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Understanding Ratio Through the Pirie-Kieren Model

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ABSTRACT

Background: Research in educational mathematics has shown that elementary school students poorly understand the concept of ratio due to the difficulties that emerge from its interpretation as a fraction. **Objective**: Therefore, based on the Pirie-Kieren theoretical model, we intend to analyse the comprehension process that appears when students solve tasks on ratio. **Design**: The approach is qualitative, and the research design was a case study. We used the field observation technique. **Context and participants**: The study was done in a primary school in the state of Guerrero, Mexico. The cases involved four students (11-12 years old) enrolled in the 6th grade. **Data collection and analysis**: The data was collected through a questionnaire (task) and an interview. For data analysis, the eight levels of understanding of the theoretical model were used. **Results**: The results indicate that the students do not manage to formalise their comprehension of the concept of ratio because of difficulties in applying mathematical strategies correctly when solving the proposed tasks. **Conclusion**: The results indicate that for students to reach higher levels of comprehension, it is necessary to carry out the dynamic process of repeating between levels as it makes one reflect on what is required to advance comprehension.

Keywords: Educational mathematics; Comprehension; Ratio; Fraction; Basic education.

Compresión del concepto razón a través del modelo de Pirie y Kieren

RESUMEN

Contexto: Investigaciones en Matemática Educativa, han puesto de manifiesto que estudiantes de primaria tienen una deficiente comprensión sobre el

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concepto razón debido a las dificultades que emergen desde su interpretación como fracción. **Objetivo**: Por lo que es necesario analizar el proceso de comprensión fundamentado desde el modelo teórico de Pirie y Kieren, que emerge al resolver tareas sobre el concepto razón. **Diseño**: El enfoque es de corte cualitativo, y el diseño de la investigación fue un estudio de caso. Se usó la técnica de observación de campo. **Contexto y participantes**: El estudio se llevó a cabo en una escuela primaria en el estado de Guerrero-México. Los casos se conformaron por cuatro estudiantes (11 – 12 años) inscritos al sexto grado. **Recolección de datos y análisis**: Los datos se recolectaron a través de un cuestionario (tarea) y una entrevista. Para el análisis de datos se usaron los ocho niveles de comprensión del modelo teórico. **Resultados**: Los resultados indican que los estudiantes no logran formalizar su proceso de comprensión en relación con el concepto de razón, debido a, las dificultades que presentan para aplicar correctamente estrategias matemáticas al resolver las tareas propuestas. **Conclusión**: Los resultados indican que, para alcanzar altos niveles de comprensión, es necesario realizar el proceso dinámico de redoblar entre niveles, puesto que se reflexiona sobre lo necesario para avanzar en el proceso de comprensión mismo.

Palabras claves: Matemática Educativa; Comprensión; Razón; Fracción; Educación básica.

Compressão do conceito de razão através do modelo de Pirie e Kieren

RESUMO

Contexto: Pesquisas em Matemática Educacional mostraram que alunos do ensino fundamental têm uma compreensão deficiente do conceito de razão devido às dificuldades que emergem de sua interpretação como fração. **Objetivo:** Portanto, é necessário analisar o processo de compreensão com base no modelo teórico de Pirie e Kieren, que emerge ao resolver tarefas sobre o conceito de razão. **Desenho:** A abordagem é qualitativa, e o desenho da pesquisa foi um estudo de caso. Foi utilizada a técnica de observação de campo. **Contexto e participantes:** O estudo foi realizado em uma escola primária do estado de Guerrero-México. Os casos foram compostos por quatro alunos (11-12 anos) matriculados na sexta série. **Coleta e análise dos dados:** Os dados foram coletados por meio de questionário (tarefa) e entrevista. Para análise dos dados, foram utilizados os oito níveis de compreensão do modelo teórico. **Resultados:** Os resultados indicam que os alunos não conseguem formalizar seu processo de compreensão em relação ao conceito de razão, devido às dificuldades que apresentam em aplicar corretamente as estratégias matemáticas na resolução das tarefas propostas. **Conclusão:** Os resultados indicam que, para atingir níveis elevados de compreensão, é necessário realizar o processo dinâmico de redobrar entre os níveis, pois reflete sobre o que é necessário avançar no próprio processo de compreensão.

Palavras-chave: Matemática educacional; Entendimento; Razão; Fração; Educação básica.

INTRODUCTION

Mathematics is a subject of great importance because it is linked to other sciences. However, part of its abstract content makes it an unwanted subject matter for many students due to their difficulties learning it. This situation has triggered multiple investigations in mathematics education, from the historical, cognitive, and didactic parts. For example, from the cognitive side, a topic that has become relevant is the study or assessment of the understanding of a mathematical concept, where several models or theories that point to obtaining specific categories of understanding have been generated, aiming to strengthen the teaching or learning of mathematical concepts (Çalişici, 2018; Rodríguez-Vásquez & Arenas-Peñaloza, 2021).

Particularly, in basic primary education (6-12 years of age), many investigations in the field of mathematics education have indicated that mathematics topics are difficult for students to understand (Lamon, 2007; Fernández, Figueras, Monzó, & Puig, 2009; Fernández & Llinares, 2010; Buforn, Llinares, & Fernández; 2018). One of them is proportional reasoning, which covers the concepts of ratio, proportion, and fraction, and which students and teachers alike find complicated to understand (Sanchez, 2013; Arıcan, 2019; Lamon, 2020; Wahyu, Kuzu, Subarinah, Ratnasari, & Mahfudy, 2020). At this educational level, ratio and proportion in fraction-related topics constitute the basis for understanding transcendent and cross-curricular concepts such as percentage, equations, and speed problems. For this reason, it is crucial to study ratio and detect the difficulties that students express when developing activities that involve that topic.

Monteiro (2003) determined that primary school math teachers have difficulties solving ratio and proportion problems because they did not understand the concepts when they were elementary and high school students. Also, Fernandez et al. (2009) stated that both students and teachers find it hard to correctly construct and appropriate the ratio concept. One of the obstacles is that they cannot separate the ratio from the arithmetic process of operating the numerator with the denominator. At the same time, Ramírez and Block (2009) stated that the topic of proportionality is confused in the curriculum since the study plans and reference textbooks for teachers do not present a relevant relationship or difference between the notions of ratio and fraction. Therefore, to improve the understanding of the concept, a curricular reorganisation is required.

In turn, in the Mexican primary school curriculum, the teaching of ratio is not clearly defined, nor is the link between the concept of fraction, showing it only as a relationship of representation (Ramirez & Block, 2009). For example, in its 2011 syllabus and programme, the notion of ratio is evidenced in block III on proportionality and functions. Its content refers to the comparison of ratios in simple cases. The 6th-grade teacher handbook introduces one characteristic of the concept: “A ratio can be represented by an integer, a fraction, a decimal, or a percentage”. Some examples they present are: 2 out of 5 students are men, which can be represented as 2/5, 0.4, or 40% (Rosales et al., 2015, p.162).

On the other hand, Çalışıcı (2018) says that students find it tough to apprehend because they must memorise formulas and algorithms instead of grasping the ratio concept; and more, that they perceive fraction numerator and denominator as two different integers, not as a relationship of magnitude between two quantities. All the shortcomings evident in the results of standardised math tests at the international and national levels (PISA and PLANEA, respectively) reveal that, in Mexico, 57% of students do not reach the basic level of mathematics skills and are below the average (490 points) of the Organization for Economic Cooperation and Development (OECD). These numbers show that students do not reach the basic level in mathematics, i.e., they can perform ordinary procedures (arithmetic operations in simple situations) but find it hard to identify how a simple situation in the real world can be represented mathematically (OECD, 2016).

At the national level, it is convenient to highlight the results of the tests of the national plan for learning assessment (PLANEA by its acronym in Spanish), applied to 6th-grade primary school students in 2015. It reveals that six out of ten students have not managed to acquire the key learnings of mathematics at the end of that school year. Specifically, 60% of the 6th-grade primary school students reached the level I (insufficient) in mathematics, where they work with the set of natural numbers. And the lowest percentage of students reached level IV (outstanding), in which they work with proportional reasoning (ratio, proportion, and fraction) (INEE, 2015).

Since the results of the research mentioned and standardised tests show that basic education students could not acquire learning that allows them to understand proportional reasoning, it is natural to ask: what is the reason for that lack of understanding? What factors prevent a good understanding of the concept of ratio? How do students proceed when developing an activity that demands the use of proportional reasoning?

Particularly in this investigation, we sought to know the process students follow to construct the ratio concept and used Pirie and Kieren’s (1994) theoretical model to know in-depth where the primary school students' lack of understanding lies. Thus, the objective of this research was to analyse the comprehension of 6^{th}-grade basic level students (11-12 years old) when they solve tasks on ratio, represented by fractions, based on Pirie and Kieren's comprehension levels. In other words, we will analyse students' continuous process to specify the ratio object in tasks that promote the use of this concept.

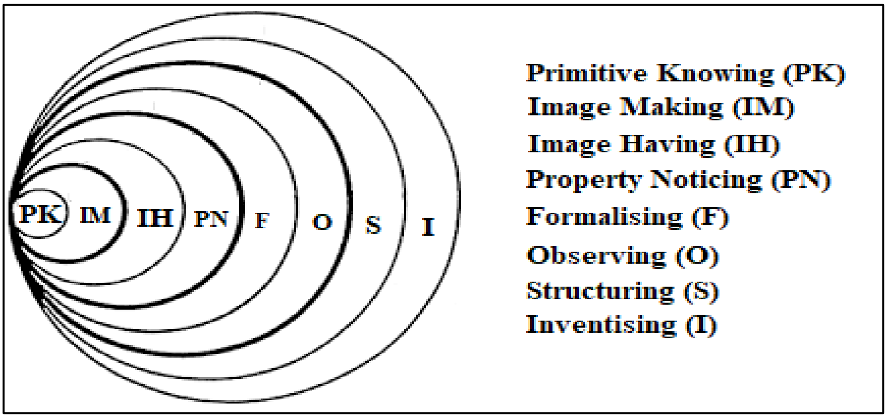
THEORETICAL FOUNDATIONS

This research used Pirie and Kieren's (1994) model, which arose from a constructivist perspective. The model describes the process of mathematical understanding in a dynamic, recursive, levelled but non-linear way (Pirie & Kieren, 1989) and also recognises understanding as *a continuous process carried out by a subject to specify an object, which is built iteratively from their experiences to build, strengthen, or modify their knowledge.*

The model is structured in eight levels (see Figure 1), used to describe students' understanding of a mathematical concept. One can progress by going forward and going backwards to a previous level to reflect on or rework earlier understandings of a mathematical concept (Delgado, Codes, Monterrubio, & Gonzalez, 2014). The eight Pirie and Kieren's understanding levels are:

Figure 1

Levels of the Pirie and Kieren 1994 model (Pirie & Kieren, 1994)



Primitive knowing (PK). At this first level, the process of mathematical understanding begins. The PK level refers to everything the student knows and knows how to do before confronting the new mathematical concept to study. It is important to note that the term *primitive knowing* should not be confused with a low level of mathematical knowledge. As Pirie and Kieren (1994) state: "Primitive here does not imply low-level mathematics, but rather is the starting point for the growth of any particular mathematical understanding" (p.170).

Image making (IM). The second level shows when a student can make distinctions based on their previous abilities and knowledge; in addition, they perform physical or mental actions to create an idea of the new mathematical concept. The images are not always pictorial representations; rather, they can be expressed through the language or specific actions of the students.

Image having (IH). At this level, the student can use a mental construction on the mathematical concept but without the need to work with particular examples or make an abstraction of the concept itself; likewise, the student sees the need to replace the images associated with the notion with a mental picture of it. Specifically, this level is reached, for example, when a representation (symbolic, pictorial, graphic, among others) of the situation associated with the mathematical object is established.

Property noticing (PN). This level is reached when the students can use or combine aspects of the mental images they already have, to build specific properties of the concept and try to generalise them.

Formalising (F). Gives evidence of when the student knows the properties and can abstract common features of that image; moreover, at this fifth level, the student works on the mathematical concept as a formal object and does not refer to a particular action or image.

Observing (O). It is achieved when the student uses his thinking and formal mathematical language, reflecting on formal statements and establishing connections between mathematical concepts that allow him to deduce patterns and regularities when expressing algorithms and theorems.

Structuring (E). At this level, the students must reflect on their formal observations as a theory and justify or verify statements through a logical or meta-mathematical argument.

Inventising (I). The last level is reached when the students can detach themselves from the concrete and determined situations of the concept, since they will have a complete understanding of it, and then undertake other perspectives that lead them to hypothesise another problem or concept. As Pirie and Kieren (1994) put it, "At this level, a person has a complete structured understanding and, therefore, may be able to break down the preconceptions that prompted this understanding and create new questions that could turn into an entirely new concept" (p. 171).

Concerning the characteristics of the theoretical model, one of the most important is the dynamic process of redoubling (MEEL, 2003). *Folding back* (redoubling or falling back) is identified when someone is at a higher level of understanding and returns to a lower level to re-examine that understanding, but in a more enriched way than before.

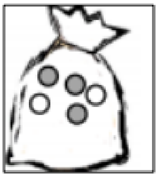
Mathematical object

The ratio concept has appeared in two contexts throughout history: first, among numbers, and second, among magnitude quantities. In both, relationships have been established with other concepts, for example, fraction and quotient (Gairín & Oller, 2012). However, one of the difficulties in understanding the concept of ratio is the ambiguity that it has precisely with the fraction and quotient concepts (Ramírez & Block, 2009).

Definition: A ratio is an abstract number that expresses only the relationship between two magnitudes, for which it lacks a species. Geometrically, a ratio is a number that results from comparing two magnitudes of the same species by quotient. Generally, if a and b are quantities of the same magnitude, their ratio is the quotient or indicated quotient that results from dividing the measure of a by the measure of b, it is called the ratio between a and b, and it is written $\frac{a}{b}$, i.e., $a{:}\,b$ (Caballero, Martinez, & Bernárdez, 1970).

Figure 2

A pictorial example of the concept of ratio (Rojas, 2010)



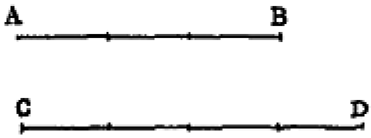
For example, given a collection of objects (marbles) (Figure 2), we can say that the black and white marbles are in a ratio of 2 to 3 or $\frac{2}{3}$, i.e., for every two white marbles, there are three black marbles; we can also interpret that the black and white marbles are at a ratio of 3 to 2 or $\frac{3}{2}$, i.e., for every three black marbles, there are two white ones.

So, the ratio is an ordered pair of numbers a and b, that can be interpreted in various ways. For example, given the following situation: "In a fruit bowl, there is one apple and three pears", the ratio can be given as follows:

- $1:3$, which means that for every apple, there are three pears.
- $\frac{1}{4}$ are apples, and $\frac{3}{4}$ are pears.
- $0{,}25$ are apples (1 divided by 4, i.e., $\frac{1}{4}$).
- 25% are apples (0.25 as a percentage).

Figure 3

Explanation of the ratio concept in Book V of Euclid's Elements (Casey, 1885)



4. If we consider two magnitudes of the same kind, such as two lines AB, CD, and if we suppose that AB is equal to $\frac{3}{4}$ of CD, it is evident, if AB be divided into 3 equal parts, and CD into 4 equal parts, that one of the parts into which AB is divided is equal to one of the parts into which CD is divided. And as there are 3 parts in AB, and 4 in CD, we express this relation by saying that AB has to CD the ratio of 3 to 4; and we denote it thus, $3 : 4$. Hence the ratio $3 : 4$ expresses the same idea as the fraction $\frac{3}{4}$. In fact, both are different ways of expressing and writing the same thing. When written $3 : 4$ it is called a ratio, and when $\frac{3}{4}$ a fraction. *In the same manner it can be shown that every ratio whose terms are commensurable can be converted into a fraction; and, conversely, every fraction can be turned into a ratio.*

From this explanation we see that the ratio of any two commensurable magnitudes is the same as the ratio of the numerical quantities which denote these magnitudes. Thus, the ratio of two commensurable lines is the ratio of the numbers which express their lengths, measured with the same unit. And this may be extended to the case where the lines are incommensurable. Thus, if a be the side and b the diagonal of a square, the ratio of $a : b$ is

$$\frac{a}{b}, \text{ or } \frac{1}{\sqrt{2}}.$$

The meaning is undoubtedly connected to the order in which the given numbers are considered. The foundation of the meaning of ratio can be seen in Book V of *Euclid's Elements* (Casey, 1885), where *ratio* is defined as the mutual relationship of two magnitudes of the same type concerning quantity. And it is exemplified considering two segments divided equally (see Figure 3).

METHODOLOGY

The research is qualitative because we interpret the process of students' understanding of a mathematical concept. We use the field observation technique, which allows direct and face-to-face observation of the students' actions in class, reflected in detailed field notes (Mcmillan & Schumacher, 2005).

This research is a case study involving four students (11-12 years old) enrolled in the 6^{th} grade of a primary school in Chilpancingo de Los Bravo, state of Guerrero, Mexico. Two students presented high academic performance (Case I) and the other two (Case II), low academic performance. We assigned codes to each student according to their Case. Case I was assigned codes E1 (student 1) and E2 (student 2), and Case II, codes E3 (student 3) and E4 (student 4).

The cases had already worked on block III of the study program (SEP, 2011), which characterises the ratio concept. To collect the data, we used a questionnaire (Figure 4) that was applied collectively, developed in two sessions, one for each case study. The sessions were video-recorded, and class notes were made. Likewise, we conducted an interview to clarify the process of solving the task. Finally, the data were transcribed and confronted, interpreted, and analysed regarding the characteristics and levels of action of Pirie and Kieren's (1994) theoretical model.

Those responsible for the educational institution and the students were informed of the study and of the responsible practices for data publication, such as guaranteeing anonymity, non-profit, and collecting data for research purposes only (Wager & Kleinert, 2011).

The questionnaire

The questionnaire consisted of a seven-item task (Figure 4), in general terms. With the first five, we want students to reflect on two situations to decide which is optimal, according to the comparison of magnitudes, i.e., ratio comparison. The last two paragraphs require the student to use their prior knowledge about percentages to identify that Baltazar's market and Mr José's vegetable garden are both optimal situations. The questionnaire was adapted from an activity in a 6th-grade book *Desafíos Matemáticos* [Mathematical Challenges], teacher's edition (Rosales et al., 2015). For that activity, materials and resources the students used were sheets of paper, pencils, erasers, and pencil sharpeners.

Figure 4

Task: "How many oranges I want" (Adapted from Rosales et al., 2015).

In Baltazar's market, a kilogram of oranges has 9 pieces and costs $10. In Mr José's market, 7 oranges weigh a kilo and cost $8.

a) Can we compare both situations? Why?
b) Represent both situations graphically.
c) Where would you buy the oranges, based on the representation above? Why?
d) Express the relationship between the quantities verbally in each situation and write both in fraction form.
e) Compare the former fractions to determine where we should buy the oranges. Justify your answers.

If in Baltazar's market the kilo of oranges is offered with a 20% rebate from its usual price,

f) What would be the new cost of a kilo of oranges? Detail your answer.
g) With this discount in Baltazar's market, where would you buy now the kilo of oranges, in Mr José's vegetable garden or the market? Why?

Particularly, with subsection a), we expected students to identify that both Baltazar's market and Mr José's vegetable garden situations are expressed with the same starting unit; therefore, they can compare ratios to decide the best situation to purchase the oranges. According to the theoretical model, in this section, students can advance their comprehension, from the *PK* or *IM* level, depending on how deep is the analysis of the statement.

In section b) we wanted to know the transcendence of the verbal representation of the statement to another type of representation (numerical, pictorial, or graphic) that the students manifested. Specifically, from the theoretical model, we relate it to the *IH* level. Subsection c) was designed to allow students, based on representations, to create, interpret, and compare situations to select the ratio that will enable them to opt for the best situation to buy oranges. Then, the objective of d) was for students to model a real-life situation in a mathematical language (representation of fractions). About the model, we sought that the students reflected on the image they created of the situation to advance to the *PN* level.

Subsection e) aimed for students to put into play their abilities to interpret and argue the comparison strategies used to choose the optimal situation to buy a kilo of oranges. Based on the theoretical model, with this section, we expected the students to advance to the *F* level, managing to detach from the image they made, providing arguments to reach generalisation.

Finally, in sections f) and g), students were expected to consider a different condition to reinterpret the starting problem as -based on their former knowledge- they must establish the new cost of a kilo of oranges in Baltazar's market (*PK*), establish the new mathematical ratios (*IM* or *IH*), start working on them (*PN*), and select the best purchase option (*F*).

Data analysis

We followed Pirie and Kieren's theoretical model to analyse the data. Then, based on field observation, we identified and described the specific characteristics of some of the comprehension levels according to the problem situation (see Figure 4).

On the first level, *primitive knowing*, we focused on identifying and describing the prior knowledge of the cases related to the comparison and equivalence of measurement units (mass) since both situations have the same unit (see Figure 4). In the second level, *image making*, we identified whether the cases could identify the unit of measurement of both situations of the task (1 kilo of oranges) and whether they related their former knowledge with the characteristics of the ratio concept. Nest, on the *image having* level, we identified whether the cases could establish a representation (symbolic, pictorial, graphic, for example) of the situation associated with the concept of ratio. On the fourth level, *property noticing*, the capacities of the cases to use strategies that would allow comparing a mathematical ratio (presented as a fraction) were determined. Last, on the *formalising* level, we identified: when the cases recognised the unit of measure (1 kilo of orange) correctly, when they established the representation of the situation related to the concept of ration, and when they could detach themselves from the representation and use generalisation properties.

RESULTS

This section is organised into two sections, Case I's understanding process and Case II's understanding process, which, in turn, were divided into two differentiated categories according to the answers given by the students and the objective of the task: *choosing the best ratio* and *ratio comparison*.

Case I’s understanding process (C1)

Based on Pirie and Kieren’s (1994) theoretical model, this section presents the comprehension analysis, i.e., the analysis of the knowledge structures of Case I (E1 and E2) when solving the task (Figure 4).

Choice of the best ratio (category 1)

Case I students initially read the task statement separately and carefully, paying attention to the data presented by the two situations (regular and rebate price) to establish that the two situations are comparable since they share the same unit of measurement. But the students related the task to the search for the cost per unit of orange in each situation, revealing that they did not identify or interpret the problem situation. Hence, they began facing the problem from the *image making* level.

> E2: *We want to know how much each orange costs here and here...* [Points to the data presented in the task in both situations].
>
> E1: *Yes! We must do the fractions* [referring to representations 9/10 and 7/8].

The students made a symbolic representation of both situations, which gave evidence that they advanced to a level outside the i*mage having (IH)* model*,* where they sought to know the value of each unit of orange. So, they expressed the ratio of the total cost by the number of oranges (see Figure 5).

Figure 5

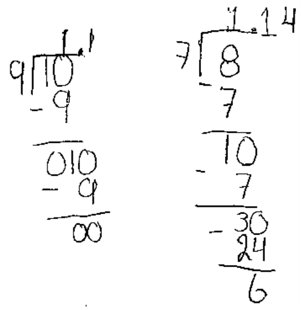
Case I’s symbolic representation, category 1 (Case I’s elaboration).

At this level, the students had the data and began to relate it to a mathematical strategy that allowed them to solve the task. The system they used was the arithmetic operation of division (Figure 6) to find the value of each unit of oranges in each situation. This procedure showed that they reached the *property noticing (PN)*, as they worked on the image they had.

E1: *We must divide 10 per 9 and 8 per 7.*

Figure 6

Case I's mathematical process, category 1 (Case I's elaboration).



Once the value per unit of each orange in both situations was found, the students constructed a pictorial representation of the number of oranges (*image having*), associating it with the unit cost they established with the mathematical division process. We can perceive a *folding back* because the students returned to a prior level; however, as shown in Figure 7, the students reached (by returning) a more solid understanding. Later, they associated the value found with the representation pictorial.

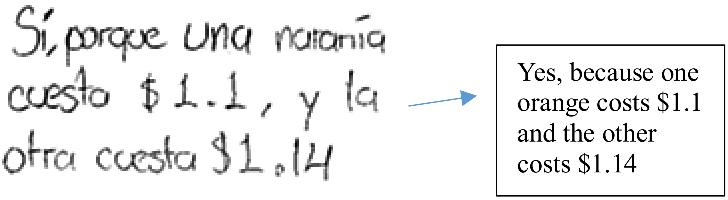
Figure 7

Pictorial representation, situation 1 and 2, respectively, Case I (Case I's elaboration).

The students made abstractions of the new image, which indicated that they advanced again to the *property noticing (PN)* level. On this level, the students conjectured about the best mathematical ratio that they should choose to buy the oranges in the market or the vegetable garden.

Figure 8

Abstractions made by Case I, category 1 (Case I's production).

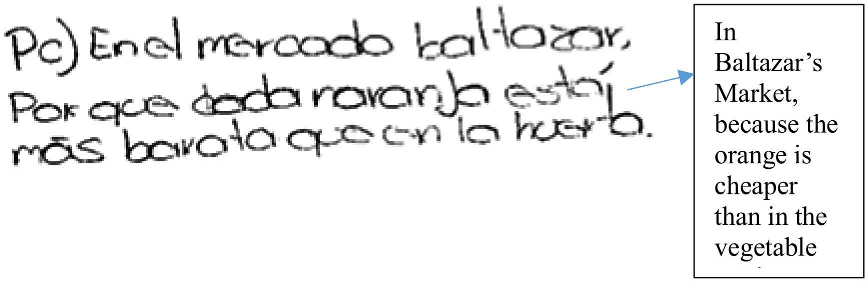


E1: *I say that in Baltazar's market, it is cheaper, because each orange costs $1.1, and in Mr José's vegetable garden, each orange costs $1.14. So, I think this is better* [points to the operation performed with 10/9].

E2: *Yes! Each orange is cheaper here* [indicates the process carried out for the market situation] *than here* [points to the process carried out for Mr José's vegetable garden].

Figure 9

Case I's conclusion, category 1 (Case I's production).

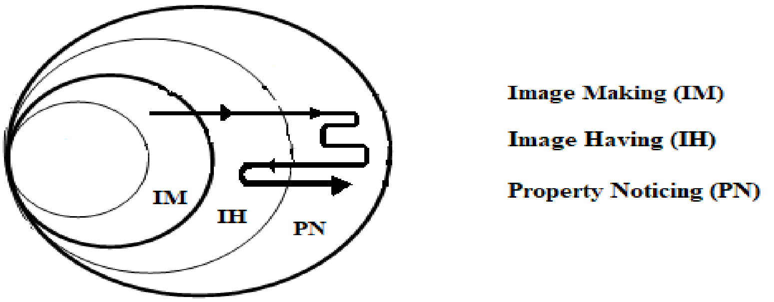


However, the students got incorrect conclusions, which means that they did not reach the *formalising* (F) level. This happened because when they started the task, they did not identify the starting unit in which both situations occurred.

Figure 10 schematises the students' path until they determined the choice for the best mathematical ratio (referring to the comparison of ratios on the price of oranges). In the following figure, the arrows indicate that Case I achieved that level in their understanding process, and the thickness of the line represents the development of the comprehension process.

Figure 10

Case I's knowledge structure to choose the best ratio.



Comparison of ratios (category 2)

The students first found the value of the discount according to the offer, so they started tackling the problem from the *primitive knowing* level, since that knowledge was their starting point in the process.

E1: *The first thing that we have to do is…*

E2: *Take the percentage of the previous price.*

E1: *Yes, aha… Of this price?* [Points to the cost of Mr José's vegetable garden, $8].

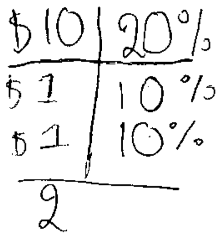
E2: *No, of ten, right? Because it says: If in Baltazar's market the kilo of oranges is offered with a 20% discount...*

E1: *So, it is this* [points to the cost at Baltazar's market] *Ah, yes, it is ten.*

Since they already knew what data to use, they immediately chose a strategy to calculate 20% of $10. During the interview, Case I stated that the process they used to obtain the percentage of a numerical quantity, an additive decomposition into percentages that were easier to calculate, as shown in Figure 11, was explained by their teacher. The process they carried out shows that, with their formed knowledge, they reached the *property noticing (PN)* level without going through the previous two, *image making* and *image having*. This is due to the complementarity of knowledge to build properties specific to the concept, managing to generalise them.

Figure 11

Process to calculate 20% of $10, Case I, category 2 (Case I's elaboration).



From this process, they obtain the new cost of the oranges in Baltazar's market, with the rebate offered.

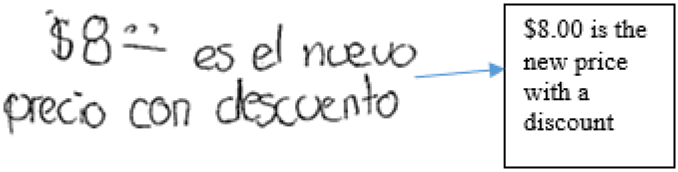
E1: *It's a $2 discount… and then we subtract it?*

E2: *Yes, we subtract two from ten.*

E1: *The new price is $8.*

Figure 12

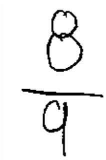
Case I's answer based on the new cost of a kilo of oranges, category 2 (Case I's elaboration).



The students then had a *folding back* with a symbolic representation of the situation (discount in the market), expressing a relation of the cost of the kilo with the total number of oranges. So, now they are on the *image having (IH)* level.

Figure 13

Case I's symbolic representation of the new situation, category 2 (Case I's elaboration).

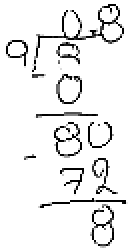


> E2: *But we have to do this* [points to the process they carried out previously to find out the value of each unit of orange] *because here it costs this, already* [points to the new cost per kilo of oranges with the discount given] *and here it costs less.*
>
> E2: *So, it would be eight ninths, right? Let's do the fraction and division again.*
>
> E1: *Aha.*

From the interaction, the students related the activity with the previous division procedure (Figure 6) to find the value per unit of each orange and use the same strategy to solve the task, i.e., they apply the division operation (Figure 14). This process shows that Case I students have once again reached the *property noticing (PN)* level, which confirms that the comprehension process is built from experiences and is carried out iteratively to build, strengthen, or modify knowledge.

Figure 14

Case I's mathematical process to calculate the cost of the oranges per unit, task 1 (Case I's elaboration).



Finally, the students synthesise the information on the process carried out and completely detach themselves from the specific image, considering the concept as a formal object, determining where it is convenient for them to buy the oranges after the discount given at Baltazar's market, which indicates that they are in the *formalising (F)* level.

> *E1: Now, each orange costs $0.8, so…*
>
> *E2: At the market, oranges are cheaper than before.*
>
> *E1: Aha. Yes, at the market.*

Figure 15

Case I's answer, category 2 (Case I's elaboration).

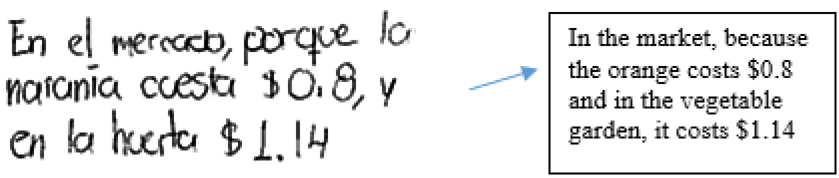
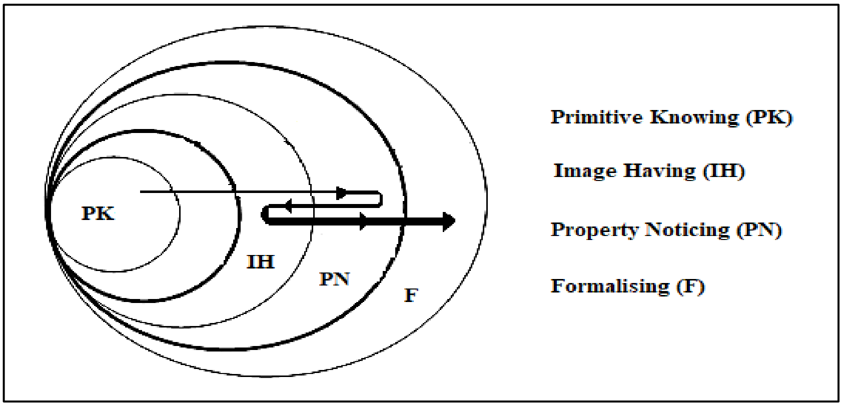


Figure 16 schematises the comprehension process outlined by the students when solving the task. In the following figure, the arrows indicate that

Case I students achieved that level in their comprehension process, and the thickness of the line represents the development of the comprehension process.

Figure 16

Case I's knowledge structure to compare the ratios.



Case II's understanding process (C2)

This second section brings the analyses of Case II's understanding, i.e., based on Pirie and Kieren's (1994) theoretical model, it presents the analysis of the process of the students' knowledge structures when solving the task (Figure 4) on ratio.

Choice of the best ratio (category 1)

Case II students began by reading the task statement together, identifying the data related to the price that both situations (regular price and rebated price) pose. Thus, they organised a worksheet in two columns (data and operation). However, the students could not identify or interpret the starting unit of both situations (a kilo of oranges).

Figure 17

Case II's s data organisation, category 1 (Case II's elaboration).

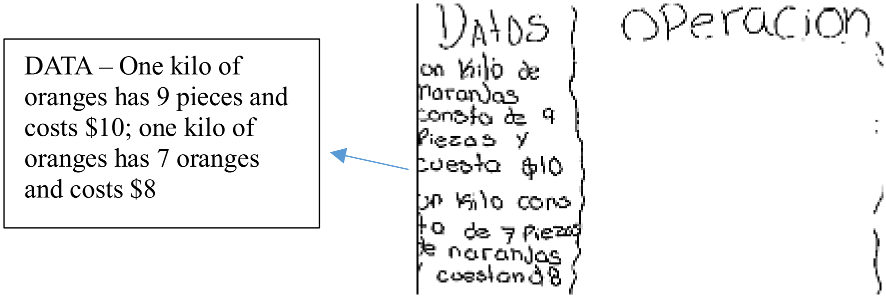
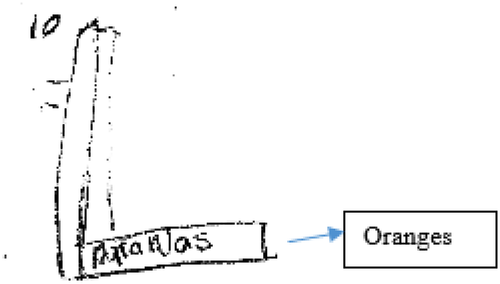


Figure 18

Second graphic representation, case II students, category 1 (Case II's elaboration).



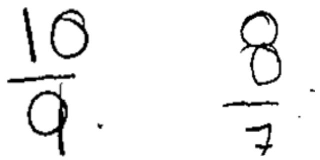
The students began to solve the task at the *image having (IH)* level, since they had a mental picture of the situation. Later, they changed this representation to a Cartesian graph, staying at the same level as the model. The students found it more difficult to relate the data of the task with a mathematical object if compared to Case I.

> E3: *But it's a graph, maybe we could do the graph like this...* [E3 builds a Cartesian graph on a sheet].
>
> E4: *No, it's not like that.*

The students found it tough to represent the situation of the oranges in the graph they constructed (Figure 17), so they made another representation of the situation (Figure 18), which indicated that the students remained at the same level (*image having*).

Figure 19

Case II's symbolic representations, category 1 (Case II's elaboration).

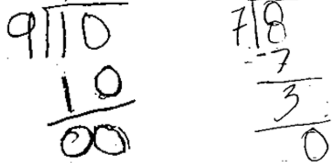


> E4: *But we have to do the fractions first... to find out how much each orange costs* [E4 erases the two circles she had already built].
>
> E3: *Aha.*
>
> E4: *It would be 10 among 9 and we have to divide.*
>
> E4: *each orange costs $1.*
>
> E4: *It costs the same at both places.*
>
> E3: *The result is the same* [referring to the division process they carried out].

Case II students, just like Case I, could not see that both situations are comparable because they have the same starting unit (one kilo); however, they could determine the cost per unit of orange in both situations by dividing. For that end, they worked on the new image they built (Figure 19), using the same strategy as Case I students, the arithmetic operation of division (Figure 20). This procedure shows that the students advanced to the *property noticing (PN)* level.

Figure 20

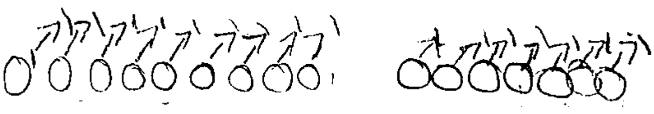
Case I's mathematical process, category 1 (Case I's elaboration).



As we can see in Figure 20, the students reported problems in performing the arithmetic operation of division, which highlights their necessities regarding their prior knowledge. Later, they made another pictorial representation of both situations (Figure 21). This process again triggered in the students a *folding back*, since they regressed to a lower level which they had already overcome (*image having*); however, they could relate the abstractions made at the *property noticing (PN)* level. For example, they represented the unit value of each orange with a circle (Figure 21).

Figure 21

Pictorial representation of the Case II students, category 1 (Case II's elaboration).



E4: *So? They are the same...*

E3: *That is what I meant… the number of pieces will increase the cost.*

E3: *We can put that at both places, they cost the same.*

E4: At both places, it suits us, according to the operations.

E3: Yes.

The students' interaction evidence that they returned to the *property noting (PN)* level, i.e., the process of determining in which situation (Baltazar's market and Mr José's vegetable garden) it would be convenient to buy the oranges. Moreover, they concluded erroneously, since they pointed out that it is convenient to buy at both places since it is the same thing. However, this conclusion owes to the difficulties in the division arithmetic process (Figure 20), which leads them not to overcome the difficulties in comparing and choosing a mathematical ratio (Figure 22).

Figure 22

Case II's abstractions, category 1 (Case II's elaboration).

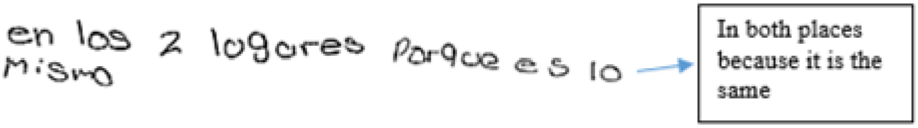
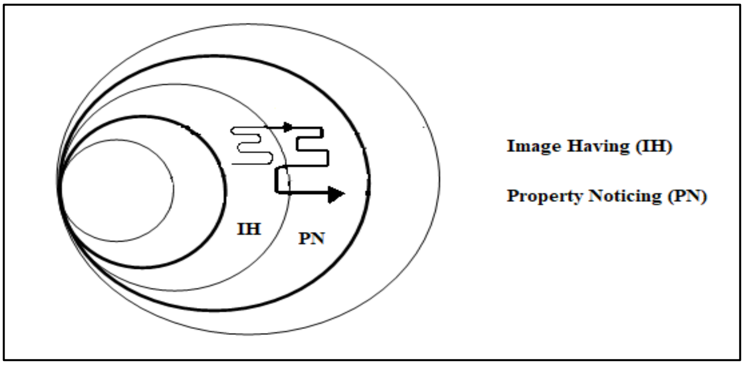


Figure 23 presents the process that the students followed until now to determine the choice of the best mathematical ratio (referring to the comparison of ratios on the price of oranges). The arrows indicate that the Case II achieved that level in their comprehension process, and the thickness of the line represents the development of the comprehension process.

Figure 23

Case II's knowledge to choose the best ratio.



Comparison of ratios (category 2)

At first, the students discussed which path they had to follow to find the value of the rebate that would be made with the offer, so they started at the *primitive knowing (PK)* level.

E4: *It says it's on sale at a 20% discount.*

E3: *The new cost is taken from the price that was previously stated.*

E3: *This, you mean?* [Points to worksheet].

E4: *Yes. But I don’t know which one it is.*

E3: *Hmm… It is the one that best suits us. This* [points to the situation of Mr José’s vegetable garden].

E4: *It is not the most convenient place; both are the same. But it would be cheaper… you would pay less for more* [referring to the situation of Mr. José].

E4: *Then, this* [points to Mr José’s vegetable garden situation].

The students found it difficult to relate the data with some solution strategy to the task. First, they did not correctly interpret which situation the mentioned discount should be applied, and they chose to apply it to the one that best suited them according to their abstractions (Mr José’s vegetable garden). Also, they did not recall how to obtain the percentage of a numerical value.

E4: *How is it done?*

E3: *Well, nothing else we took away, only this [points to* $8], *we took a discount of 20%.*

E3: *We learned that in class, but I don’t remember.*

E4: *Ah...now, now, now… lend me the pencil.*

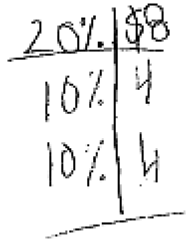
E4: *We put this here* [performs the same process that Case I peers did].

E3: *Oh, yeah. It’s true* [expresses when observing the process carried out by E4].

Calculating the percentage consisted of decomposing the amount to which the percentage had to be calculated into addends to obtain the requested rate (Figure 24). This process shows that students continue to use their former knowledge to obtain the percentage of a numerical value (*primitive knowing*).

Figure 24

The process to calculate 20% of $8, case I students, category 2 (Case I's elaboration).



From Figure 24, we observe that the students made mistakes when calculating the discount; however, they expressed that the new cost of a kilo of oranges in Mr José's vegetable garden is $6 with the discount (Figure 25), moving towards the *property noticing (PN)* level.

Figure 25

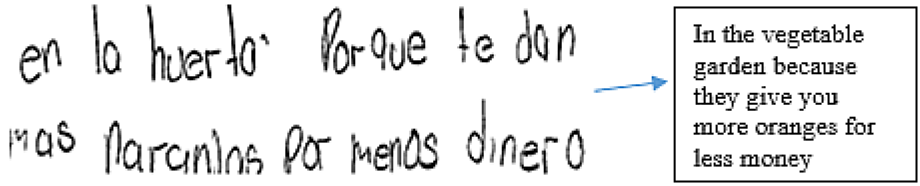
Case I's answer based on the new cost of a kilo of oranges, category 2 (Case I's elaboration).

Since they had obtained the new price, applying the discount to the price of a kilo of oranges at Mr José's vegetable garden, they had to decide again where to buy the oranges, whether at Mr José's place or the market, concluding that they would buy at Mr José's. Yet, they continued to argue that they offer more oranges for less money (Figure 26). Even though the deductions in the comparison of ratios are correct (lower price, same quantity of oranges), the lack of reading comprehension and the difficulties in the arithmetic processes that they carried out are evident.

> E4: *Here is cheaper* [points to the situation of Mr José's vegetable garden].
>
> E4: *Because they give you more oranges…*
>
> E3: *So, for less price.*

Figure 26

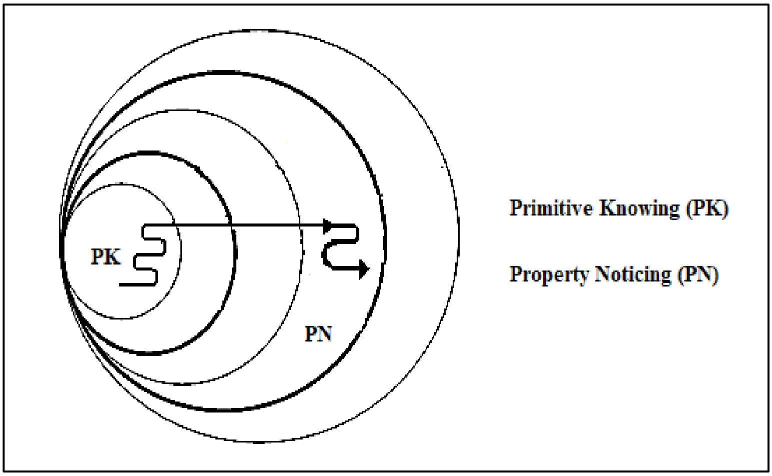
Case II students' final answer, category 2 (Case II's elaboration).



The following figure outlines the process followed by Case II students in the comparison of ratios with the same unit of measure. The arrows indicate that the students achieved that level in their comprehension process, and the thickness of the line represents the development of the comprehension process.

Figure 27

Case I's knowledge structure to compare the ratios.



Case II students reached a lower level in the comprehension process, according to Pirie and Kirien, due to the difficulties they had from the beginning in identifying data and arithmetic calculations. Consequently, they failed to formalise their comprehension process in ratio comparison.

What could explain the differences in the comprehension process between the cases would be that Case II students are still in the process of achieving a conceptual understanding of the ratio concept and some other mathematical processes, as revealed by their academic history. Regarding the theoretical model, to understand the concept, students should at least have reached the *formalising (F)* level. Therefore, their comprehension process was low.

It is clear that the students, in both cases, despite presenting different academic performances, carried out the same procedure in their comprehension process when establishing comparisons of mathematical ratios. But the difference is found in the difficulties that the Case II students manifested compared with Case I students. Namely, difficulty in understanding and relating the data with a mathematical object and difficulty in the strategies implemented (pictorial representation and symbolic, division algorithm, percentage calculation).

CONCLUSIONS

In this research, we analysed the understanding of four 6th-grade basic education students of the concept of ratio when solving a task that requires choosing the mathematical ratio to decide the best situation of where to buy at the best price. Pirie and Kieren's (1994) theoretical model demonstrated that knowledge comprehension is not linear (from lower level to higher level). Rather, it is an iterative construction of the process, which involves forward-backward (*folding back*) cycles to advance to a higher level. The *folding back* allowed students to re-examine their concepts and advance knowledge and skills in the comprehension process in an enriched way.

From the concept of comprehension that the theoretical model manifests, it was possible to establish that the case studies failed to formalise the mathematical ratio object since comprehension is a process that is built from their experiences. Evidently, they found the strategies used to solve the task too difficult. In the task, the students were expected to identify that both situations had the same starting unit, so that they could compare ratios and choose the best situation to buy a kilo of oranges, but Case I and Case II students were unable to establish that relationship, which confirms that they could interpret or decide which unit of measurement was at stake in a comparison of mathematical ratios.

According to the theoretical model and the development of the activities by the students, the comprehension process related to the concept of ratio is described as follows: first, they associated the ratio concept with an image *CIM*) or immediately created and represented it (*IH*). The key representations in their comprehension process were pictorial and symbolic, where they presented errors to establish them. We recall that the symbolic representation (fraction) is the one with which they first related the situations of the task. Subsequently, in their comprehension process, we allowed the students to work on the image they had (*PN*), for example, using the division algorithm to compare mathematical ratios, a process in which Case 2 presented errors. Then, they tried to abstract a hypothesis from the results obtained from the implemented strategies, and this is how they managed to reach their highest level of understanding (*PN*). However, the students failed to detach themselves from the image they created of the mathematical object. That is why they could not formalise the concept of ratio as a formal object (*F*).

In the comprehension process, we noticed the following difficulties: non-identification of the starting unit in a comparison of mathematical ratios, manifesting weak reading comprehension and data interpretation; difficulty in translating from a verbal to a symbolic language, shown during the development of the activity; to perform arithmetic operations with fractions, evidenced in Case II students, who failed to develop simple divisions; difficulty in comparing mathematical ratios, the students just established the comparison based on the result of their arithmetic processes, which for Case II was incorrect and; difficulty identifying and working with mathematical ratios, students were unable to establish the relationship between the magnitudes of the numerator and denominator in a mathematical ratio.

In this sense, the theoretical model allowed us to delve into the students' understanding process when solving tasks collectively (in groups) related to the concept of ratio. Furthermore, we could see the difficulties students manifest in their comprehension process when they solve tasks that allude to the concept of ratio. The above helps teachers implement strategies that allow their students to overcome difficulties that may impair them from adequately understanding the concept of ratio. Given that it is necessary to continue with empirical studies that report on student comprehension when solving tasks related to ratio, future research could focus on deepening this interpretation with more participants to generalise the results obtained in this case study.

AUTHORSHIP CONTRIBUTION STATEMENT

JAAP and FMRV conceived the presented idea and developed the theory. JAAP adapted the methodology for the research, developed the activities, and collected the data. FMRV and JAAP analysed the data. All the authors actively participated in the discussion of the results, reviewed, and approved the final version of the paper.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, [JAAP], upon reasonable request.

REFERENCES

Arıcan, M., (2019). A diagnostic assessment to middle school students' proportional reasoning. *Turkish Journal of Education*, *8*(4), 237-257. https://doi.org/10.19128/turje.522839

Buforn, A., Llinares, S., & Fernández, C. (2018). Características del conocimiento de los estudiantes para maestro españoles en relación con la fracción, razón y proporción. *Revista mexicana de investigación educativa*, México, *23*(76), 229-251.

Caballero, A., Martínez, L., & Bernandez, J. (1970). *Matemáticas, tercer curso*. Esfinge.

Çalişici, H. (2018). Middle school students' learning difficulties in the ratio-proportion topic and a suggested solution: envelope technique. *Universal Journal of Educational Research*, *6*(8), 1848-1855. http://dx.doi.org/10.13189/ujer.2018.060830

Casey, J. (1885). *The first six books of the elements of Euclid*. Longmans, Green.

Delgado, M., Codes, M., Monterrubio, M., & González, M. (2014). El concepto de serie numérica. Un estudio a través del modelo de Pirie y Kieren centrado en el mecanismo "folding back". *Avances de Investigación en Educación Matemática*, *6*, 25–44.

Fernández, A., Figueras, O., Monzó, O., & Puig, L. (2009). *Competencias en razón y proporción en la escuela primaria*. Universidad de Valencia. PUV.

Fernández, C. & Llinares, S. (2010). Relaciones entre el pensamiento aditivo y multiplicativo en estudiantes de educación primaria. El caso de la construcción de la idea de razón. *Horizontes Educacionales*, *15*(1), 11-22.

Gairín, J. & Oller, A. (2012). Análisis histórico sobre la enseñanza de la razón y la proporción. In: *Investigación en Educación Matemática XVI.* (pp 249–259). Sociedad Española de Investigación en Educación Matemática.

Instituto Nacional para la Evaluación de la Educación (INEE). (2015). *Resultados nacionales 2015. Área: Matemáticas. Plan Nacional para la Evaluación de los Aprendizajes (PLANEA)*. http://www.inee.edu.mx/images/stories/2015/planea/final/fasciculosfinales/resultadosPlanea-3011.pdf

Lamon, S. (2020). *Teaching fractions and ratios for understanding. Essential content knowledge and instructional strategies for teachers.* Routledge.

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In: *Second handbook of research on mathematics teaching and learning.* (pp. 629-667). National Council of Teachers of Mathematics.

Mcmillan, J., & Schumacher, S. (2005). *Investigación educativa: Una introducción conceptual*. Pearson.

Meel, D. (2003). Models and theories of mathematical understanding: Comparing Pirie and Kieren's model of the growth of mathematical understanding and APOS theory. In*: Research in collegiate Mathematics Education: Issues in mathematics education*. (pp. 132-181).

Monteiro, C. (2003). Prospective Elementary Teachers' Misunderstandings in Solving Ratio and Proportion Problems. In: *27a Conferencia del Grupo Internacional de Psicología de la Educación Matemática*. (pp. 317-324). https://eric.ed.gov/?id=ED501033

Organización para la Cooperación y el Desarrollo Económico (OCDE). (2016). *Informe de resultados de México en la evaluación 2015 de PISA*. https://www.oecd.org/pisa/PISA-2015-Mexico-ESP.pdf

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For the learning of mathematics*, *9*(3), 7-11.

Pirie, S.; Kieren, T. (1994). Growth in mathematical understanding: how can we characterise it and how can we represent it? *Educational Studies in Mathematics*, *26*, 165-190.

Ramírez, M., & Block, D. (2009). La razón y la fracción: un vínculo difícil en las matemáticas. *Educación Matemática*, *21*(1), 63-90.

Rojas, N. (2010). *Conocimiento para la enseñanza y calidad matemática de la instrucción del concepto de fracción: Estudio de caso de un profesor chileno.* Tesis de Máster en Didáctica de la Matemática, Universidad de Granada, Granada.

Rosales, M., Barrientos, J., Issa, E., López, M., Tovilla, M., & Velázquez, L. (2015). *Desafíos matemáticos. Libro para el maestro. Sexto grado.* Secretaria de Educación Pública.

Rodríguez-Vásquez, F.; & Arenas-Peñaloza, J. (2021). Categories to Assess the Understanding of University Students about a Mathematical Concept. *Acta Scientiae*, *23*(1), 102-135. https://doi.org/10.17648/acta.scientiae.5892

Sánchez, E. (2013). Razones, proporciones y proporcionalidad en una situación de reparto: una mirada desde la teoría antropológica de lo didáctico. *Revista Latinoamericana de Investigación en Matemática Educativa*, *16*(1), 65-97.

Secretaría De Educación Pública (SEP). (2011). *Programas de estudio 2011. Guía para el maestro. Educación básica. Primaria. Sexto grado*. Gobierno federal de México. http://edu.jalisco.gob.mx/cepse/sites/edu.jalisco.gob.mx.cepse/files/sep_2011_programas_de_estudio_2011.guia_para_el_maestrosexto_grado.pdf

Wager, E., & Kleinert, S. (2011) Responsible research publication: international standards for authors. In: *The 2nd World Conference on Research Integrity*. (pp 309-16). Imperial College Press/World Scientific.

Wahyu, K., Kuzu, T., Subarinah, S., Ratnasari, D., & Mahfudy, S. (2020). Partitive Fraction Division: Revealing and Promoting Primary Students' Understanding. *Journal on Mathematics Education*, *11*(2), 237-258. http://doi.org/10.22342/jme.11.2.11062.237-258