, ex-scholarship holders of the Institutional Scholarship for Teaching Initiation Program (Pibid) of the Degree in Mathematics at the Federal University of Sergipe. **Data collection and analysis**: Data collection took place through interviews and protocols answered by teachers during the interviews; regarding beginning teachers, the study had as theoretical references the works of Huberman and Garcia and that of Ball, Thames, and Phelps about the necessary teachers' knowledge for teaching. **Results**: It was possible to conclude that the teachers in the process of teaching the multiplicative conceptual field did not master the didactic knowledge essential for teaching operations. Therefore, it is necessary to expand the teachers' knowledge base for teaching operations.

Keywords: Early career teachers; Mathematical knowledge for teaching; Multiplication conceptual field; Pibid

Corresponding author: Rafael Neves Almeida. Email: rafael@mat.ufs.br

Conhecimentos de Professores de Matemática em Início de Carreira Sobre o Campo Multiplicativo

RESUMO

Contexto: As operações com números naturais têm grande destaque nos cinco primeiros anos do ensino fundamental, segundo os currículos prescritos e praticados. O professor de Matemática, segundo os recentes currículos prescritos, deverá retomar esse tema no 6.º ano de modo que os alunos consolidem e ampliem esses conhecimentos. Esse fato pode se constituir em um grande desafio para esse docente que pode não ter tido uma formação adequada para esse ensino e também desconhecimento do trabalho realizado nos anos iniciais. **Objetivos**: investigar os Conhecimentos Didáticos e Curriculares de professores de Matemática sobre o ensino de problemas envolvendo a multiplicação e divisão – o campo multiplicativo, segundo os Parâmetros Curriculares Nacionais e Vergnaud. **Design**: observou-se os princípios de um estudo qualitativo. **Cenário e Participantes:** cinco professores de Matemática, em início de carreira, ex-bolsistas do Programa Institucional de Bolsa de Iniciação à Docência (Pibid) do curso de Licenciatura em Matemática de um campus da Universidade Federal de Sergipe. **Coleta e análise de dados**: A coleta de dados se deu por meio de entrevistas e protocolos respondidos pelos professores durante as entrevistas; a análise teve como referenciais teóricos, no tocante a professores em início de carreira, os trabalhos de Huberman e Garcia e o de Ball, Thames e Phelps relativamente aos conhecimentos de professores necessários à docência. **Resultados**: Foi possível concluir que os professores no processo de ensino do campo multiplicativo não dominavam conhecimentos didáticos essenciais para o ensino das operações. Portanto, faz-se necessário ampliação da base de conhecimentos desses docentes para o ensino das operações.

Palavras-chave: Professores em Início de Carreira; Conhecimento Matemáticos para o Ensino; Campo Multiplicativo; Pibid.

INTRODUCTION

Concerning the topic of Operations, the National Curricular Parameters (Brasil, 1998) and the National Common Core Curriculum - BNCC (Brasil, 2018) expect that students graduating from elementary school will solve problem situations with natural, integer, and rational numbers that involve different meanings of the four fundamental operations, through different strategies, including the exact calculation or by estimation, written or mental calculation, understanding the processes involved in them.

Teaching and learning operations with natural numbers occupy a good part of the time teachers dedicate to mathematics from the 1st to the 5th grades of elementary school. However, that task is not exhausted in that period. For this reason, the teaching of operations is usually prescribed for the 6th grade too. The PCNs' indications (Brasil, 1998) for the 3rd cycle (former 5th and 6th grades) and BNCC's (Brasil, 2018) for the 6th grade attest to this fact, as presented below.

> For the study of the contents presented in the Numbers and Operations block, teachers must propose problem situations that allow students to develop the numerical sense and the meanings of the operations. (Brasil, 1998, p. 66)

> Regarding elementary school - final years (middle school), the expectation is that students solve problems with natural, integer, and rational numbers, involving fundamental operations, with their different meanings, and using different strategies, understanding the processes involved. (Brasil, 2018, p. 269)

These guidelines justify our purpose of investigating the didactic and curriculum knowledge of mathematics teachers about the teaching of problems involving multiplication and division - the multiplicative conceptual field- because, in the 6th grade, which we call here the year of transition for students between two stages of elementary education, the mathematics teacher must resume and deepen the teaching of operations so that their students consolidate and expand skills related to the theme.

Therefore, didactic and curricular knowledge about teaching operations with natural numbers is necessary for the middle-school mathematics teacher. However, our experiences with the mathematics teachers' initial and continuing education say that they usually know little about how to teach those contents. However, did teachers who experienced innovative activities in their initial education, such as practice as a curricular component and the Institutional Scholarship Program for Teaching Initiation – Pibid – develop skills to teach this topic, especially different meanings of operations?

For this reason, this study presents the results of our investigation of the didactic and curricular knowledge of five mathematics teachers, participants of Pibid, graduates from a federal university, on problems students of the early years solved in the multiplicative conceptual field[1].

---

[1] The multiplicative conceptual field involves different meanings of multiplication or division. In another article, we discussed the didactic knowledge of mathematics

We chose a group of early-career mathematics teachers because, besides the curricular innovations they experienced in their degrees, it is a consensus that they must face challenges, such as developing a repertoire to build a teaching practice.

THEORETICAL FRAMEWORK

About early career teachers

The beginning of a teacher's career is a decisive phase for constructing their professional identity. In that phase, they identify their peers' roles, values, and attitudes in the profession and develop a self-image as a teacher, establishing meanings for the situations they experience (Oliveira, 2004).

For Huberman (1995), the beginning of the teaching career is a process that involves regressions and discontinuities, including two moments that can occur concomitantly: survival and discovery. Survival means the struggle to overcome problems intrinsic to the school institution when the teacher experiences the shock of reality. This expression, evidenced by Veenman (1988), refers to the differences between the teachers' expectations before initial training and their experiences when they begin teaching. The moment of discovery can be determined by the pleasure and enthusiasm in organising or creating specific learning situations for their students or when conducting their classes. The feelings experienced in the discovery contribute to developing attitudes and energy to overcome the obstacles inherent to the profession – survival.

Garcia (1998), when debating teacher training, says that the beginning of teaching is a time characterised by tensions and intensive learning. He considers this a very relevant phase, during which teachers build and acquire fundamental knowledge to develop professional competencies. Garcia (2010), Mariano (2006, 2012), Mizukami (2004), and Nono and Mizukami (2006) discuss the teachers' professional development as an ongoing process that begins before the prospective teachers graduate.

According to Ponte, Galvão, Trigo-Santos, and Oliveira (2001), the initial period of teaching is marked by several difficulties that could be grouped as follows: concerning students, such as indiscipline and lack of motivation;

---

teachers at the beginning of their careers for teaching the different meanings of addition and subtraction.

concerning working conditions, such as the excessive number of classes and the lack of teaching materials; concerning the inadequacies of professional knowledge, such as not having available a repertoire of strategies to teach what they are expected to teach.

Professional knowledge for teaching has characteristics that guide and regulate the professional practice. According to Ponte et al. (2001),

> The teacher's professional knowledge is decisive for the performance in their professional activity. This knowledge has numerous facets and dimensions, guiding and regulating the professional practice. It is situated knowledge (as, indeed, all knowledge) and, therefore, closely linked to the teacher's context. It is largely implicit knowledge, marked by a set of images, conceptions, and values that determine its fundamental structure. (p. 3)

Our experience in teacher education allows us to observe a fundamental fact: more experienced mathematics teachers, who can choose classes first, prefer to teach more advanced grades, perhaps because they feel inadequate to teach children and pre-adolescents. Therefore, in general, those who take more mathematics classes in the 6th grade are not the most experienced teachers.

Mathematics teachers working in the 6th grade – transition year – have an extra problem because, to plan their classes, they must know the curriculum of the initial years, identify students' abilities, and assess specific difficulties in relation to the topics that they will teach, and learn about specific strategies, especially operations.

Thus, the authors mentioned above can justify our choice to investigate a group of early career teachers about their teaching knowledge.

About Teacher Knowledge

We share with Shulman (1986) and Ball, Thames, and Phelps (2008) that teachers lack the training to develop knowledge about the content they will teach. Therefore, we consider in this article the categories discussed by those researchers on the knowledge the teacher should master.

According to Shulman (1986), investigations on teaching did not take into account questions related to teachers' justifications for the approaches adopted in their classes and on the situations and metaphors chosen. The absence of those questions proved that the researchers did not consider relevant research on the teaching of specific contents of the disciplines, which Shulman called a lost paradigm.

The knowledge base for teaching adopted by Shulman (1986) is composed of three categories of knowledge considered fundamental for the teacher: content knowledge, pedagogical content knowledge, and curriculum knowledge.

Ball et al. (2008) reinterpreted Shulman's (1986) ideas in the field of mathematics and refined their categories. These authors brought significant contributions to discussing the knowledge necessary for teaching mathematics: mathematical knowledge for teaching (MKT).

Shulman's (1986) content knowledge was separated into common knowledge, specialist knowledge, and horizontal knowledge. Regarding pedagogical content knowledge, Ball et al. (2008) indicate the subcategories: knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum.

Common content knowledge is necessary for the mathematics teacher, but it is not exclusive to him/her. The teacher must understand the concepts and procedures to teach, master the situations and tasks proposed to students, and use notations and representations correctly.

Specialised content knowledge is closely associated with teaching practice and is distinct from the mathematical knowledge required in other professions. It consists of the ability not only to point out students' errors but to analyse and identify their probable causes and give students convincing justifications. This category includes knowledge about different procedures and reasoning to solve a problem and the formulation of questions that lead students to relate their knowledge to new facts, concepts, and procedures that will be addressed.

> This category of knowledge also includes the skills necessary for proposing works to students, classifying them, confronting different strategies and solutions, and identifying lines of reasoning that would be mathematically correct (or not) or that would always work (or not). These are specific requirements of the teacher's job – they are not necessary, for example, for a person trying to solve an everyday situation. They are not necessary knowledge for the teacher because they must be

> taught to the students. They are necessary for the teacher to perform his teaching role effectively. (Corbo, 2012, p. 4)

Horizontal content knowledge deals with how content is connected, enabling the teacher to choose how to teach a concept or procedure to contribute to the formation of a basis for the study of other topics in the future.

The knowledge of content and students associates the understanding of mathematics with the understanding of the students' mathematical thinking - arising from experience - allowing the teacher to predict and interpret typical errors and search for strategies to overcome them.

Moreover, knowledge of content and teaching must articulate the understanding of mathematical content with pedagogical topics that can intervene in the teaching and learning processes. It concerns the consideration of the possibilities of successful learning when choosing strategies to approach a concept, selecting representations, contexts, problem situations, and examples. Finally, we have content and curriculum knowledge that is linked to knowing the curriculum and how the content to be taught at the moment fits into it.

Ball et al. (2008) consider that the boundaries between the proposed categories are fine lines that allow different explanations regarding the knowledge necessary for mathematics teaching. Despite this, how can we ignore questions such as:

> Where, for example, do teachers develop explicit and fluent use of mathematical notation? Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept? Do they learn definitions for fractions and compare their utility? Where do they learn what constitutes a good mathematical explanation? Do they learn why 1 is not considered prime, or how and why the long division algorithm works? Teachers must know these sorts of things, and engage in these mathematical practices themselves in order to teach and they must also learn to teach them to students. Explicit knowledge and skill in these areas is vital for teaching. (Ball et al., 2008, p.12)

Thus, the option of basing our investigation on the ideas defended by Shulman (1986) about the knowledge needed for teaching, in general, and by Ball et al. (2008) on the knowledge needed specifically for the teaching of mathematics, is justified by the interest of our study to investigate the knowledge the mathematics teacher must have to teach the multiplicative conceptual field.

About the multiplicative conceptual field

About Numbers and Operations, recently prescribed curricula (Brasil, 1998; Brasil, 2018) expect that, upon reaching the 6th grade of elementary school, students solve problems with natural and decimal numbers involving different meanings of operations, argue and justify the procedures used, and check the plausibility of the results found. However, several studies indicate that students still do not have this mastery at that stage.

The references to the National Curricular Parameters (Brasil, 1998) in this text are justified because, during the development of this research, the BNCC was still under construction, and no preliminary version of the document had yet been released. However, although those two documents have different natures, the BNCC reiterates the importance of learning the operations given by the PCNs for the 6th grade. When consulting the documents that guide mathematics teaching in the state of Sergipe - the PCN and the Curricular Reference of the State Teaching Network of Sergipe (Sergipe, 2011) -, we can notice the vital role of operations, whose study begins in the 1st grade and extends to the 7th grade of elementary school.

The PCNs (1998) emphasise how important it is that the teacher explores the different meanings of operations in other contexts in the final years of elementary school. This document indicates for the 3rd cycle (6th and 7th grades):

> Analysis, interpretation, formulation, and resolution of problem situations, comprising different meanings of operations, involving natural, integer, and rational numbers, recognition that different problem situations can be solved by a single operation and that eventually different operations can solve the same problem. (p. 51)

The PCN explain, in the "Orientações Didáticas" [Didactic Guidelines] section, the expression **additive field,** for addition and subtraction operations; and the expression **multiplicative conceptual field,** for multiplication and division, although the meanings are not clearly discussed. However, the justification for this adoption can be identified in the following text.

> The development of research in didactics of mathematics brings new references to the treatment of operations. Some of them indicate the additive and subtractive problems as a first aspect to be addressed in class, concomitantly with the work of construction of the meaning of natural numbers. The justification for working together on additive and subtractive problems is based on the fact that they compose the same family, i.e., there are close connections between additive and subtractive situations. [...] As in the case of addition and subtraction, the importance of working together on problems that explore multiplication and division is highlighted, since there are close connections between the situations that involve them and the need to work on those operations based on in a wider field of meanings than has usually been realised. (Brasil, 1997, p. 69-72)

This adoption is based on Vergnaud's (1983) theory of conceptual fields. For this researcher, a conceptual field is a grouping of problems whose progressive domain presupposes concepts, procedures, and symbolic representations in close connection. From this perspective, the construction of a concept involves a series of sets: a set of situations, which give meaning to the object in question; a set of invariants, which indicate properties and procedures necessary to define this object; and a set of symbolic representations, which allow associating the meaning of the object to the respective properties. A mathematical concept is constructed articulated with other concepts, through a series of corrections and generalisations. Thus, we can say that the student builds a field of concepts that makes sense in a field of problems, and not an isolated concept in response to a particular problem.

Regarding the multiplicative conceptual field, Vergnaud (1983) considers that

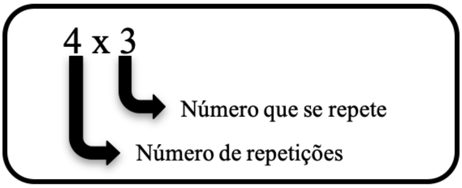
> the conceptual field of multiplicative structures is, at the same time, the set of situations whose treatment implies one or more multiplications or divisions, and the set of concepts and theorems that allow analysing those situations: simple proportions and multiple proportions, linear and n- linear function, direct and inverse scalar ratio, quotient and product of dimensions, linear combination and linear application, fraction, ratio, rational number, multiple and divisor, etc. (p.127)

Regarding multiplication and division, the PCN emphasise the importance of exploring situations that allow students to understand the following meanings: addition of equal parts, combinatorics, rectangular configuration and comparison of ratios that cover the idea of proportionality. We will describe each of them in more detail.

The first is multiplication as the addition of equal parts. In this case, we associate the writing 4×3 to the following addition $3 + 3 + 3 + 3$. (Figure 1).

Figure 1

Addition of equal parts



When we take multiplication as the addition of equal parts, we need to be careful, since the operations 7×2 and 2×7, despite presenting the same numerical result, can be interpreted differently. Let us see the following situation: "Joseph needs to take care of his sick son. The doctor prescribed 2 tablets a day for 7 days. How many pills will his child need to take?" Which of the two writings would be the most appropriate?

Regarding the combinatorial meaning of the multiplicative conceptual field, we refer to situations in which the fundamental principle of counting is applied. Problems of this type can be solved initially without direct calculation but with double-entry tables or tree diagrams.

Exploring situations that involve multiplication with the meaning of comparison of ratios – which includes proportionality – favours the development of proportional thinking, i.e., finding proportionality constants and solving different problems. See the example: "João bought 3.5 kg of tapioca for R$ 21.00. How much will he pay for 7 kg?" The meanings of the division with actions associated with "apportion equally" and "determine how much fits" are related to proportionality.

Another meaning of the multiplicative conceptual field appears in problem situations, whose data can be represented by a rectangular configuration, as in the example: "In a room, there are 20 chairs in rows. Each row has the same number of chairs. Knowing that there are 4 rows, what is the number of chairs per row?" Problems of this type are relevant to developing the concept of the area of a rectangular surface.

RESEARCH SETUP

This article presents the description and analysis of qualitative research, as attributed by Bogdan and Biklen (1999), on a group of beginning teachers' didactic and curriculum knowledge about the teaching of the multiplicative conceptual field.

We reiterate that this study was carried out with the collaboration of five mathematics teachers from the public network, all graduates of the mathematics degree course of the same *campus* from a federal public university in the state of Sergipe. All those teachers participated in the Pibid during graduation. The teachers were between 25 and 31 years old, taught in the final grades of elementary public schools in the countryside of Sergipe and had no more than four years of career. For this study and to preserve the participants' identity, we gave them numbers from 1 to 5.

The data exposed in this text come from individual interviews, recorded in audio and video, and questions that they answered in writing. The interviews allowed us to clarify doubts and justifications for their written answers (which we call protocols) and identify each one's didactic and curricular knowledge about the topic under study. This study was approved by the Research Ethics Committee (CEP) and registered on Plataforma Brasil under the Certificate of Presentation of Ethical Appreciation (CAAE) No. 23543113.4.0000.5546.

We proposed to the participants to analyse how the 5th graders of elementary school solved the problems of the multiplicative conceptual field. The students' problems and answers the teachers should analyse were based on the 2015 Pedagogical Report of the School Performance Assessment System of the State of São Paulo – SARESP (São Paulo, 2015). We iterate that our purpose was to investigate the mathematics teachers' didactic and curriculum knowledge about the teaching of problems involving multiplication and division, the multiplicative conceptual field.

In this text, we discuss the analysis of the resolutions of only two of the proposed problems of the multiplicative conceptual field, involving the meanings of division and comparison of ratios, as we believe that the teachers' answers are sufficient for the reader to understand the participating teachers' didactic knowledge about the theme.

DATA ANALYSIS ON THE MULTIPLICATIVE CONCEPTUAL FIELD

In this topic, we analysed the teachers' knowledge about operations, especially the multiplicative conceptual field, in the context of the comparison of ratios, in which the notion of proportionality is inserted. In this way, we observed the teachers' answers to problems 1 and 2 present in the protocols, which allowed us to investigate the teachers' knowledge of this content.

By solving the problem in different ways and analysing the students' answers, we hoped that the teachers would draw on their knowledge base about the meanings of multiplication and division.

We started by analysing the protocols of Problem 1 (see Figure 2).

Figure 2

The first question proposed to teachers (Protocols)

Uma professora propôs as seguintes situações-problemas para seus alunos do 5º ano do EF:

Situação A)
Ana tem 24 bombons e deseja reparti-los igualmente para quatro crianças. Quantos bombons cada criança deverá receber?
Situação B)
Paula tem 24 bombons e deseja guardá-los em caixas de modo que cada caixa tenha exatamente quatro bombons. Quantas caixas serão necessárias para guardar todos os bombons?

A professora percebeu que 82% dos seus alunos acertaram o problema A, ao passo que apenas 34% acertaram o problema B. Você saberia explicar a razão (ou razões) desses índices? Explique.
Explique como você resolveria concretamente (ou seja, por meio de desenhos) esses dois problemas. As ações que descrevem ou resolvem cada problema são iguais? Explique.

For this question, as seen above, teachers were asked to analyse the discrepancy observed in the hit rates of situations A and B. However, the teachers were unable to explain the probable reason for this divergence.

We noticed that Teacher 1 could not explain why the students had different rates of hits in each question.

> *The first problem... Well, perhaps, the explanation, I still can't say, talk about the indices, how I would answer.* (Teacher 1)

Teachers 3 and 4 also did not explain why students got one of the questions right more than they did with the other: for them, the problems are equivalent.

> *80%, 82% got problem A right, and 34, only 34 got problem B? So it's pretty similar, right? It's the same problem actually, only written differently. You have 24 chocolates you want to divide equally into 4, it is a division, how many chocolate bonbons should each child receive? The other, Paula has 24 bonbons and wants to store them in boxes so that each box has exactly 4 bonbons. How many boxes will be needed to store all the bonbons? We eventually return to the same problem, the question is that the form of the question here is more direct [referring to question A], here it is more implicit [referring to question B], the question, I believe that the students found it more difficult. Because one is more direct and the other is not.* (Teacher 3)

> *Is that not the same sort of thing? No, Ana has 24 bonbons and wants to divide them equally into 4, equally for four children, it's true, it's not the same thing, is it? For me, it would be the same.* (Teacher 4)

Perhaps this misguided way of classifying problems as equivalent originated from the most widely used procedure to solve the problem, which is the division algorithm. As Teacher 2 expressed:

> *The solving method is the same.* (Teacher 2)

Teacher 5 even noticed a greater difficulty in situation B, but could not explain it. He demonstrates interpreting from the same perspective as Teacher 3, by stating that situation A is written in a way that makes it easier for the student to interpret it.

> *Looking at it as a teacher, it seems that problem A is clearer, in the sense of thinking like this, "how many bonbons will each child receive?" It is logical that if you had 4 children, 24 divides by 4, in the student's mind this count is easier, now let us go to letter B, "how many boxes will be needed to store all the bonbons", I think it is easier intuitively you interpreting the problem, I think it is a matter of intuition, of interpreting the problem, it is easier for the student to interpret the A than the*

> *B, now, I don't know why, I am saying this according to myself.* (Teacher 5)

This speech seems to show that the teacher perceives differences between the resolution procedures, which may indicate that he does not distinguish the meanings involved in the problems.

All teachers solved the questions mentally, interpreted the two problems and chose a correct method to find their solution. This knowledge, for Ball et al. (2008), is the so-called common content knowledge. However, they were unable to identify the difference between the meaning of the division involved in situation A and the meaning of situation B: they did not explain, for example, that in the first situation, the number of bonbons in each of the four groups (children) should be calculated and in the second, the number of bonbons was given, but the number of groups (boxes) was not known. None of them explicitly identified the two meanings present: equitable distribution (A) and measure or quota (B), demonstrating that for the division, they do not master the specialised content knowledge that the teacher who teaches mathematics needs.

Wanting them to elucidate the difference between the two meanings, we suggested that teachers respond to situations in other ways other than calculating. We hoped that through pictorial representation, they would identify the difference between the meanings or, at least, amplified their previous answers. A part of their resolutions is written in the protocols, and another part is what they stated in an audible voice so that we were able to capture and later transcribe.

For situation A, Teacher 1 drew the four children and handed them the bonbons, one by one (see Figure 3).

> *Then, [he draws the four children and begins apportioning the candies]. Then, he starts counting, 1, 2, 3, 4. Until getting to 24. 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21. 22, 23, 24. Thus, problem A could be solved or presented in this way.* (Teacher 1)

When proceeding to find the solution to situation B (see Figure 4), he realises that the resolution method is not the same; it is not distribution. Through the drawings, the teacher identifies the difference between the actions of the two situations, although he does not fully justify it.

Figure 3

Resolution - Teacher 1 - Situation A, Problem 1 (Protocols)

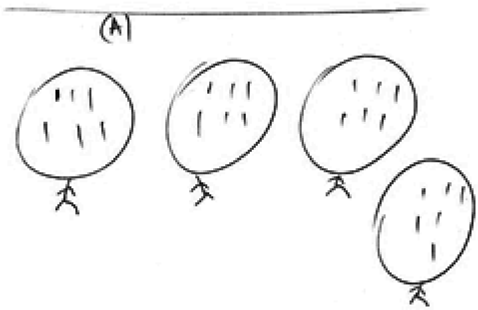
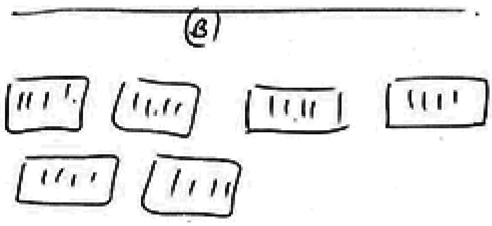


Figure 4

Resolution - Teacher 1 - Situation B, Problem 1 (Protocols)



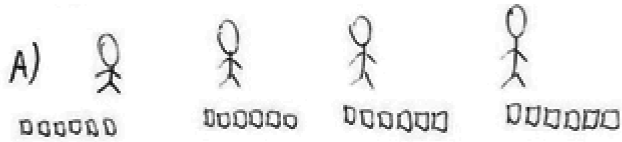
> *Now, problem B is as if the question were like this: how many children will be covered, the issue of the boxes, it is as if it were going to give... I want to keep the bonbons in boxes, and each box must have 4 bonbons for each child, how many will I need? It is rewriting the problem. Then we could do the counting. A box, four bonbons. The other box, four, to do this sum until it reaches 24. Then, in a problem with a higher number, we would give an example with a smaller number, to follow the example. There is a difference in the way of solving it. Here [item a] the number of children is already established, I will distribute the bonbons and here [item b] will be the box, but I don't know the reason for this difference.* (Teacher 1)

Teacher 2 solved the same way as teacher 1 (see Figure 5).

> *For the letter A, I would draw children. Dividing into 4 children, I put the 4 children and I wento on distributing the bonbons. For the children from letter A, that would be it!* (Teacher 2)

Figure 5

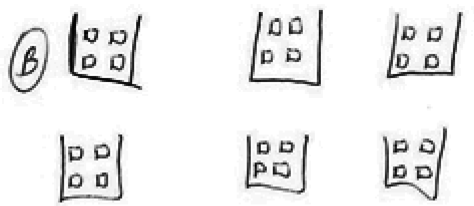
Resolution - Teacher 2 - Situation A, Problem 1 (Protocols)



The resolution of situation B is shown in Figure 6, and Teacher 2 adopted different reasoning from the previous one.

Figure 6

Resolution - Teacher 2 - Situation B, Problem 1 (Protocols)



This teacher grouped chocolates four at a time until he reached 24. He solved the two problems in different ways, but, as he cannot perceive this, he claims that the letters A and B would be the same thing.

> *With the letter B, it would be the same thing! When explaining [it] to them [the students], what would I put? Four in each, I would put four and add four, four, until reaching 24*. (Teacher 2)

Teacher 3 solved situation A through an equitable distribution (Figure 7), as shown in the following fragment.

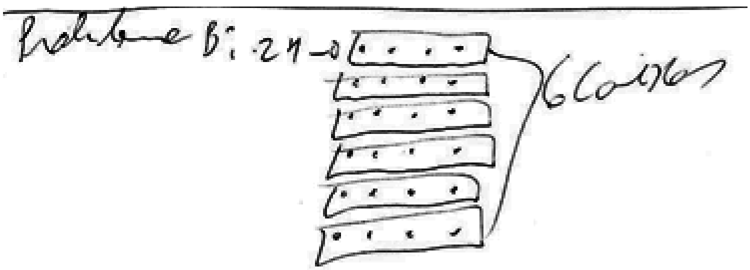
I would solve it like this: The A, I would have 24 bonbons... and they will be divided by 4, so it would be 1,2,3,4,5,6, so each child will receive 6 bonbons. For every six bonbons, I would have a child. (Teacher 3)

For him, situation B is equivalent, as it is enough to put four bonbons in each box and then count the number of boxes.

Problem B is the same, similar, only that here you would have 24 bonbons. I'm going to put 4 bonbons in each box, I put a box. So, 6 boxes would be used, more or less like this, in an explanatory way. (Teacher 3)

Figure 7

Resolution - Teacher 3 - Situation B, Problem 1 (Protocols)



Although teacher 4 considers the two situations equivalent, he indicated in his statement that the students would have more difficulties in the second but did not explain why. To solve situation A, he did not make drawings, he just justified his answer by talking and gesturing:

24 bonbons and [you] want to share them equally among four children. Ready! I have those 24 bonbons, I have 4 children here and I would start distributing the bonbons to the children. Then it would be 6 for each, it's six, right? (Teacher 4)

When giving his answer to situation B, Teacher 4 reaffirmed that the questions are equivalent:

That second one is the same thing, it only changes that they will be kept in boxes, no, and here [problem A] they will be shared among 4 children, ok, there will be 6 for each. Paula has 24 *bonbons and wants to store them in boxes so that each box has four bonbons, how many boxes? 6.* (Teacher 4)

For him, answering directly, i.e., using the division algorithm, would be the way he would solve situation B:

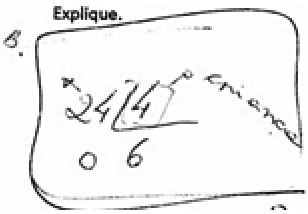
> *I would answer directly! Now, to explain to the students? You have to make the 24 bonbons. I used to do it like this, in fours.* (Teacher 4)

Teacher 4's drawings are very similar to those of Teachers 1, 2, and 3.

Teacher 5 presents the division algorithm (Figure 8). He perceives a difference between the meanings of the answers, but he does not identify that situation A is about equitable distribution and that situation B is about measurement (how many fits).

Figure 8

Resolution - Teacher 5 - Situation A and B of problem 1 (Protocols)



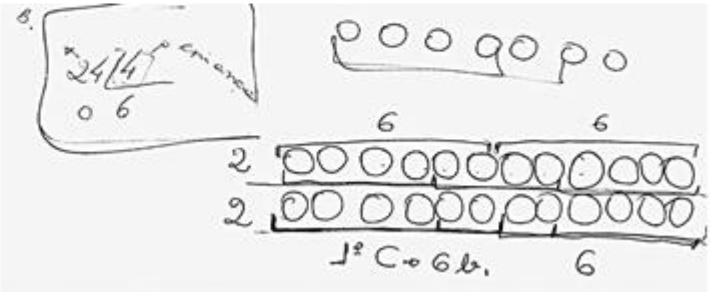
He described his resolution thus:

> *I would do the straight calculation, 24 by 4, without worries, in the letter A, 24 wants to divide [them] into 4 children, how many bonbons, 24 divided by 4, that's it! You understand, I would do the division calmly, now explaining to the student that the 24 here is the number of bobons and this 4 here is the number of children; in the letter B this 4, despite being the same number, but not represents the same thing, this 4 in the letter B represents the number of boxes, of boxes, not of bonbons, right? See that in the letter A, 4 children, in the letter B, 4 bonbons, the counting will be the same, what will be different here is what each number means.* (Teacher 5)

Teacher 5's answer in the protocol contains the same drawing to explain problems A and B. He drew two rows of marbles, each with 12 marbles (see Figure 9).

Figure 9

Resolution - Teacher 5 - Situations A and B, problem 1 (Protocols)



We can see that, for situation A, Teacher 5 separates four groups of six bonbons. Let us just see the lines on top of the marbles in the first row. Here is how he described his answer:

> *[You] draw 24 bonbons, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 [twelve on top and the other twelve on the bottom], and with 24 bonbons, we would make groups of 4 children. Ah, yes, 4 children, as I made two parts of 12, so here goes half of the children, take this half of the sweets, this other half would go with the other half of the children, as I have 4 children, here there would be two children, here, there would be two children, here divided by two, so, as I have 12 two children, it would be 6 for each, then, 1, 2, 3, 4, 5, 6, the first child would have 6 bonbons, and then the second child 6 more, the third child 6 more, the fourth child 6 more, I would draw it like that.* (Teacher 5)

Regarding situation B, he replied, using the same scheme he had used in the previous question, only this time he makes traces at the bottom, grouping them in fours.

> *The second, "Paula has 24 bonbons and wants to keep them, 4 in each box". She has 4 bonbons, this one would maybe be a little easier with the drawings I said. At this point, it becomes*

more difficult to interpret, but here it is easier to divide because I already know that there must be 4. Here, I would have to divide first to later find out that there were 6 in each, here, I already know that each one has 4, so here the, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, then 1, 2, 3, 4, 5, 6, 6 boxes. (Teacher 5)

Teacher 5 also used different reasoning for each question but did not show signs of having identified the various meanings of the questions.

In summary, all five teachers, despite indicating that the two situations of Problem 1 involved different actions, did not assertively identify the two meanings of the division.

Problem 2 (Figure 10) deals with an issue covering the comparison of ratios, which we analyse below.

Figure 10

The fourth question proposed to teachers (Protocols)

1. O professor Domingos propôs o seguinte problema para alunos de diferentes séries:

   > Em uma caixa há 5 bombons de caramelos e 13 de chocolate. Uma outra caixa tem 100 bombons de caramelo. Quantos bombons de chocolate devemos colocar para que se tenha a mesma proporção da primeira?

   a) Primeiro, você deverá resolver esse problema. Se possível, apresente diferentes maneiras de resolvê-lo.
   b) Esse problema envolve qual(is) noção(ões) matemática(s)?
   c) Esse problema pode ser proposto para alunos a partir de quais anos do Ensino Fundamental?
   d) Analise agora as resoluções de quatro alunos do professor Domingos, apresentadas na folha que será entregue.

Besides the questions in Figure 11, the following were also proposed:

Figure 11

Complementing the fourth question proposed to teachers (Protocols)

As resoluções dos alunos:

a. Analise as resoluções desse problema por 4 alunos do professor Domingos. Todos eles chegaram à resposta correta: 260. Mas, estão corretos todos os procedimentos utilizados? Explique.

b. Como você avalia cada uma dessas resoluções? Para facilitar essa avaliação, atribua pontos em uma escala de 0 a 10.

Figure 12 illustrates the students’ answers to Teacher Domingos’ question.

Figure 12

Students’ resolution proposed for problem 2 (Protocols)

Aluno	Resolução
André	5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100 13 26 39 52 65 78 91 104 117 130 143 156 169 182 195 208 221 234 247 260 260
Diego	100 : 15 = 20 (resto 00) 13 × 20 = 260
Júlia	13 : 15 = 2,6 (30, 0) 2,6 × 100 = 260
Beto	5 —— 13 100 —— x $x = \frac{13 \times 100}{5} = 260$

Problem 2 deals with the comparison of ratios. The teachers were asked to solve it in different ways and, later, to analyse the solutions of the four students to this problem (Figure 12).

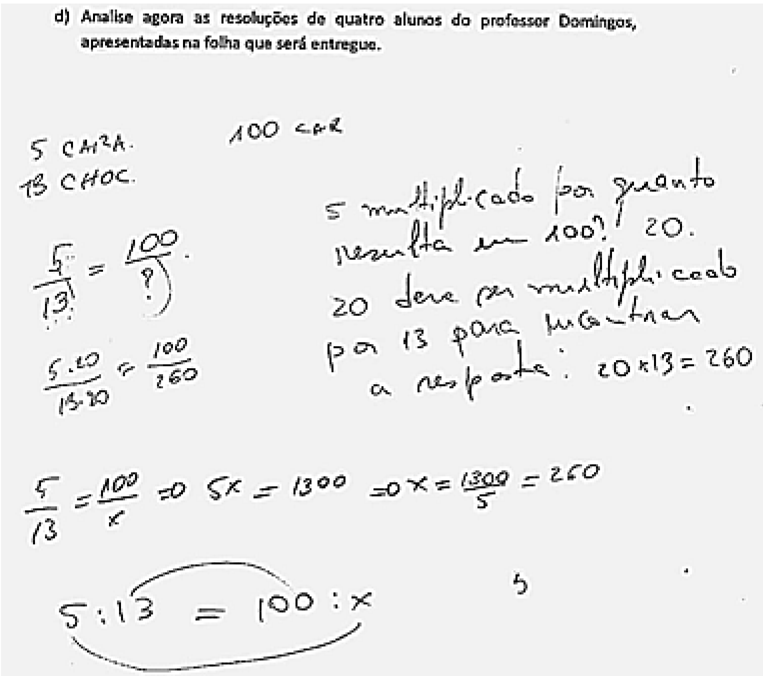
All five teachers answered that they could use the rule of three as a procedure to find the correct solution to the problem, but only four solved and recorded their answers in other ways, too, using the proportionality constant, equivalent fractions, and percentage.

Here we present copies of the protocols, selected according to the main points that can evidence the teachers’ content knowledge and excerpts from the testimonies that describe or reinforce what is registered in the protocols.

Teacher 1 (Figure 13) indicated three different methods to solve the problem. He used equivalent fractions for the first. For him, “5 is to 13, as 100 is to an unknown value”; therefore, for the fractions to be equivalent, he would have to find a number that multiplied by 5 would give 100. Finding this number, it would be enough to multiply it by 13 and find the answer to the problem. In addition to the answer common to all, he also showed, in the last line, the outline of a resolution, in which he made the traditional “extremes and means”, identifying the antecedent and consequent terms without performing the operations.

Figure 13

Resolution - Teacher 1 - Problem 2 (Protocols)



d) Analise agora as resoluções de quatro alunos do professor Domingos, apresentadas na folha que será entregue.

5 CAR A. 100 CAR

13 CHOC.

5 multiplicado por quanto resulta em 100? 20.
20 deve ser multiplicado por 13 para encontrar a resposta: $20 \times 13 = 260$

$$\frac{5}{13} = \frac{100}{?}$$

$$\frac{5 \cdot 20}{13 \cdot 20} = \frac{100}{260}$$

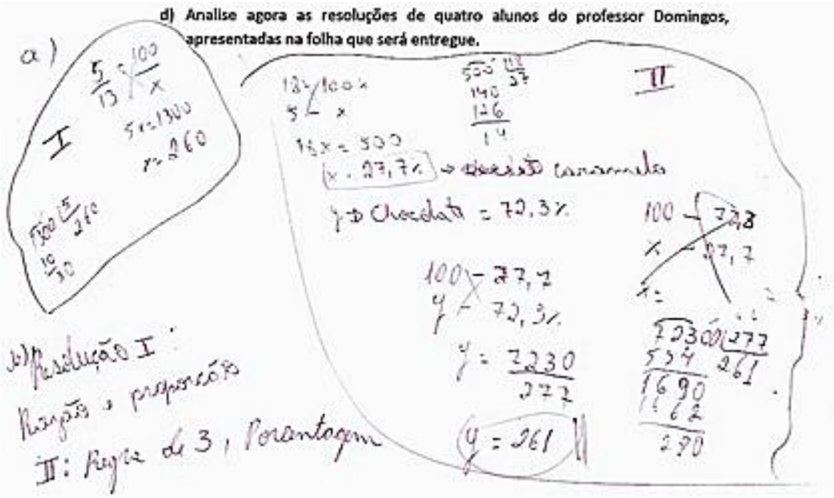
$$\frac{5}{13} = \frac{100}{x} \Rightarrow 5x = 1300 \Rightarrow x = \frac{1300}{5} = 260$$

$$5 : 13 = 100 : x$$

Teacher 2 exposed two ways to solve the problem. For him, the two boxes must contain the same percentage of caramel and chocolate bonbons to keep the proportion. So, he identifies what that percentage was in the first box and uses the rule of three to find how many bonbons would be in the second box. Note that he found 261 bonbons, an error arising from the approximation made to the percentage.

Figure 14

Resolution - Teacher 2 - Problem 2 (Protocols)



Professor 2 thus commented on his resolutions:

> *I solved it in two ways, one was simpler, it was using the ratio of the bonbons, the ratio of the caramel bonbons over chocolate, the other reason also caramel over chocolate, being that you don't know what chocolate is, I did extremes and means there, it was easy, the other one I did in a more complicated way, I was using the rule of three of percentage, but it didn't come out as exact as item a, it was more approximate, it was 261. From the first, which is simpler, is ratio and proportion, you can easily solve it, in the second, what must you use? Rule of three and percentage.* (Teacher 2)

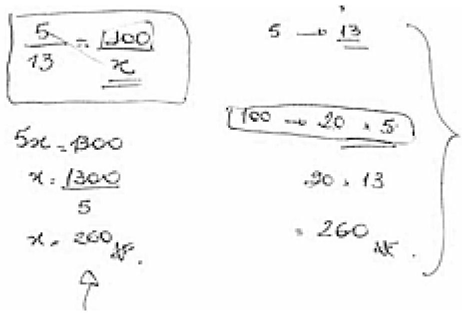
Teacher 3 used proportional reasoning and verbalised that he could also solve by percentage.

> *Yeah, I used logic, 5 caramel bonbons for 100, it's the question of multiplying, I multiplied it by 20, to give 100, 13, I multiplied the numerator, the top one by 20, so, to keep the proportion, I multiply it by 20 too, so, it will give 260, I kept the proportion. I could solve it another way using a... putting an unknown here.* (Teacher 3)

Teacher 4 presented only one way to solve the problem: the rule of three, and Teacher 5 solved the problem by using the constant of proportionality in addition to the rule of three.

Figure 15

Resolution - Teacher 5 - Problem 2 (Protocols)



The teachers' answers reveal that all of them, except for Teacher 4, are aware of more than one type of solution for this problem.

The participants' answers to question c) of problem 2, "From which grades of elementary school can this problem be proposed?" were divergent, as shown in Figure 16.

Figure 16

Answer about the grade in which problem 2 could be proposed.

	Problema 2
	Ano
Professor 1	6.º ano
Professor 2	8.º ano
Professor 3	7.º ano
Professor 4	5.º ano
Professor 5	6.º ano

With this difference in answers, we can see that there is little knowledge of the mathematics curriculum for elementary school about the notion of proportionality that defends the introduction of this type of situation from the 4th grade, but with smaller numbers. None of the participants mentioned the recommendations contained in the curriculum documents on this subject from the first years of elementary school to its formalisation in the 6th grade.

Next, we discuss how teachers analysed students' resolutions, shown in Figure 11. In Figure 17, we have the marks attributed by the teachers to the resolutions proposed by four students.

Figure 17

Grades teachers gave students for the resolution of problem 2.

	Problema 2			
	André	Diego	Júlia	Beto
Professor 1	10	10	10	10
Professor 2	7	8	8	10
Professor 3	10	0	0	10
Professor 4	6	7	7	8
Professor 5	8	10	10	10

A reading of those data reveals immediately the divergence between the marks given to Diego and Júlia by Teacher 3 and the marks given by the other teachers. In his statement, Teacher 3 reports that he did not find meaning in Diego and Júlia's resolutions. By analysing the resolutions and not just the final answers, the teacher said he would consider that only André's and Beto's answers were correct.

> *Look, this one! [referring to André] 10, 15, 20... several equivalent fractions, you can see that he was multiplying by*

two by two... the most important thing is that he got it. Here he [Diego] divided the proportion always like this, it was 20 then multiplied it by 13, here he divided the 13 by 5, multiplied by 100... I believe that Diego and Julia are not right about the procedures. In this case, this answer is already proven. I would consider André and Beto, that they used the logic of keeping the proportion of both terms. Now, here, Diego, for example, added the 100, which was the value, the 100 caramels, with the 5 caramel bobons, then he made a relation between caramel bonbons and caramel bonbons, from that ratio and multiplied by the amount of chocolate bonbons, I did not see logic not here. Maybe I'm wrong, but I would only consider André's and Beto's answers as correct. The others, I would consider [them]wrong. (Teacher 3)

Based on his testimony, we can deduce that Teacher 3 did not understand the resolutions elaborated by Diego and Júlia.

The justifications by Teacher 2 clearly indicate that he had difficulties in analysing the reasoning used by Diego and Júlia.

If you observe here, Beto has used it in a simpler and easier way to understand, which was what I did here, which is by ratio and proportion, Diego and Júlia did it in a similar way, but not so simple to understand, if they were students mine I was going to call them and ask them to explain how they got this here. André was a lot more laborious, he was taking [what], the multiples of 5 and 13, until he reached the number of exact numbers up to 100, and seeing how many were the multiples of 13.

All so right here, only that who would the simplest and easiest here? Beto! That's why I gave him 10 too.

As André had a lot of work, I gave 7, and Diego and Júlia were similar, 8, but they would have to explain to me how they did it here. (Teacher 2)

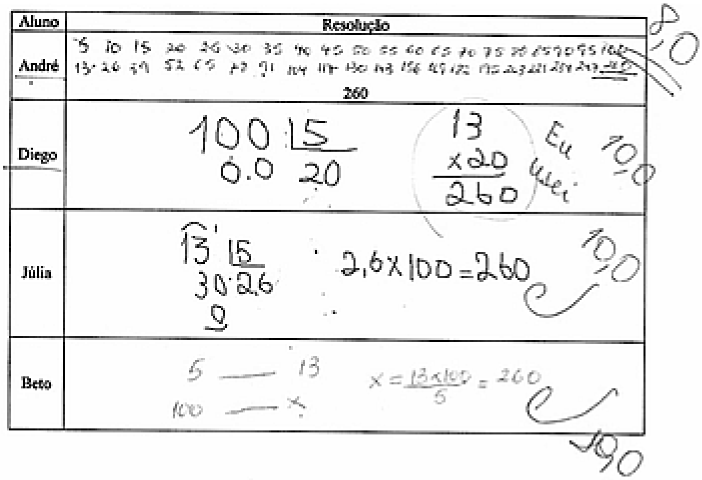
Teacher 4, likewise, seems not to have understood Julia and Diego's resolutions. André's resolution was understood, but the lack of a conventional answer was strange, as there was no "math counting", as shown in the excerpt

He [André] puts it in 5 by 5 until it reaches 100, out of 13, puts it in 13 by 13...correct, this one, this one, and this one is right, because he puts it in 5 by 5 then it gets to 100 because it's 5 in a box and 100 in the other, then one box had 13, how many chocolates were in the other box to be proportional, this one makes sense too... But those here [Júlia's and Diego's] were more, they did not present any calculations here... It's bad to determine like that without knowing who these students are. In Beto's case, we have that he answered the way I did, so I gave him an extra point. André's resolution, it is not wrong. But, it's because, like, he did more by deduction, I don't know, he didn't do any math. (Teacher 4)

Teacher 5 only did not give André the maximum grade (Figure 18).

Figure 18

Notes - Teacher 5 - Problem 2 (Protocols)



In his statement, Teacher 5 says that he seeks to understand the student's thinking better. He can understand how the students may have thought, how everyone got the correct answers, and finally gave everyone a top grade.

Until he finds out that the proportionality constant exists, that when multiplied, it will give the same value, but that's ok too,

> *now let me see this one, this one here is the student who just learned to do too much math, let me see, how it went that he used this method, 5, 10, 15, 20, 25, 1, 2, 3, 4, 5, 6, 7, 8, 20, hum, yes, he did it, André did the same method as Diego, only Diego did the calculation using the algorithm correctly, André thought the same way, but he didn't use the algorithm, see, 20 that divides by 5, or 100 that divides by 5, 20, that is, he discovered that within 100 he has 20 portions of 5 and that this should happen the same way, with 13 he multiplied 20 times, only that, instead of it multiplying algorithmically, he added 20 times the amount 13 until reaching 260.* (Teacher 5)

The teacher justifies having given a grade 10 to all as follows:

> *I am more driven by whether he was right or if he was wrong, I always thought that way, because as I worked in scientific initiation with error analysis and also in the master's dissertation, we also worked with the issue of interpretation, reading, and meaning, and not if the student got it right or if he got it wrong. That's why I said I would give everyone a 10 if it was a matter of hits and misses, it only decreases in this one [pointing to André's answer] because of the way he did it.* (Teacher 5)

There seems to be an inconsistency between his willingness to give everyone full marks and what he actually did. Although he says that he seeks to see if the student was right or wrong, André's answer was considered less elaborate reasoning and, consequently, he received a lower grade than the others. This teacher understood all the resolutions, but, as an unconventional strategy was used, he assigned a lower grade, perhaps disregarding official guidelines for proportionality.

We could see that only one teacher assigned the maximum grade to all. Three other participating teachers had difficulties understanding the students' responses.

Those teachers presented common content knowledge, as they used correct reasoning to solve the proposed problems. It was possible to realise that these teachers, except for Teacher 4, know different ways of solving the problem situations presented. However, they seemed to ignore the curriculum knowledge that, according to the prescribed curricula, the notion of proportionality must be present from the early years and that there is a progression of complexity from grade to grade of elementary school.

Finally, given the teachers' difficulties in understanding the different meanings present in the problem situations and the presented resolutions, we could conclude that the teachers did not master the specialised content knowledge and the content knowledge for teaching well enough. Neither did they show knowledge of content and students.

CONCLUDING REMARKS

This investigation involved five early-career mathematics teachers (four years or less in the profession) who participated in the Pibid and graduated at the same *campus* from a federal university. Our purpose was to identify their knowledge for teaching problems in the multiplicative conceptual field – multiplication and division – in the 6th grade, which we call the transition between the two stages of elementary school. Our strategy of asking the teachers to analyse students' problem solving in the multiplicative conceptual field was revealing in assessing the specific and pedagogical knowledge of the content in question.

Based on Ball et al. (2008), despite the teachers' good mathematical training, our observations of their' analysis of the three proposed situations - two meanings of the division and comparison of ratios - point out that they did not master the knowledge required for the teaching of operations. Teachers, for example, did not differentiate the meanings of division as a partition and as a measure (quotas). In addition, the analyses of problem resolutions involving the comparison of ratios were not all adequate. They showed a lack of knowledge of curricular guidelines published since the PCN (1998), such as the importance of teaching the different meanings and, in particular, the idea of proportionality, indicated throughout elementary school, including in the early years when multiplication is introduced.

Therefore, we reiterate that those teachers participating in Pibid, whose initial training included teaching practice as a curricular component, did not develop all the relevant knowledge for teaching operations to 6^{th}-grade students. Our research emphasises how urgent it is to open up discussions in the teaching degree courses and projects, such as the pedagogical internship and Pibid. It is necessary to discuss difficulties children in the early years of schooling find in learning the concepts and procedures related to numbers and operations since they will continue studying the topic in the second stage of elementary school.

AUTHORSHIP CONTRIBUTION STATEMENT

The authors R.N.A. and R.C.P. were responsible for the development and application of the questionnaires and, later, they discussed the results, structured, and wrote the article.

DATA AVAILABILITY STATEMENT

Data supporting the results of this investigation will be made available by the corresponding author, R.N.A., upon reasonable request, via email.

REFERENCES

Ball, D. L., Thames, M. H., & Phelps, G. (2008, November/December). Content knowledge for teaching: what makes it special? *Journal of Teacher Education*, Washington, *59*(5), 389-407.

Bogdan, R., & Biklen, C. (2003). *Investigação qualitativa em educação: uma introdução à teoria e aos métodos*. Porto.

Bogdan, R., Biklen, C. (1999). *Investigação qualitativa em educação: uma introdução à teoria e aos métodos*. Porto.

Brasil. (1997). Ministério da Educação. Secretaria da Educação Básica. *Orientações Parâmetros Curriculares Nacionais (1.ª a 4.ª série): Matemática*. 142 p

Brasil. (1998). Ministério da Educação. Secretaria da Educação Básica. *Orientações Parâmetros Curriculares Nacionais(5.ª a 8.ª série): Matemática*. 148 p.

Brasil. (2018). *Base Nacional Comum Curricular (BNCC). Segunda versão revista*. MEC/CONSED/UNDIME

Corbo, O. (2012). *Um estudo sobre os conhecimentos necessários ao professor de matemática para a exploração de noções concernentes aos números irracionais na educação básica*. Tese de Doutorado em

Educação Matemática, Universidade Bandeirante de São Paulo, São Paulo..

Garcia, C. M. (1998). *Formação de professores - Para uma prática educativa*. (Trad. Isabel Narciso). Porto.

Garcia, C. M. (2010). O professor iniciante, a prática pedagógica e o sentido da experiência. *Revista brasileira de pesquisa sobre formação docente*, *3*(3), 11-49.

Huberman, M. (1995). Professional careers and professional development: Some intersections. In TR Guskey, & M. Huberman (Eds.), *Professional development in education: New paradigms and practices*. (pp. 193- 224). Teachers College Press.

Mariano, A. L. S. (2006) *A construção do início da docência: um olhar a partir das produções da Anped e do Endipe*. UFSCar.

Mariano, A. L. S. (2012) A aprendizagem da docência no início da carreira: Qual política? Quais problemas? *Revista Exitus*, *2*(01), 79-94.

Mizukami, M. G. N. (2004). Aprendizagem da docência: algumas contribuições de LS Shulman. *Educação* (UFSM), *29*(2), 33-50.

Nono, M. A.; & Mizukami, M. G. N. (2006). Processos de formação de professoras iniciantes. *Revista Brasileira de Estudos Pedagógicos, Brasília*, *87*(217), 382-400.

Oliveira, H. (2004). *A construção da identidade profissional de professores de Matemática em início de carreira*. Tese de Doutorado,Universidade de Lisboa, Lisboa, Portugal.

Ponte, J. P., Galvão, C., Trigo-Santos, F., & Oliveira, H. (2001) O inicio da carreira profissional de jovens professores de Matemática e Ciências. *Revista de Educação*, *10*(1), 31-45.

São Paulo (Estado). Secretaria da Educação (SEE). (2015). *Relatório Pedagógico. SARESP 2015*. SEE.

Sergipe. Secretaria de Estado da Educação do Esporte e da Cultura (SEED).(2011). *Referencial Curricular*. SEED.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, *15*, 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*, 1-22.

Veenman, S. (1988). El proceso de llegar a ser profesor: un análisis de la formación inicial. In A. Villa (Coord.), *Perspectivas y problemas de la función docente*. (pp.39-68). Narcea.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. Carpenter, J. Moser, & T. Romberg. *Addition and subtraction. A cognitive perspective*. (pp. 39-59). Lawrence Erlbaum.