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Independent and Mutually Exclusive Events: Analyzing the Difficulties of Higher Education Students

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ABSTRACT

Background: The concepts of mutually exclusive and independent events are sources of didactic phenomena because university students when developing activities addressing these concepts, present spontaneous erroneous ideas, indicating the need for studies that identify these difficulties. **Objective**: We seek to understand what are the confusions between these concepts to provide elements for their better treatment and implementation in teaching. **Design**: The present study analyzed, through the Ontosemiotic Approach - OSA, the knowledge of higher education students related to a problem proposed in the classroom and to identify the semiotic conflicts. **Setting and Participants**: A group of students from a federal university in the state of São Paulo, Brazil, was invited to participate in the study, voluntarily and anonymously, and all 74 students from an introductory course to probability theory of a science and technology course at this university participated in the research. **Data collection and analysis**: Students responded to the problem in writing and, through their answers, were analyzed, classifying them as correct, partially correct, incorrect and unanswered. **Results**: We identified the difficulty in interpreting the problem statement, the inappropriate use of common language (terms and expressions) and the lack of clarity in exposing the arguments to solve the problem. **Conclusions**: Based on the difficulties encountered by this group of students, it is suggested that these concepts be worked on in the classroom using different teaching tools, such as, for example, the creation of learning environments based on research processes and real situations.

Keywords: teaching of Probability; higher education; independent and mutually exclusive events; ontosemiotic focus.

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Evaluar conceptos de eventos independientes y mutuamente excluyentes en el pensamiento del estudiantado de la educación superior

RESUMEN

Fundamento: Los conceptos de eventos mutuamente excluyentes e independientes son fuentes de fenómenos didácticos porque los estudiantes universitarios al desarrollar actividades que abordan estos conceptos, presentan ideas erróneas espontáneas, lo que indica la necesidad de estudios que identifiquen estas dificultades. **Objetivo**: Buscamos comprender cuáles son las confusiones entre estos conceptos para brindar elementos para su mejor tratamiento e implementación en la enseñanza. **Diseño**: El presente estudio analizó, a través del Enfoque Ontosemiótico - AOS, el conocimiento de estudiantes de educación superior en relación a un problema propuesto en el aula e identificar los conflictos semióticos. **Ámbito y Participantes**: Un grupo de estudiantes de una universidad federal en el estado de São Paulo, Brasil, fue invitado a participar en el estudio, de forma voluntaria y anónima, y todos los 74 estudiantes de un curso de introducción a la teoría de la probabilidad de la Licenciatura en Ciencia y Tecnología. en esta universidad participó en la investigación. **Recopilación y análisis de datos**: Los estudiantes respondieron el problema por escrito y, a través de sus respuestas, fueron analizadas, clasificándolas en correctas, parcialmente correctas, incorrectas y sin respuesta. **Resultados**: Se identificó la dificultad en la interpretación del planteamiento del problema, el uso inadecuado del lenguaje común (términos y expresiones) y la falta de claridad en la exposición de los argumentos para solucionar el problema. **Conclusiones**: A partir de las dificultades encontradas por este grupo de estudiantes, se sugiere trabajar estos conceptos en el aula utilizando diferentes herramientas didácticas, como, por ejemplo, la creación de ambientes de aprendizaje basados en procesos de investigación y situaciones reales.

Palabras clave: enseñanza de la probabilidad, enseñanza superior, eventos independientes y mutuamente excluyentes, enfoque ontosemiótico.

Eventos independentes e mutuamente exclusivos: analisando as dificuldades de alunos do Ensino Superior

RESUMO

Contexto: Os conceitos de eventos mutuamente exclusivos e independentes são fontes de fenômenos didáticos, pois alunos universitários ao desenvolverem atividades abordando esses conceitos, apresentam ideias errôneas espontâneas, indicando a necessidade de estudos que identifiquem essas dificuldades. **Objetivo**: Buscamos compreender quais são as confusões entre esses conceitos a fim de fornecer elementos para seu melhor tratamento e implementação no ensino. **Design**: O presente estudo analisou, por meio do Enfoque Ontosemiótico - EOS, o conhecimento de alunos do ensino superior relacionado a um problema proposto em sala de aula e identificar os conflitos semióticos. **Ambiente e participantes**: Um grupo de estudantes de uma universidade federal no estado de São Paulo, Brasil, foi convidado a participar do estudo, de forma voluntária e anônima, sendo que todos os 74 alunos de uma disciplina de introdução à teoria de probabilidade de um curso de ciências e tecnologia desta universidade participaram da pesquisa. **Coleta e análise de dados**: Os alunos responderam ao problema por escrito e, por meio de suas respostas, foram analisadas classificando-as em corretas, parcialmente corretas, incorretas e sem resposta. **Resultados**: Identificamos a dificuldade na interpretação do enunciado do problema, o uso inadequado da linguagem comum (termos e expressões) e a falta de clareza na exposição dos argumentos para resolver o problema. **Conclusões**: Partindo das dificuldades encontradas por esse grupo de alunos, sugere-se que esses conceitos sejam trabalhados em sala de aula utilizando diferentes ferramentas de ensino, como, por exemplo, a criação de ambientes de aprendizagem baseados em processos de pesquisa e situações reais.

Palavras-chave: ensino de probabilidade; educação superior; eventos independentes e mutuamente exclusivos; enfoque ontosemiótico.

INTRODUCTION

This study is based on our belief and of Batanero (2000), Lopes (2003), Barragues Fuentes and Guisasola Aranzabal (2009), and Brasil (2018) that probabilistic training is vital for the formation of adult citizens prepared to navigate an environment of strong social, political and economic interdependencies and, when necessary, to interpret situations that require probabilistic elements such as, for example, the results of electoral polls, which often present the decisions made based on these studies.

In addition, Probability contributes to a much more balanced image of science, which traditionally presents a markedly deterministic character for students, in which everything is explainable in terms of causes and effects, and which indicates the importance of reinforcing their general mathematical skills through specific skills in Probability (Fischbein, 1975; Batanero, 2000; Lopes, 2008; Brazil, 2018).

The question of the teaching of Probability is not only conditioned to basic education and, therefore, must also be analyzed in higher education. For Ara (2006), the practice of the teaching team for the teaching of statistics and Probability for Engineering courses, for example, has identified that they present understanding difficulties in relation to the concepts involved in the statistical and probabilistic contents, which leads students to a lack of motivation for their learning, and even generates high failure rates.

According to Coutinho (2001) and Batanero and Godino (2003), the construction of probabilistic concepts must be based on the understanding of three basic notions: the perception of chance, the idea of a random event, and the notion of Probability.

Thus, Lopes (2003) and Kataoka, Rodrigues and Oliveira (2007) state that such concepts are addressed through activities in which students can perform experiments and observe events, promoting the intuitive manifestation of chance and uncertainty, building these results, mathematical methods for the study of such phenomena.

In addition, educational research indicates that they have difficulties in understanding the concepts and formal procedures related to chance (Borovcnick and Peard, 1996; Batanero, Navarro-Pelayo and Godino, 1997; Sánchez, 2000; D'Amelio, 2004; Barragues Fuentes and Guisasola Aranzabal, 2009; Batanero, 2016).

We consider that statistical reasoning, according to Makar, Bakker and Ben-Zvi (2011), is defined as the way in which individuals reason with statistical ideas and make sense of statistical information, having the conceptual understanding of essential ideas, such as variation, the underlying distribution, centre, spread, association, and sampling or the combination of ideas about data and uncertainty that lead to inferences.

For Jolliffe (2005), probabilistic reasoning can be defined as the way in which people attribute meaning to probabilistic information. Therefore, to reason means to understand and be able to explain and justify probabilistic processes. The author also points out that making the classroom space a research environment requires that students actively participate in terms of communication and expression of solutions to probability problems. This can make it easier for teachers to follow the path taken by students to solve the problem and understand their probabilistic reasoning.

Specifically in this work, we will approach the conceptual errors that generally occur due to the sole use of common sense to give an interpretation of the independence of events (Nabbout and Maury, 2005; Cordani and Wechsler, 2006).

The confusion of the word independence with exclusion can be an example of this, promoting difficulties in understanding two different probabilistic concepts, independence and incompatibility (Cordani and Wechsler, 2006; D'Amelio and Diblasi, 2006).

Another error associated with the use of common sense is to consider only the definition of independence for independent chronological events, which, according to Steinbring (1986), are associated with the occurrence of successive experiments. As this author states, the other definition of independence is called stochastic and independent events. It is based on the mathematical formula $P(A \cap B) = P(A) * P(B)$, and its understanding is restricted to mathematical proof.

For Kataoka, Trevethan and Silva (2010), this mathematical formula for the independence of events comes from the conditional probability expression and, for this reason, a parallel study is necessary for both concepts. Furthermore, according to Díaz and Batanero (2009), the importance of building knowledge and concepts related to conditional Probability allows us to change the degree of confidence in random events when new information is available.

In this context, research was carried out with some university students from a federal university in Brazil to understand the difficulties experienced in understanding the concepts of mutually exclusive and independent events using the tools of the Ontosemiotic Approach (OSA) of knowledge and mathematics teaching (Godino, Batanero and Font, 2007; Godino, 2009; Godino, Wilhelmi, Blanco and Contreras, 2016).

THEORETICAL REFERENCES

D'Amelio (2004) deepened the study and characterization of errors in higher education students on the concepts of mutually exclusive and independent events. He believes that although the concept of independent and mutually exclusive events is seemingly simple, people's spontaneous ideas lead to incorrect answers.

Therefore, D'Amelio (2004) considers that the errors of the subject are analyzed in front of certain situations when there is a focus for the discussions on the definition of the concept of independent and mutually exclusive events in a course of Probability, how persistent are these ideas, what happens in the process in which the subject confronts his errors with the results of the application of theoretical concepts, to provide elements for better treatment and implementation in teaching.

And so, in his study, D'Amelio (2004) concluded that the difficulties and confusion regarding the concepts of mutually exclusive and independent events were confirmed for higher education students, considering that a possible cause of this confusion is the lack of references to face these problems and thus propose situations appropriate to the objectives of the introductory courses of Probability and statistics.

D’Amelio (2013), like D’Amelio (2004), considers that the population of university students who study Probability in Argentina in activities that involve these concepts present erroneous and spontaneous ideas, which indicates confusion and incorrect associations.

In the study by D´Amelio (2013), a test with problems related to the concepts of mutually exclusive and independent events was applied to 97 statistics students after the contents were taught. They were applied to study the reasoning developed by the students in the resolution strategies, and, in an a priori analysis, the students’ responses were explained and subsequently analyzed.

The results indicate that these have the theoretical status of the premises but are not separated from the status of the content. By not performing calculations in the proposed situation, they only apply definitions and assign the property of independence to mutually exclusive events; that is, they use irrelevant premises. Thus, the authors consider that deductive reasoning is incomplete because it remains in the status of the content and not in the theoretical status of the premises: he knows it because he writes it, but he confuses it (D’Amelio, 2013).

In a study with Mexican high school mathematics teachers, Sánchez (2000) and Guzmán and Inzunsa (2011) found that they exhibited various confusing ideas when faced with tasks that involved the independence of events. Among the main difficulties encountered is the lack of a clear distinction between independent experiences and independent events, the belief that independent events are synonymous with mutually exclusive events, and the belief that only the concept of independent events can be applied to sequences of experiences.

Furthermore, Chernoff (2009) classifies the tasks used in research on the perception of randomness into two types: 1) prediction tasks and 2) recognition tasks. These tasks also allow us to analyze the understanding of independence (Batanero, 2016).

For this study, we consider Meyer’s (1982) conceptualization: If events are independent within the same experiment, the product of probabilities rule is considered: the events A_1, ..., An are independent if $P(A_1 \cap .. \cap A_n) = P(A_1)$

x ... x $P(A_n)$. Therefore, independence is considered to occur when the outcome of one event does not alter the probabilities of other events (previous, simultaneous, or future).

Therefore, when two or more events are mutually exclusive, the occurrence of one excludes the occurrence of the others. If two events are mutually exclusive, the probability that one or the other occurs is equal to the sum of the probabilities that each of them occurs; that is, the elements of these events do not repeat, $P (A \cup B) = P (A) + P(B)$.

METHODOLOGY

Approach

The focus of the work deals with the analysis of the conceptual problems observed in the solution presented by the students who participate in the research on the proposed problem, using the Ontosemiotic Approach (OSA), which allows defining different categories, each of which encompasses responses based on a common idea. We seek to understand what are the confusions between these concepts to provide elements for their better treatment and implementation in teaching.

Analysis Units

A group of students from the Federal University of ABC (UFABC) in the city of Santo André, São Paulo, Brazil, was invited to participate voluntarily and anonymously in the study. Therefore, in 2020, the 74 students of a class with introductory contents of the theory of the Probability of a science and technology course at this university participated in the investigation.

We inform that the study was carried out in a classroom, and although the university councils carried out no ethical evaluation, students were asked to sign and authorize the use of the solutions to the problem under analysis in a model similar to a Term of Free and Informed Consent (ICF). Therefore, the authors explicitly exempt Acta Scientiae from any consequences arising, including full assistance and possible compensation for any damage to any research participants, per Resolution No. 510, of April 7, 2016, of the National Health Council of Brazil.

Recollection Techniques

This study analyzes, through the OSA, the institutional and personal knowledge (described in item 3.4) related to the problem shown in Figure 1 to identify the distinctions between them that result in semiotic conflicts. Therefore, this discrepancy will be verified in resolving a problem that focuses on independent and mutually exclusive events. The student responded in writing to a problem related to these concepts.

Figure 1

A problematic situation involving the concept of independent events

Given that the Probability that travellers visiting Antarctica and observing albatrosses using a company affiliated with the International Association of Antarctic Tour Operators (Iaato) is P (Catch sight of Wandering Albatross) = 28.57% and P (Catch sight of Southern Royal Albatross) = 14.29 % and considering that three albatrosses were detected between these two types, what is the probability that at least two are Wandering Albatrosses?

An epistemic configuration was elaborated (task resolution by an ideal subject) to carry out this analysis, which served as a reference to study the cognitive configurations of the student body (the answers given). According to Blanco, Godino and Pegito (2007), by preparing these configurations, it became possible to identify primary objects and relationships (related to semantics) and secondary (related to the context).

Analysis processing

Godino (2009) understands as a concept the formulations introduced through definitions, descriptions and complements in the OSA, which are also the practices carried out by the student body to solve a mathematical problem with the implicit or explicit use of mathematical objects since they have to remember and apply the definition.

In the OSA model of mathematical knowledge and instruction, according to Godino, Batanero and Font et al. (2007), Godino (2009), and Godino et al. (2016), it is assumed that the mathematical practices (operative and discursive) used to solve a problem situation communicate the solution to others or allow the solution to be validated and generalized to other problems and contexts.

For the authors, problem situations are additional mathematical applications, exercises, problems, and actions that induce a mathematical activity, such as problems involving independent events.

The practices have a double character, and their meaning can be considered from an institutional point of view (in our case, the university, the teaching team and the study plan) or a personal point of view (a person facing a problematic situation, the student) (Godino and Batanero, 1994; Godino et al., 2007).

The teaching in this ontosemiotic structure is based on the student body's participation in the community of practices that share its institutional meaning, while learning would be seen as the appropriation of this meaning (Godino et al., 2007).

If the systems of practices are shared within an institution, the emerging objects are considered institutional objects; if these systems are specific to a person, they are considered personal objects (Godino and Batanero, 1994).

In addition to the aforementioned institutional-personal duality, other dualities are recognized in the OSA (Godino et al., 2007), which are also relevant for our study, namely the expression-content duality, which allows confronting the meanings of the objects that intervene in semiotic functions (understood as correspondences established by a person or institution between an antecedent, expression and a consequent content) with the institutional meanings of reference.

According to Godino et al. (2007), these coincidences can occur in many ways: 1) representational, in which one object is placed in the place of another for a given purpose; 2) instrumental, in which one object uses another as an instrument; 3) structural in which two or more objects form a system from which more objects emerge.

In this process of comparison, verifying the discrepancies between these meanings, institutional and personal, leads to identifying semiotic conflicts (Godino and Batanero, 1994).

This study focuses on the epistemic and cognitive facets, which are the critical facets of teacher training from the perspective of the OSA (Godino, 2009), in which an anthropological and semiotic point of view is postulated for them, in which human activity acquires meaning from the actions of people to solve the problems they face.

For Godino (2009), the epistemic factor refers to the mathematical knowledge of the institutional context in which the study process takes place, that is, the school, the teachers and the textbooks in consideration of different components (problems, languages, procedures, definitions, properties and arguments) and the cognitive refers to the personal knowledge of the students, that is, the progression of their learning. According to the OSA, these content components are called core objects (Pino-Fan, Godino and Font, 2011).

RESULTS

Figure 2 shows two strategies (E1 and E2), identifying the possible paths students must follow to solve the problem based on the study by Fernandes, Serrano and Correia (2016).

Figure 2

Possible strategies to solve the proposed problem

E1 - Recognize the possible events that occur, identify and list the favourable events, assign them the correct values and perform the calculations based on the sum of the conjunctive probabilities favourable to the requested one. E2 - Recognize the possible events that will occur, identify and list the favourable events, assign them the correct values and carry out the calculations taking into account the definition of complement to subtract from the Universe the conjunctive probabilities that represent the unfavourable cases.

The analysis of the responses allowed each response to be classified into one of the following four ordinal levels:

(1) No response (SR): This category refers to the lack of response.

(2) Incorrect answer (I): Appropriate techniques are not used to solve the problem.

(3) Partially correct answer (PC): They lack an explicit technological environment or present some inadequate aspects to arrive at the answer.

(4) Correct answer (C): show the use of an adequate technique to solve the problem and in which the theoretical and technological environment justifies the solution form is explicit.

Table 1 shows the frequency of the responses of the research participants at each of the following ordinal levels; that is, no response (SR), wrong answer (I), partially correct answer (PC), and the correct answer (C).

Table 1

Frequency of student responses to the problem

Type of response	Number of students	Percentage (%)
Correct (C)	15	20.27
Partially Correct (PC)	3	4.05
Incorrect (I)	53	71.63
No Response (NR)	3	4.05

Figure 3

Posible solución al problema propuesto a partir de E1

Catch sight of Wandering Albatross = AWA y Catch sight of Southern Royal Albatross = ASRA

$P(AWA) = 0{,}2857$ y $P(ASRA) = 0{,}1429$

$$\begin{aligned} &P(AWA \cap AWA \cap ASRA) + P(AWA \cap ASRA \cap AWA) \\ &\quad + P(ASRA \cap AWA \cap AWA) + P(AWA \cap AWA \cap AWA) \\ &\quad = (0.2857 \times 0.2857 \times 0.1429) + (0.2857 \times 0.1429 \times 0.2857) \\ &\quad + (0.1429 \times 0.2857 \times 0.2857) + (0.2857 \times 0.2857 \times 0.2857) \\ &\quad = 0.01166 + 0.01166 + 0.01166 + 0.02332 = 0.0583 = 5.83\% \end{aligned}$$

In the data presented in Table 1, it can be identified that only 15 students (20.27%) responded adequately to the problem and the following one shown in Figure 3; that is, it is the easiest way to solve the problem.

It is noteworthy that the algebraic development presented is only one way of expressing and developing the problem since there are other elements that can make up the language, as pointed out by Blanco et al. (2007), such as the representation of images that, in this case, can be the visual representation of the problem.

It is also observed that 53 students (71.63%) presented the solution incorrectly, and another three students (4.05%) presented some type of conceptual error, notation or calculation error.

Initially addressing the students' responses that we consider incorrect (I), Table 2 shows the frequency of errors identified and associated with strategies E1 and E2, presented in Figure 2 of this work, through which we will perform the analysis using OSA.

Table 2

Type of strategy (E1 or E2) used by the students associated with the incorrect answer category (I)

Type of error by strategy	Number of students	Percentage (%)
E1	37	69.82
E2	7	13.20
Unidentifiable	9	16.98
Total	53	100.00

From the reading of Table 2, it can be identified that the solution presented by nine students (16.98%) could not be associated with any of the two strategies defined in this work, which indicates deficiencies of the students in the development of the problem and consequent lack in the mastery of concepts.

We take the study by Kelly and Zwiers (1986) to justify why this group of students has difficulty solving the problem: of distinguishing between independent and dependent events. The identified problem arises from not knowing the concept of mutually exclusive events, that is, not realizing what independent or dependent events are in a real problem.

For example, if we roll a pair of dice, the result on one die does not influence the outcome on the other die. In other cases, knowledge of a specific area is often needed to make an informed judgment, regardless of whether or not the particular events are independent.

Another point noted in the analysis is that, in the real world, it is not always clear whether two events are mutually exclusive. For example, not all students know that "Clark Kent" and "Superman" are not mutually exclusive since they are the same person. Therefore, it would be helpful to draw examples from nature, where the distinction between contradictory, contrary, mutually exclusive and not mutually exclusive is not always clear (Kelly and Zwiers, 1986).

In addition, it was found that the strategy most used by the students was E1, Table 2. However, it is verified that, among those who chose to solve the problem through this strategy, 69.82% made some type of error.

In the case of using E1, students should have associated the following concepts based on Meyer (1982): The multiplication rule assuming that a procedure, designated by a, can be carried out in n_1 ways and that a second procedure, designated for b, it can be done in n_2 ways. Furthermore, each way of doing a can be followed by any of the ways of doing b. Then the procedure formed by a followed by b can be done in $n_1.n_2$ ways.

So they should have considered that the multiplication rule can be extended to any number of procedures, i.e., if there are k procedures and the i-th procedure can be performed in no way, i = 1, 2, ..., k, then the procedure formed by 1, followed by 2, ..., followed by procedure k, can be performed in $n_1 \times n_2 \times \ldots \times n_k$ ways (Meyer, 1982).

In the case of the solution presented in E1, we consider "P (AWA $\cap$ AWA $\cap$ ASRA)", where we would have three different events indicated by AWA and AWA and ASRA.

They should still have considered the addition rule, assuming that one event, designated, say, by a, can be performed in n_1 ways and that a second procedure, designated by b, can be performed in n_2 ways. Also, suppose that it is not possible for both events a and b to occur together. So, the number of ways we can do a or b is $n_1 + n_2$.

In sequence, consider that the addition rule can be generalized as follows: if there are k procedures and the i-th procedure can be performed in n_i ways (i = 1, 2, ..., k), then the number of ways we can perform event a, or event b, or ..., or event k, is given by $n_1 + n_2 + \ldots + n_k$, assuming that no two of them can be performed together.

In the case of the solution presented in E1, for example, "P (AWA $\cap$ AWA $\cap$ ASRA) + P(AWA $\cap$ AWA $\cap$ ASRA)", we would have two different events indicated by P(AWA $\cap$ AWA $\cap$ ASRA) or P (AWA $\cap$ AWA $\cap$ ASRA).

For strategy 2 (E2), Figure 4, the student body should have associated, in addition to the concepts expressed in E1, the following concepts that are also based on Meyer (1982).

They should have started from the idea that two events, A and B, are said to be mutually exclusive if they cannot occur together, i.e., the intersection of A and B is the empty set (A ∩ B = ∅). Also, consider that the same two events A and B are collectively excluded if the union between them is the same sample space; that is, A ∪ B = S.

Consequently, given any event A, then its complement $\bar{A}$ will be the event that will occur if, and only if, A does not occur. What we infer from this is that, considering two events A e $\bar{A}$, then A ∩ $\bar{A}$ = ∅ (mutually exclusive) and A ∪ $\bar{A}$ = S (collectively exclusive).

Figure 4

Possible solution to the problem proposed from E2

Catch sight of Wandering Albatross = AWA, catch sight of Southern Royal Albatross = ASRA and Catch sight of other types of Albatrosses = AOA

$$P(AWA) = 0.2857, P(ASRA) = 0.1429 \; y \; P(AOA) = 1 - (0.1429 + 0.2857) = 0.5714$$

$$\begin{aligned} 1 - [&P(AWA \cap ASRA \cap ASRA) + P(ASRA \cap ASRA \cap AWA) \\ &+ P(ASRA \cap AWA \cap ASRA) + P(AWA \cap AWA \cap AOA) \\ &+ P(AWA \cap AOA \cap AWA) + P(AOA \cap AWA \cap AWA) \\ &+ P(ASRA \cap ASRA \cap AOA) + P(ASRA \cap AOA \cap ASRA) \\ &+ P(AOA \cap ASRA \cap ASRA) + P(AWA \cap AOA \cap AOA) \\ &+ P(AOA \cap AOA \cap AWA) + P(AOA \cap AWA \cap AOA) \\ &+ P(ASRA \cap AOA \cap AOA) + P(AOA \cap ASRA \cap AOA) \\ &+ P(AOA \cap AOA \cap ASRA) + P(ASRA + ASRA + ASRA) \\ &+ P(AOA \cap AOA \cap AOA) + P(AWA \cap ASRA \cap AOA) \\ &+ P(AWA \cap AOA \cap ASRA) + P(ASRA \cap AWA \cap AOA) \\ &+ P(ASRA \cap AOA \cap AWA) + P(AOA \cap AWA \cap ASRA) \\ &+ P(AOA \cap ASRA \cap AWA)] = 1 - 0.9417 = 0.0583 = 5.83\% \end{aligned}$$

Therefore, if they choose to use the solution for complementary events, they would have to think of a third group (event) to represent the composition of the sample space (S), i.e., "see other types of albatrosses - AOA" and determine its probability: P(AOA) = 1 - [P(AWA) + P(ASRA)] = 1 - [0.2885 + 0.1429] = 0.5714. The events “AWA”, “ASRA”, and “AOA” would be mutually exclusive and AWA ∪ ASRA ∪ AOA = S (sample space).

We point out that all the students who opted for strategy 2 (E2) highlighted identifying the AOA event that generated P (AOA) = 0.5714. They even thought about the complementarity between the events, but created probabilities for the AWA and ASRA events, forcing them to be mutually and collectively exclusive.

Table 3

The individual and joint occurrence of errors present in the answers to the proposed problem using strategies E1 and E2

Occurrence of Errors	E1		E2	
	n	%	n	%
Not identifying all the events that should constitute the solution to the problem (NIE), Identifying problems in the calculations (C) and little clarity in the solution of the problem (PC)	11	29.72	2	28.57
Not identifying all the events that should constitute the solution to the problem (NIE) and Identifying problems in the calculations (C)	12	32.46	2	28.57
Not identifying all the events that should constitute the solution to the problem (NIE)	9	24.32	2	28.57
Not identifying all the events that should constitute the solution to the problem (NIE) and little clarity in the solution to the problem (PC)	1	2.70	-	0.00
Not identifying all the events that should constitute the solution to the problem (NIE) and misinterpretation of the problem (IEP)	1	2.70	-	0.00
Identify problems in the calculations (C) and lack of clarity in the solution of the problem (PC)	1	2.70	-	0.00
Identify problems in calculations (C)	1	2.70	1	14.29
Little clarity in the solution of the problem (PC)	1	2.70	-	0.00

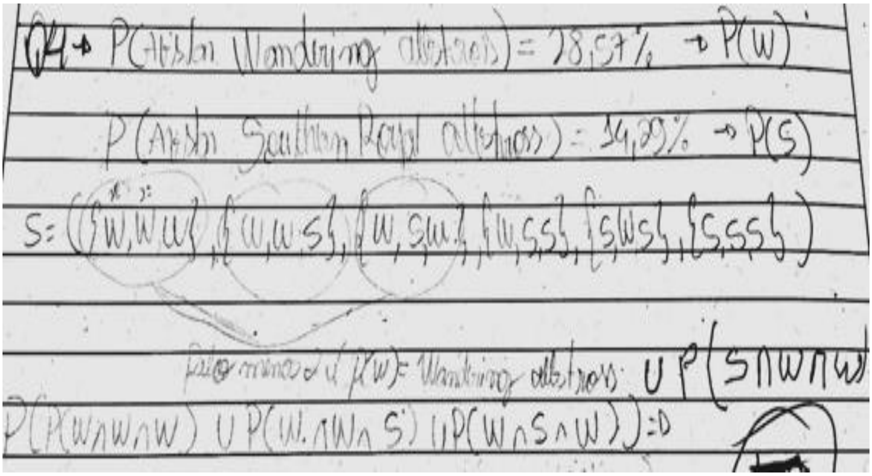
Students' responses were analysed to identify errors taking these resolutions as a reference. Four types of errors present in the resolutions were identified: 1) Not identifying all the events that should constitute the solution to the problem (NIE); 2) Identifying problems in the calculations (C); 3) Little clarity in the solution of the problem (PC); 4) Wrong interpretation of the problem (IEP). Table 3 presents the occurrence of errors in the resolutions present in each strategy.

It can be seen in Table 3 the occurrence of more than one type of error when solving the problem (considering the categories created in this work), observing that in 70.27% of the answers, the students used the E1 strategy and 57.14%, the E1 strategy. E2. The most common error was that not all events were listed; therefore, the probability requested by the exercise was not calculated correctly.

In both strategies (E1 and E2), the NIE type error predominated both jointly (with other errors) and individually. Figure 5 represents a typical response to the NIE type error of one of the students.

Figure 5

Error made by one of the students and classified as identification of the probabilities of the conjunction of events (IP)



The proposed problem considered the event E = {sighting of three albatrosses between Wandering albatrosses - W and Southern Royal - S} and requested that the Probability that at least two albatrosses Wandering albatrosses be determined.

In the case of figure 5, we consider that the conceptual errors led the student not to reach the proposed solution, that is: (1) the sample space was incorrect since it was not considered that the sum of the probabilities of the events that they formed it was different from 100%; (2) there are no options in the description of all the elements contained in the sample space and, consequently, the requested event, that is, (S, W, W); (3) due to the lack of the event (S, W, W), the Probability of the proposed event was underestimated.

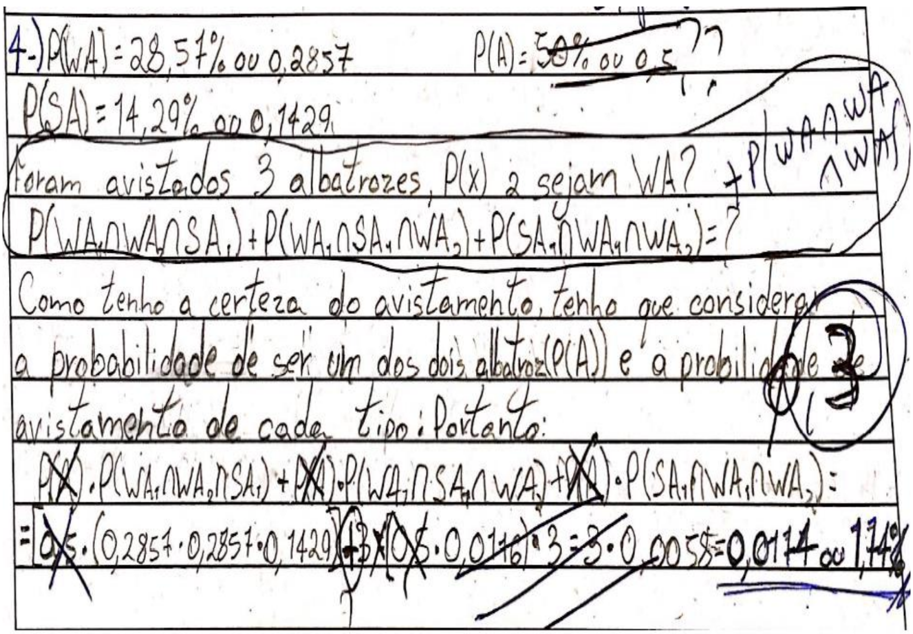
The errors related to the calculation (C) presented three main factors: 1) performing operations incorrectly; 2) erroneous transcription of data; 3) transforming a decimal number into a percentage in the wrong way.

It was found that the calculation errors occurred mainly by not identifying all the events that should constitute the solution to the problem, as exemplified in Figure 6.

The problem presented in figure 6 is similar to the error indicated in figure 4, that is, the lack of one of the elements of the requested event, in this case, the event (WA, WA, WA).

Figure 6

The response of a student with calculation error (C) and identification of probabilities (IP)



4-) $P(WA) = 28,57\%$ ou $0,2857$ $P(A)$: ~~50% ou 0,5~~

$P(SA) = 14,29\%$ ou $0,1429$

Foram avistados 3 albatrozes, $P(x)$ q sejam WA? $+ P(WA \cap WA \cap WA)$

$P(WA_1 \cap WA_2 \cap SA_1) + P(WA_1 \cap SA_1 \cap WA_2) + P(SA_1 \cap WA_1 \cap WA_2) = ?$

Como tenho a certeza do avistamento, tenho que considerar a probabilidade de ser um dos dois albatroz ($P(A)$) e a probabilidade de avistamento de cada tipo: Portanto:

$P(A) \cdot P(WA_1 \cap WA_2 \cap SA_1) + P(A) \cdot P(WA_1 \cap SA_1 \cap WA_2) + P(A) \cdot P(SA_1 \cap WA_1 \cap WA_2) =$

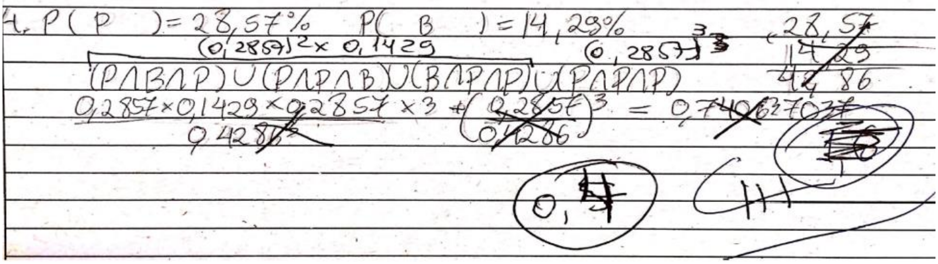
$= [0,5 \cdot (0,2857 \cdot 0,2857 \cdot 0,1429)] \cdot 3 = (0,5 \cdot 0,0116) \cdot 3 = 3 \cdot 0,0058 = 0,0174$ ou $1,74\%$

In addition, a probability indicated by P(A) = 0.5 was presented without justification based on probability theory. It was considered that since two albatrosses were detected, each of them would have an equal chance of being selected, which is the classic probability approach and would not fit in this context.

In addition, when calculating, the multiplication operation is exchanged for addition and addition for multiplication. We also identify the use of the value “0.0116”, which is a truncated value of the multiplication of “0.2885 * 0.2857 * 0.1429”, resulting in a reduction in product size and Probability.

Figure 7

La respuesta de un estudiante que contiene los errores: poca claridad (P) e identificación de probabilidades (IP)



Regarding the type of error, unclear problem solving (PC) refers to unclear answers when the wrong notation is presented concerning probability theory. In figure 7, we show an example of a solution that brings the type of error of little clarity in the solution of the problem (PC) and not identifying all the events that should constitute your solution (NIE).

La figura 7 muestra la siguiente representación: $(P \cap B \cap P) \cup (P \cap P \cap B) \cup (B \cap P \cap P) \cup (P \cap P \cap P)$ que debería haberse presentado como: $P(P \cap B \cap P) \cup P(P \cap P \cap B) \cup P(B \cap P \cap P) \cup P(P \cap P \cap P)$.

La indicación no influiría en el cálculo, sin embargo, demuestra la falta de preocupación por expresar correctamente en términos notacionales y conceptuales haciéndonos pensar si fue sin querer la indicación incorrecta o la falta de comprensión de la representación correcta.

Además, se generó la probabilidad de ver cada uno de los dos tipos de albatros, generando la probabilidad (0.2885 + 0.1429 = 0.4286). Luego, cada una de las partes de la representación de los elementos del evento se dividió entre esa cantidad y aún aumentó al exponente tres.

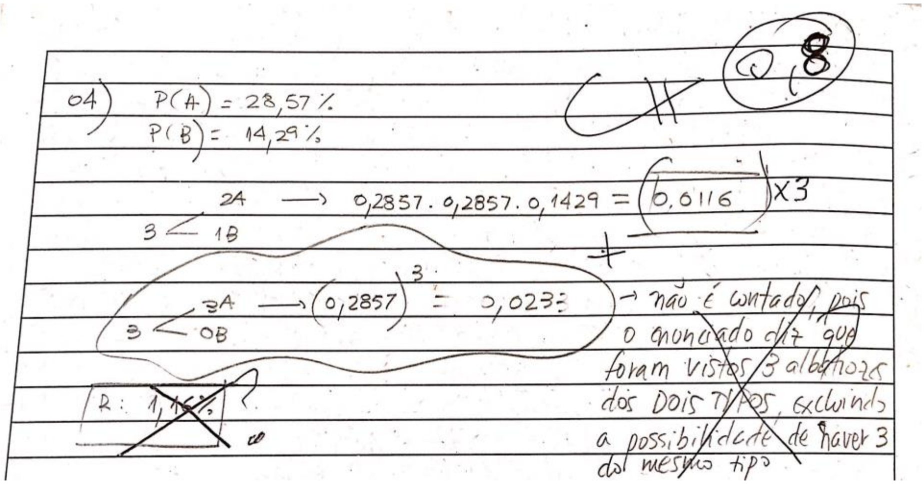
En los errores de tipo NIE, identificamos que es difícil para el estudiantado representar el lenguaje simbólico (notaciones). Los errores de tipo C, de manera similar, presentaron un conflicto que impregna el uso del lenguaje simbólico, ya que estos se equivocaron en la manipulación de símbolos matemáticos para el desarrollo de la resolución de problemas.

Otro conflicto semiótico observado fue la falta de comprensión encontrada en la interpretación del enunciado del problema (IEP) y el uso inapropiado del lenguaje ordinario (términos y expresiones) para la composición del argumento indicado por la falta de claridad en la presentación de los argumentos para resolver el problema (PC)

El error menos recurrente en ambas estrategias (E1 e E2) fue la interpretación errónea del problema (IEP), es decir, la adopción de la premisa errónea de lo que indicaba el problema.

Figure 8

The response of a student that contains the errors: lack of clarity (P) and misinterpretation of the problem (IEP)



En la Figura 8 se puede ver la respuesta de un alumno que se refiere a la posibilidad de detectar los albatros, en función de su justificación sobre una falsa premisa: " no contados, ya que el planteamiento del problema dice que se vieron 3 albatros de ambos tipos, excluyendo la posibilidad de que haya 3 del mismo tipo".

This type of error has the immediate consequence of identifying the events incorrectly (IEP) since, by establishing this justification, one of the possible events is eliminated from the sample space, that is, $(A \cap A \cap A)$. It is interesting to note in the solution that the student indicated the probability $P(A \cap A \cap A) = (0.2857) * 3 = 0.0233$ and did not use it.

In addition, he only considered one of the possibilities of the possible events where there would be two albatrosses of type A and one albatross of type B. He considered the event $(A \cap A \cap B)$ and did not consider the events $(A \cap B \cap A)$ and $(B \cap A \cap A)$. Such neglect of events resulted in an incorrect composition of the probabilities of the conjunction of the event of interest.

Semiotic conflicts were therefore found to penetrate a primary object (language) directly and a primary object (argument) indirectly. The constituent components of language that were significant for the discussion were ordinary, graphic, and symbolic language.

Kahneman and Tversky (1979) show that people with little or no knowledge of statistics estimate the Probability of events through certain heuristic judgments, such as representativeness and availability. According to the representativeness heuristic, people estimate the probabilities of events based on how well an outcome represents some aspect of its original population.

For D'Amelio (2013), the student body has the theoretical status of the premises, but they are not separated from the status of the content. When performing calculations in the proposed situation, they only apply definitions and assign the property of independence to mutually exclusive events; that is, they use irrelevant premises.

Finally, we take D'Amelio (2013) when he says that the main didactic problem consists of diverting the students' attention from the content and focusing it on the form. But since, in reality, the semantic content and the form cannot be separated in the register of natural language, it would then be necessary to neutralize the semantic content by proposing to reason with absurd propositions: it is needed to lead the student to dissociate the logical form and the semantic content.

FINAL CONSIDERATIONS

The research methodology in this article allowed us to analyze the productions carried out by the student body faced with a problem of Probability. In this research, the object studied refers to mutually exclusive events and independent events. The problem situation confronts the student with a demonstration in which its resolution is requested.

The most frequent error when it was decided to solve the problem using strategy 1 (E1) was related to not identifying all the events that should constitute the solution to the problem, where not all the elements of the event were listed and, therefore, the Probability requested by the problem was not calculated correctly.

Regarding the students who chose to solve the problem considering strategy 2 (E2), they considered the complementarity between the events. Still, they erroneously created probabilities for the AWA and ASRA events, forcing them to exclude each other mutually.

Therefore, although the students understand in which contexts to use the concept of complementary events, they made a mistake when adjusting the probabilities of the events presented in the problem and that they were not collectively exhaustive; that is, the sum of their probabilities is the Probability of the sample space.

Thus, after performing the data analysis (solving the problems presented by the student body), we judge that the OSA analysis tools (Godino and Batanero, 1994; Godino et al., 2016) are characterized as a possibility to analyze the process of learning of fundamental elements of the theory of Probability, allowing to highlight the pertinence and relevance of the actions carried out by them, as well as the knowledge presented, allowing with the analysis of the answers to show the approximation or distance between the personal meanings reached and the expected institutional meanings, which is the appropriation of the basic concepts of Probability and training for citizenship.

With this study, we realize that the population of university students who participate in this research and who study Probability in activities in which concepts of mutually exclusive events and independent events are involved present spontaneously erroneous ideas, confuse both and associate them incorrectly. These concepts seem to be straightforward or simple in their definition. Nevertheless, after evaluating the solutions they indicated, we recognize that they have deficiencies in the concepts.

In addition, the investigation of the performance of this student body in the probability calculation tasks warns us of difficulties and biases that generally come from understanding the conditions that govern the experience and the set of possible results related to it.

Based on the studies carried out by Sánchez (2000), D'Amelio (2004), Guzmán and Inzunsa (2011), D'Amelio (2013) and Fernandes, Serrano and Correia (2016) with Argentine, Mexican and Portuguese students, we consider that this study with Brazilian students is similar to the results obtained in other countries in that it has shown the difficulties and confusion of concepts of mutually exclusive and independent events. One possible cause of such confusion is the lack of an adequate theoretical basis for addressing these issues, and proper planning should be considered to fit the objectives of introductory probability courses.

Therefore, it is suggested that these concepts be reworked in the classroom using different teaching tools, such as creating virtual learning environments based on research processes and real situations. And then, evaluate again if the concepts were learned that contribute to improving the learning of the various introductory concepts of Probability and extend the analysis that is also carried out here for the reading and interpretation of problematic situations with which the students can appropriate greater depth of concepts.

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AUTHORS' CONTRIBUTION STATEMENTS

APOJ conceived the ideas for the tasks presented. APOJ and DFBN developed the theory, adapted the methodology to this context, carried out the activities, and collected the data. APOJ and DFBN analyzed the data in the first round. All authors actively participated in the discussion of the following rounds of analysis and results, revised, improved the theoretical discussion included and approved the final version of the work.

DATA AVAILABILITY STATEMENTS

Data supporting the results of this study will be made available by the corresponding author APOJ, upon reasonable request via email.

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