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Teacher Training and the use of the Abacus in the Practice of Quantifying Physical Quantities

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ABSTRACT

Background: Several studies in mathematics education show the use of the physical abacus in the activities of teachers in initial graduation, emphasising, in their manipulation, practices with addition or subtraction operations, or yet, which express a direct translation of the numeral to the physical abacus and, vice versa, without, however, explaining part of the complexity involved the social practice of quantifying physical quantities, which even makes possible the structuring of the numeral. **Objective**: To show that the use of the abacus by teachers in initial training may not ensure the realisation or mastery of the practice of quantifying physical quantities with discrete units. **Design**: In this sense, a study and research path (SRP) was carried out based on the training of early years teachers. **Environment and participants**: 25 students in initial training of a degree course at a public educational institution participated in an activity on decimal number system (DNS), but largely positional number system (PNS), proposed based on a problem in an unusual context (for them), they mobilised to face and answer questions. **Data collection and analysis**: We present an excerpt from the empirical research forwarded by Ferreira (2020) with teachers in initial training. **Results**: The data observed in the empirical confirms the hypothesis of the existence of a problem regarding the use of the abacus as a possible facilitator in the structuring of non-decimal numbers through the practice of quantifying physical quantities. **Conclusions**: The results found with the teachers revealed, in addition to the problem of teacher training on what to teach and how to teach numerals, that using the abacus made it difficult, if not hindered, the practice of quantifying physical quantities with discrete units. We also found changes in the quality of the teachers' relationships with structuring decimal and non-decimal numbers.

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Keywords: Teacher training; Decimal number system; Abacus; Numeral; Anthropological theory of the didactic.

A formação de professores e o uso do ábaco na prática de quantificação de grandezas físicas

RESUMO

Contexto: Diferentes pesquisas na Educação Matemática evidenciam o uso do ábaco físico nas atividades de professores em formação inicial, enfatizando, em sua manipulação, práticas com operações de adição ou de subtração, ou ainda, que expressam uma direta tradução do numeral para o ábaco físico e, vice-versa, sem, no entanto, explicitar parte da complexidade que envolve a prática social de quantificação de grandezas físicas que, inclusive, torna possível a estruturação do numeral. **Objetivo**: evidenciar que o uso do ábaco por professores em formação inicial pode não assegurar a realização ou domínio da prática de quantificação de grandezas físicas com unidades discretas. **Design**: Nesse sentido, foi realizado um Percurso de Estudos e Pesquisas (PEP) orientado a partir da formação de professores dos anos iniciais. **Ambiente e participantes**: 25 alunos em formação inicial de um curso em licenciatura de uma instituição pública de ensino participaram de uma atividade sobre Sistema de Numeração Decimal (SND), mas amplamente Sistema de Numeração Posicional (SNP), proposta a partir de um problema em contexto não usual (para eles), eles se mobilizaram em enfrentar e responder a questões. **Coleta e análise de dados**: apresentamos um recorte a partir da pesquisa empírica encaminhada por Ferreira (2020) com professores em formação inicial. **Resultados**: os dados observados na empiria ratificam a hipótese de existência de uma problemática quanto ao uso do ábaco como possível facilitador na estruturação de numerais não decimais por meio da prática de quantificação de grandezas físicas. **Conclusões**: os resultados encontrados com os professores revelaram, além do problema de formação docente sobre o que ensinar e como ensinar os numerais, que o uso do ábaco dificultou, senão impediu a realização da prática de quantificação de grandezas físicas com unidades discretas, além é claro, de mudanças na qualidade de relações dos professores com a estruturação dos numerais decimais e não decimais.

Palavras-chave: Formação de Professores; Sistema de numeração Decimal; Ábaco; Numeral; Teoria Antropológica do Didático.

La formación del profesorado y el uso del ábaco en la práctica de cuantificar cantidades físicas

RESUMEN

Contexto: Diferentes investigaciones en Educación Matemática muestran el uso del ábaco físico en las actividades de los docentes en formación inicial, destacando, en su manipulación, prácticas con operaciones de suma o resta, o todavía, que expresan una traducción directa del numeral al ábaco físico y, viceversa, sin embargo, explicar parte de la complejidad implicando en la práctica social de cuantificar cantidades físicas, que, incluyendo posibilita la estructuración del numeral. **Objetivo**: demostrar que el uso del ábaco por parte de los profesores en la formación inicial puede no asegurar la realización o el dominio de la práctica de cuantificación de cantidades físicas con unidades discretas. **Diseño**: En este sentido, se llevó a cabo una Ruta de Estudios e Investigación (PEP) basada en la formación de los docentes en los primeros años. **Entorno y participantes**: 25 alumnos en formación inicial de una carrera de grado en una institución educativa pública participaron en una actividad sobre Sistema de Numeración Decimal (SND), pero mayoritariamente Sistema de Numeración Posicional (SNP), propuesta a partir de un problema en contexto inusual (para ellos), se movilizaron para afrontar y responder preguntas. **Recogida y análisis de datos**: presentamos un extracto de la investigación empírica remitida por Ferreira (2020) con docentes en formación inicial. **Resultados**: los datos observados en el empírico confirman la hipótesis de que existe un problema en cuanto al uso del ábaco como posible facilitador en la estructuración de numerales no decimales a través de la práctica de cuantificar cantidades físicas. **Conclusiones**: los resultados encontrados con los profesores revelaron, además del problema de la formación del profesorado sobre qué enseñar y cómo enseñar numerales, que el uso del ábaco dificultaba, si no obstaculizaba, la práctica de cuantificar cantidades físicas con discretas. unidades, por supuesto, a partir de cambios en la calidad de las relaciones de los profesores con la estructuración de números decimales y no decimales.

Palabras clave: Formación del profesorado; Sistema de numeración decimal; Ábaco; Número; Teoría Antropológica de la Didáctica.

INTRODUCTION

The systems of numeration come from prehistoric civilisations, and each of them, through their practices, built registers of representation of quantities. To refer to some numeral systems, we can highlight the Egyptian, Greek, Roman, and Mayan civilisations, among others, which, for a long time, contributed in some way to the construction of the Hindu - Arabic number system, also known as the decimal number system, henceforth DNS.

Ferreira and Guerra (2020) highlight, based on Ifrah (1985), that:

> This system would have been brought by the Arab civilisation to Europe, in the middle of the 7th century, from where it was spread to other civilisations, becoming dominant today in the world. Perhaps that is why its teaching has become indispensable in schools, initially, to meet the craft of different social practices among human activities that were later

> included in schools of fundamental knowings. (Ferreira & Guerra, 2020, p.2)

This social and cultural importance on the DNS to meet different social practices, among them the basic school, for example, presents in some way the relevance of knowledge as a "*wise knowledge*" legitimised by culture (Chevallard, 2005). In this sense, the DNS teaching is interesting for educational institutions, as it cuts across the basic school curriculum, which, according to Terigi and Wolman (2007), occupies a strategic place in the curriculum of all countries. For this reason, they consider that DNS teaching is one of the main factors in school failure.

Some mathematics education literature deals with the teaching-learning process of the DNS, as it is content provided for in the school curriculum. Since it is considered indispensable for the teaching of the initial years of basic education, several researchers were willing to study and investigate the problems found in school on the topic, such as Lerner and Sadovsky (1996), Terigi and Wolman (2007), Itzcovitch (2008), Lendínez, Garcia, and Sierra (2017), and Sierra and Gascón (2018).

According to Sadovsky (2010, p.13), we must "review the mathematics that lives in the school, interrogate it, analyse it to conceive other (teaching) scenarios".

Many researchers share this same thinking and assume that it is a problem related to teacher training involving the DNS. Among them, we have Carvalho (2007), Terigi and Wolman (2007), Sadovsky (2010), Cenci, Becker and Mackedanz (2015), Ferreira and Guerra (2020), and Ferreira (2020).

Cenci, Becker, and Mackedanz (2015), for example, carried out research in Brazil from 2008 to 2015 on early years teachers' training with interventions on DNS teaching. These authors, besides reporting the non-learning of the DNS, highlight it as a problem for these teachers, perhaps due to their little familiarity with this knowledge (Chevallard, 2005, 2019).

For Sadovsky (2010), mathematics and its teaching are a problematic core in schools for the early years, as it is responsible for children's school success and failure. He considers that the vital point is to focus on the issues of a very particular intersection between the contents and the mathematical update with central and concrete problems for teaching.

Terigi and Wolman (2007) propose to understand the social and educational processes that contribute to the production of school failure, focusing on analysing how the usual teaching of the number system may be contributing to the production of school failure and showing how it is possible, in specific conditions, to generate teaching proposals that place children in a position of increasing mastery of this cultural tool, which is the basis for learning and mathematical knowledge at school.

Carvalho (2007) understands that the mathematical contents learned in teacher education courses for the early years have contributed little to and in pedagogical practices, as prospective teachers bring with them, since basic education, a didactic of reproduction of models, repeating what they saw in a system of mimicry. In this sense, they reveal that teachers in training feel insecure about mathematics teaching. Thus, the challenge is to teach mathematics that allows "to assume the contents to be taught safely and in such a way that they are considered satisfactory" (Cenci, Becker, & Mackedanz, 2015, p. 33).

Ferreira (2020) considers that factors external to the classroom act to make teaching difficult, one of which is the low relationship of these teachers with the mathematical content that corroborates unsatisfactory teaching for early years children. Another important point is the need for a more complete education for teachers to endow them with a theoretical-methodological infrastructure that will allow them to build a relationship of complicity with the mathematical contents, in our case, the decimal numbers. From their realities (concrete situations), they must come to understand them and thus reconstruct their practices, considering their needs and difficulties. This leads us to affirm that more comprehensive and specific mathematical studies are needed in the early years teacher education courses.

Following and using Ferreira and Guerra (2020), initial and continuing education promote the encounter or re-encounter of new issues about teacher education for the production of new teaching practices with decimal numbers, taking into account the mobilisation of old and new knowledge in interactions with each other to meet the teaching of a given school year in line with the world, requiring educators and education centres to look at mathematical content, in our case decimal numbers, as indispensable in teacher education.

Still about the teaching of the DNS, the problem may originate from the treatment given to the notion of numeral, which seems to be naturalised (Itzcovich, 2008, Ripoll; Rangel & Giraldo, 2016) mainly by teachers, who possibly do not understand their complexity and everything that involves it, such as the construction of Hindu-Arabic numerals as social practices of quantification of physical quantities that can lead to understandings opposed to what it is actually intended for in the social context.

For Ferreira and Guerra (2020):

> Decimal numerals are objects of social knowledge that are accessed by social agents in an almost transparent way, as paramathematical knowledge (Chevallard, 2005), which, in teaching, is knowledge used in an unquestionable and indispensable way, but which are never objectively taught. They are knowledge institutionalised by our civilisation, culture, and society, and, for this reason, they can prevent the structure of decimal numbers from not being, or not being visible at school (Ferreira & Guerra, 2020, p. 6).

Regarding more specific knowledge in this PNS case, Sierra and Gascon (2018) point out that most of the works uncritically assume the organisation of mathematical knowledge about the DNS and, more broadly, the positional number system, henceforth PNS, for teaching in school institutions. Those authors raise the need to problematise institutionalised mathematical knowledge as it is conceived by the anthropological theory of didactics, henceforth ATD (Chevallard, 1999, 2005, 2019, 2020), which highlights that human activity, in particular mathematical activity, can be described through praxeological organisations, i.e., "the notion of a praxeology was introduced as an essential means of analysing human activity – whether mathematical or otherwise"[1] (Chevallard, 2019, p. 83, our translation).

Questioning our daily practices seems necessary or perhaps essential for the construction of significant knowledge, especially about the DNS-PNS school knowledge, considering that the construction of numerals has its historical-epistemological genesis in the social practices of quantification of physical quantities, although, in general, the "discourse" of the academic mathematics institution on the study of natural or integer numbers seems to prevail in basic school.

In this sense, Chevallard (2019), at the heart of the didactic transposition subtheory, highlights more comprehensive unpretentious issues, such as:

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[1]Text snippet: *The notion of a praxeology was introduced as an essential means of analyzing human activity—be it mathematical or otherwise.*

What is this knowledge that you call 𝓀σ and claim to teach? Where does 𝓀σ come from? How is 𝓀σ legitimised - epistemologically speaking? *Is* 𝓀σ viable in the long run? Or will it have to be reprocessed or even displaced? (Chevallard, 2019, p. 76, author's emphasis, our translation).

In the wake of this construction, Ferreira and Guerra (2020, p. 6) propose for the early years teacher education the construction of the basic notion of numerals "from quantifications through successive, non-decimal clusters" that meets the recommendations by Ripoll, Rangel, and Giraldo (2016), where the goal is to (re)construct the notions of numerals from the practices of quantification of physical quantities.

Teacher education is an issue of interest to ATD, and, in this case, it stands out as a problem of the teaching profession, paraphrased here in the following terms: *what to teach and how to teach decimal numbers?* that generally seems to be faced from the use or exploitation of concrete material resources, among them, and of our interest, the physical abacus as conditioning that is supposedly seen as a "facilitator" for the teaching and learning of arithmetic operations. The hypothesis raised is that using the abacus can make it difficult, if not prevent, the quantification of physical quantities for the member structuring and registering, as discussed below.

THE RESEARCH PROBLEM

Some authors who highlight the use of the abacus for teaching, such as Viegas and Serra (2015), Cruz, Teodoro, and Bonutti (2019), Gomes, Paula, and Oliveira (2019), and Lima, Santos, and Abreu (2019), point out some potentialities of the abacus, and argue that its use can minimise difficulties in teaching and learning operations with decimal numbers.

In that regard, Gomes, Paula, and Oliveira (2019) highlight that:

> The structure of this material and its intrinsic dynamics help to understand ideas and concepts, such as positional value, one-to-one correspondence; counting by clusters; composition and decomposition of quantities; number recognition; operation recognition; numerical operability, among others (Gomes, Paula, & Oliveira, 2019, p. 27587).

The text extract clearly highlights that using the abacus helps in counting by clusters and/or composing and decomposing quantities. In addition, the authors Gomes, Paula, and Oliveira (2019) consider that materials are instruments with the potential to lead students to develop observation and reasoning, contributing to the learning of arithmetic operations.

However, for Lima, Santos, and Abreu (2019), the abacus has the potential to foster logical-mathematical development, leading students to exercise their ability to observe, perceive, and concentrate, among others. According to them, the abacus allows students to operate the four operations using concrete material, which fosters learning, as it is a resource that aims to help the teacher to complement his didactics through this approach.

This thinking highlights the possibilities of using the abacus as an instrument to facilitate school learning, including different practices of school mathematics and, in a dominant way, addition and subtraction operations, for example.

In this sense, for Cruz, Teodoro, and Bonutti (2019), using the abacus in the classroom works as a facilitator for understanding the decimal number system, by making a concrete approach to the representation of numbers, also helping in addition and subtraction operations.

Cruz, Teodoro, and Bonutti (2019, p. 7) state that "first the student must know how to represent any proposed and possible numeral on the abacus". This idea misleads possibilities of questioning it as proposed by the ATD, as a didactic instrument that may or may not make it possible to teach non-decimal numbers, in addition, of course, to make it difficult, if not to prevent, the task of quantifying physical quantities using the abacus.

It seems evident that the approach presented by those authors does not consider one of the possible tasks that can be considered with the use of the abacus, i.e., the quantification of physical quantities for structuring the decimal and non-decimal number.

Under this understanding, we presuppose that the use of the abacus seems restricted, if not limited, to the teaching of culturally instituted decimal numbers in school education, whose praxeologies that include the passage of the representation of the decimal number registration in the physical abacus and vice versa mislead possible complexities of the practice of quantification that can be evidenced in the act; in particular, the non-visibility of non-mathematical knowledge that conditions and is conditioned in the conformation of the practice at stake.

This comprehension seems to meet a methodological problem assumed by the ATD, but not only, that "arises with respect to all conditions and restrictions mentioned above and, of course, of conditions and restrictions "placed" or "created" within the didactic system *S* (*X*; *Y*; ♥), by *X* or *Y"*[2] (Chevallard, 2009a, p. 17, our translation).

The ratification, or even rectification, of our hypothesis leads us to seek to answer the following question paraphrased by the type of *Basic Problem,*[3] henceforth BP, expressed as follows:

> *BP: Given certain restrictions on the teaching institution in initial education, what is the set of conditions under which this institution can integrate the practice of quantification of physical quantities to structure the numbers?*

In this sense, we show that the use of the physical abacus here interpreted as one of the initial conditions from a problem in an unusual context about non-decimal numbers may not ensure the realisation of the practice of quantification of physical quantities by teachers in training, although part of the literature suggests the use of the physical abacus as a resource that contributes to learning, including using it in performing elementary operations. Still, we assume it is not sufficient to ensure the practice of quantifying physical quantities, in particular, if we consider that "between knowledge and practice there is a distance that is never entirely abolished"[4] (Chevallard, 2005, p.171, our translation). Otherwise:

> The knowledge on *one* reality domain of reality is knowledge about the social practices related to this domain, which undoubtedly is relevant to those practices. But its *congruence* in relation to them, what would constitute them in the knowledge of those practices is never guaranteed (Chevallard, 2005, p. 172, emphasis added by the author).

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[2] Text snippets: se pose par rapport à l'échelle complète des conditions et contraintes évoquée dans ce qui précède et, bien sûr, par rapport aux conditions et contraintes « portées » ou « créées », au sein d'un système didactique S(X ; Y ; ♥), par X ou par Y.
[3]The basic problem type is announced in the following terms: "*Étant donné certaines contraintes pesant sur telle institution ou telle personne, sous quels ensembles de conditions cette institution ou cette personne pourrait-elle intégrer à son équipement praxéologique telle entité praxéologique désigné ?*" (Chevallard, 2009a, p. 17).
[4] Text snippet: que entre un saber y una práctica hay una distancia nunca enteramente abolida.

Our assumptions are based on the absence not only of tasks of quantification of physical quantities in the school institution, but of the mathematical and non-mathematical knowledge that endows this social practice with sense and meaning, which can prevent, if not hinder, the students' and teachers' recognition of how to face the practice of quantification of physical quantities, even if they have some praxeologies of abacus use as a possible facilitating instrument.

To delimit the construction of possible answers to the questioning of the investigation mentioned, we resorted to the theoretical-methodological resources of the ATD, more precisely, the notion of study and research path (Chevallard, 2013), as we briefly present below.

RESEARCH METHODOLOGICAL DEVICE

To face this research, which is part of the problem of teacher education on what to teach and how to teach the DNS - PNS, we assume the didactics of the research path (Chevallard, 2009a, 2013), from now on, SRP [study and research path], more precisely, as a didactic-methodological device for teacher education. The purpose is to encourage coping with situations, since "knowledge is a situation"[5] (Bosch & Chevallard, 1999, p.3, our translation), and, as such, this notion constitutes the strong hypothesis of the very definition of mathematical knowledge based on the theory of didactic situations (Brousseau, 1995).

Usually, a SRP is forwarded from *undetermined questions* Q_i that are answered by *specific questions* Q_{ij} during the investigation (Chevallard, 2009a), which "can lead a class to meet again a complex of works that can vary depending on the path taken (which depends on the activity of X, of the decisions of Y, but also the praxeological resources $R_i^\lozenge$ e O_j currently accessible)"[6] (Chevallard, 2009a, p. 28, our translation), which can be modelled by the notion of a main didactic system S(X, Y, Q) capable of producing or not auxiliary didactic systems to build strong responses.

[5] Text snippet: une connaissance is une situation.

[6]Text snippet: *Une même question Q peut ainsi conduire une classe à rencontrer un complexe d'œuvres qui peut varier selon le parcours emprunté (lequel dépend de l'activité de X, des décisions de y, mais aussi des ressources praxéologiques R ◊ i et Oj actuellement accessibles)*.

From this perspective, it is necessary to consider that carrying out a “complete” SRP as per Chevallard (2013) includes the performance of five types of tasks consubstantial with the investigative situation, described in the following terms:

> H_1. *See* the answers $R^{\lozenge}$ that live in the institutions.
>
> H_2. *Analysis,* in particular, on the experimental and theoretical double plane those answers $R^{\lozenge}$.
>
> H_3. *Evaluate* those same answers $R^{\lozenge}$.
>
> H_4. *Develop* your answer$R^{\square}$.
>
> H_5. *Spread and defend* the answer R thus produced. (Chevallard, 2013, p. 3, author’s emphasis, our translation).

According to the author, the technique that consists of performing those types of tasks in a coordinated way does not necessarily follow a linear logic and, as such, demands specific requirements from the teacher and/or researcher, who goes through “incorporating” specific attitudes: problematising, Herbartian attitude, procognitive attitude, the exoteric attitude, and that of the common encyclopedist (Chevallard, 2013), as well as the “appeal to the praxeologies called dialectics of investigation that often take the school-university didactic culture against the mainstream”[7] (Chevallard, 2013, p. 4, our translation) that are interpreted as indispensable ingredients for a *functional epistemology of knowledge* (Bosch & Gascon, 2010, P. 86, emphasis added by the authors, our translation).

To meet our objective, we consider praxeological organisations carried out in an SRP by a group of twenty-five teachers in initial training, as part of a discipline of a early years teacher education course from a public institution, referred by Ferreira (2020) from a problem in an unusual context for them, involving notions about the DNS in the context of a PNS.

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[7] Text snippets: *l’appel à des praxéologies appelées dialectiques de l’enquête qui souvent prennent à contre-pied la culture didactique scolaire-universitaire aujourd’hui encore dominante.*

THE TYPE OF PROBLEM IN AN UNUSUAL CONTEXT

Specifically, we consider the following excerpt of the problem initially described by Ferreira (2020) and Ferreira and Guerra (2020):

- I belong to a people similar to humans. I have I mouth, V eyes and Z limbs, like them. But I differ by having only A, that is, Z minus I, fingers on each of those limbs, and I not have hair, i.e., O hair on the whole body. On my planet, we grow grains and tubers like Earthlings. In particular, in our last solar year AIOOO, which numerically corresponds to the Earth's Christian solar year of 2000, we obtained the following output:

Table 1 shows data regarding the proposed situation.

Table 1

Representation of grains or tubers. (Ferreira, 2020, p. 112; Ferreira & Guerra, 2020, p. 10)

PRODUCTS	PRODUCTION
Beans	AZOIO
Rice	ZVAII
Cassava	ZZAAV

On my planet, we only use the V, A, Z, I, and O representation registers to represent quantities.

Based on the information described in the text, one of the questions the teachers in training faced was: ***Q_1* - *How did the Ets get the representation of quantities as presented in the text?***

The following analyses focus on task H5 of the SRP carried out by the teachers, organised into five groups, here symbolically represented by ***FI_1***, ***FI_2***, ***FI_3***, ***FI_4*** and ***FI_5***, taking into account the synchrony of this task with the other tasks that characterise, according to Chevallard (2013), a real research activity.

ANALYSIS OF THE RESULTS FOUND IN THE TEACHER EDUCATION EMPIRICAL VIEW

Guided by task **H_5** of the SRP (Chevalard, 2013), which demands the dissemination and defense of the answers produced and/or elaborated, each FIk group of teachers[8] presented and defended their answer in front of the class [FI, D], where FI represents the set of teachers in initial training or all groups, and D represents the director of studies or the teacher educator.

The answers produced after the SRP tasks had been completed, i.e., $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_4}$ within each group, were placed by each group for class evaluation [FI, D]. In this way, we focus on the defenses of how much teachers use the abacus as an instrument for the quantification of physical quantities, as we consider it of greater relevance for the construction of the final answer approved or not by the class and for containing the defense of elements that can meet our objective of showing that the use of the abacus may not ensure the realisation of the practice of quantification of physical quantities.

Thus, we present below the situational excerpts and praxeologies revealed by teachers in training.

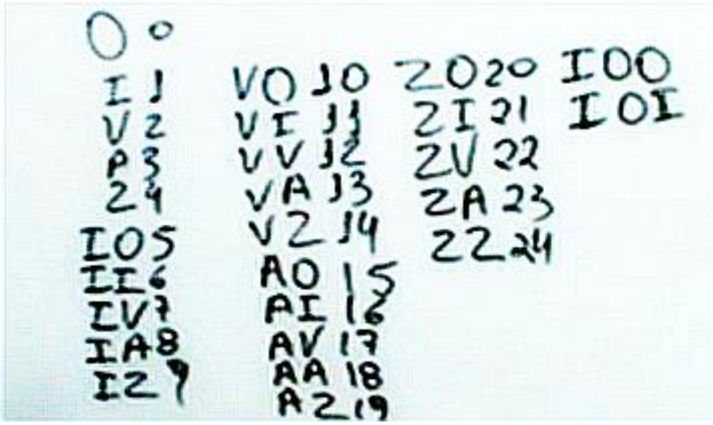
- ***FI₁*** - I was thinking a lot about those relationships, and I came to almost the same conclusion about the representation that she did, I made the same relationship here... But group 1 presented the representation of the Ets, but when it comes to 10 it passes to our reality, IO, that would be represented by 10 for us... but I believe... actually... I think that it would be more or less like this... the OI = 1, OV = 2, OA = 3 and OZ = 4, then since theirs is quinary, it only goes up to 4... so when it gets here [...] it gets more or less like this:

Figure 1 shows the register of the activity evidenced by group $\boldsymbol{FI_1}$.

[8] *The empirical data obtained from the teachers' manifestation in initial training took place in the context of a curricular subject at a public institution of higher education. In this sense, the professors' registers highlighted here do not give their identities, their images, and voices, thus ensuring the dignity and due protection of the participants in scientific research. For this reason, no prior ethical evaluation was requested by the appropriate councils of the research project from which the work arises. Thus, we assume and exempt Acta Scientiae from any consequences arising therefrom, including full assistance and eventual compensation for any damage resulting to any of the research participants, as directed by Resolution N, 510, 07 April 2016, of the National Health Council of Brasil.*

Figure 1

Register of the relationship between letters and numerals (Ferreira, 2020)



- – This means that we count 1,2,3,4 but when we get to 5, it's as if we got to 9, so it becomes IO = 5, we move to 10, and automatically 1 goes forward, and 0 zero is left behind...
- – Let me explain... when does it get to 99?
- – It goes to 100, it's like the last ten, 44 is the last, got it?

We infer from the diffusion and defense of the group $\boldsymbol{FI_1}$, the following praxeological structure:

> ***** *Task type* $\boldsymbol{T_1}$: It can be reduced by carrying out the task of "relating decimal numbers as names of quantities, with quinary numerals";
>
> ******Task technique* $\boldsymbol{\tau_1}$: It was defined as a one-to-one correspondence between decimal numerals, as the name of quantities, and quinary numerals, in which they follow the same intuitive rule of writing registers of decimal numerals, i.e., they are made up of positions occupied by digits, in the case O (zero), I (one), V (two), A (three) and Z (four) and each position, when occupied by the digit that corresponds to the maximum value, in this case, Z, must be restarted from the O (zero), taking the successor of the digit in the next left position.

Put to the test in front of the class [**FI, D**], a limitation of the type of technique $\boldsymbol{\tau_1}$ used was revealed when the director of studies D asked the group $\boldsymbol{FI_1}$:

- ***D* -** If you were to do it, would you reach the relationship between 2000 with AIOOO?
- ***FI_1* -** Ah! The teacher would take too long! The process is long...

The difficulty highlighted by the group lies in the effort required to enumerate all decimal numerals up to the decimal numeral 2000, as it is not possible to find a quinary numeral (decimal) corresponding to a given decimal numeral (quinary) in isolation.

One of the initial conditions introduced by study direction D was the following:

- ***D* -** Let's do the following... let's use the lids (from a soda bottle)?!

Faced with the condition created by the director of studies, the groups of teachers disseminated practices with concrete materials on how to face the problematised situation:

- ***FI_2* -** It's as if they had to count the ten caps, they would count, for example... taking away the logical zero, which is absence, the first cap would be I, second cap the V, third cap the A, fourth cap the Z, continuing... fifth cap the IO, sixth cap the II, the seventh cap IV, the eighth IA, the ninth cap IZ, tenth cap VO, for us there are ten caps, for them there is VO.

Figure 2 shows the counting process.

- ***FI_5*:** Then, I understood it, but...
- ***FI_2*:** You didn't understand here at the end, did you?
- ***FI_5*:** In this relationship that you brought us here, we already know that their system is quinary, right, so 24 goes to 30..., but if we were to represent what would it be like from 24 to 30?
- ***FI_2*:** Here, it would still be... IOO = 25, IOI = 26, IOV = 27, IOA = 28, then I got lost... so it continues until it reached 30 quantities.

The strategy used by teachers with concrete material did not show support to justify the use of the technique that was maintained, that is, the sequencing technique, maintaining doubts about the Ets number system. In any case, the first attempt at quantification with physical quantities was abandoned.

Faced with the difficulties expressed by the groups, one of the group members ***FI_2*** revealed a technique τ_2 through the formula investigated on internet sites, as it guides its defense and dissemination of the answer: $\boldsymbol{\tau_2} = \boldsymbol{r_N\beta^N + \cdots + r_1\beta^1 + r_0\beta^0 = \sum_{k=0}^{N} r_k\beta^k}$, according to the positional representation of base β (Ripoll, Rangel, & Giraldo, 2016, p. 25):

Figure 2

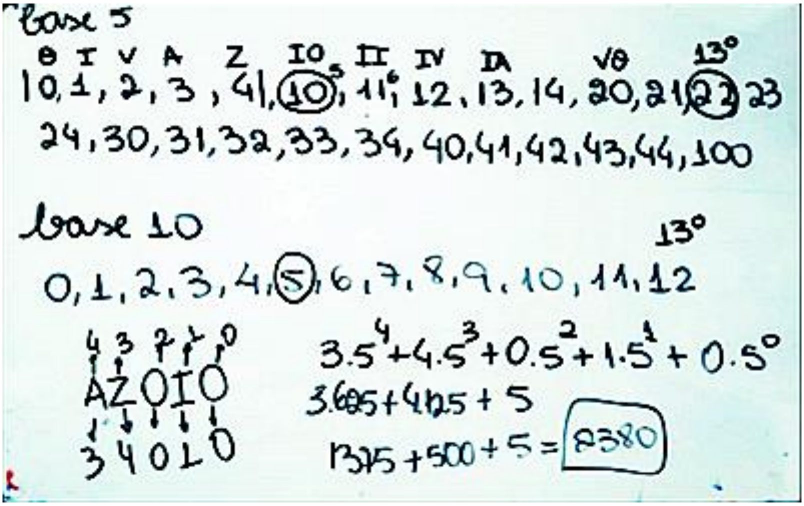
Counting with caps (Ferreira, 2020)

- $\boldsymbol{FI_2}$ **-** I'll show you what I researched, may I, teacher? Everything I researched here was taken from some Wikipedia websites (https://pt.wikipedia.org/wiki/Codifica%C3%A7%C3%A3o,) and YouTube (https://www.youtube.com/watch?v=2pGkFn4Sgao). In them, I'll find out that there is a formula, I'm going to apply this formula here, and you'll see if it works or if it doesn't, okay, then? I did it to see if it had coherence, and it was really right, because simply when I threw in the formula... in this case, the AIOOO... I wanted to know if using the formula would give 2000. Then, I hit it!!! The ET was normal, it had two eyes, lol...

Figure 3 presents the base change formula.

Figure 3

Register of use of the investigated formula (Ferreira, 2020)



The use of the mathematical technique materialised by the investigation of the formula $\boldsymbol{\tau}_2 = \boldsymbol{r_N\beta^N} + \cdots + \boldsymbol{r_1\beta^1} + \boldsymbol{r_0\beta^0} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{N}} \boldsymbol{r_k\beta^k}$ created new conditions for the study process. It not only allowed the validation of the technique $\boldsymbol{\tau_1}$ and, more broadly, on the praxeological organization at play but also expanded the possibility to answer other questions, in particular, to validate the relationship of the quinary numeral 31000 with the decimal numeral 2000. In addition, the formula allowed them to create conditions for the construction of answers to the questioning $\mathbf{Q_1}$ on “how probably the Ets arrived at the representation of quantities as presented in the text?” (Ferreira & Guerra, 2020, p. 10).

However, although these techniques reveal potential to teachers, it seems that their low relationship (Chevallard, 2005) with the technique $\boldsymbol{\tau_2}$ made it difficult for them to envision, if not prevented them from envisioning, possibilities for dealing with the problem of representing a decimal numeral in a quinary numeral using the formula.

Seeking to somehow justify the validation of the answers put to the test in front of the class [**FI, D**] built in the training course, group $\boldsymbol{FI_5}$ introduced a new condition, in particular, with the proposition of the physical abacus, as we highlighted in its defense in interaction with other groups:

- ***FI_5* -** I'll try to show it on the abacus, let's start with the unit, if our system were the quinary, right?! Let's count 1, 2, 3, 4, it passes to the next pin, when it's 5, then, it will reset the units, won't it? Let's count again: 1, 2, 3, 4, so it will become 4, and I will use a ball of another color, it fills up again, and we take it out and start again, and move to the next peg, every time the people count from 4 to 4, at 5, it resets and starts again.
- ***FI_5* -** If we take the abacus, this is interesting, taking three marbles, how many will fit in the unit of higher-order 2? how many fit? How big is this place? Up to 4, right? So I'm going to put 4 balls, let's start from scratch, how many do you need to fill the place?
- ***FI_1*** *– 4*
- ***FI_5* –** Right
- ***FI_3* -** Mine is on the right...
- ***FI_5* -** No! You can't! It maintains the same principle, from left to right... so here it only fits 4, if it were in the decimal system, it would fit 9.

Figure 4 highlights the handling of the counting process and its representation on the abacus

- ***FI_1*** *– 4*
- ***FI_5* -** Can I put another ball?
- ***FI_4* -** No!!!
- ***FI_5* -** So, it goes against the system, because it doesn't support it, the house only supports 4, so I take the unit of order 2 and go to the unit of order 3, which means that the unit of order 2 arrived at 5, so I take it and I move to the next one, in this case, the unit of order 3, and I represent it with another ball, with another color, in this case, a red one, I put one more ball in the unit of order 3, how much is it?
- ***FI_2* -** I understand that this question is clear in our minds, but when it comes to teaching children using concrete material... we can't...

Figure 4

Representation on the abacus (Ferreira, 2020)

- ***FI_5*** **-** How many can I put, then...?

The teachers' dialogues show their doubts about using the physical abacus to quantify physical quantities, mainly when ***FI2*** affirms being a problem for the teachers teaching numerals with concrete materias more precisely in the sense of the didactic problem paraphrased by Ferreira (2020, p. 201, authors' emphasis): "**What and how to teach systems of numerals to teachers in initial training to work in the initial years?**"

Following this line of thought, what did group ***FI_4*** present in their defence? Highlighting the following aspects:

- ***FI_4*** **-** Let's count 1, 2, 3, 4, it passes to the next pin, when it's 5, then, it will reset the units, won't it? Let's count again: 1, 2, 3, 4, so it will become 4, and I will use a ball of another color, fill it up again, and we take it out and start again, and move to the next peg, every time the people count from 4 to 4, at 5 it resets and starts again.

Figure 5 shows the representation of counting with the physical abacus

Figure 5

Representation of counting with the physical abacus (Ferreira, 2020)

- ***FI*$_4$** - Which number does this cluster of balls represent?
- ***FI*$_3$** - 5
- ***FI*$_2$** - It's not 4!
- ***FI*$_2$** - yes, it will be it, in fives.
- ***FI*$_1$** - So, why not fill it?
- ***FI*$_2$** - Because we are explaining it to children, and it must be slow, step by step.
- ***FI*$_2$** - If I get confused, the child will be confused too! Why are you filling it there?
- ***FI*$_2$** - You just have to think that the absence is zero, when completed, this is the quinary system... Here, the quinary system has been completed, so it fills again with 1, 2, 3, 4, 5 ... it fills again, so, one more goes here, it is going to fill up, one more, it is going to fill up again, then we empty it again, and we count until it is complete..., but from now on I got lost! Hahaha.

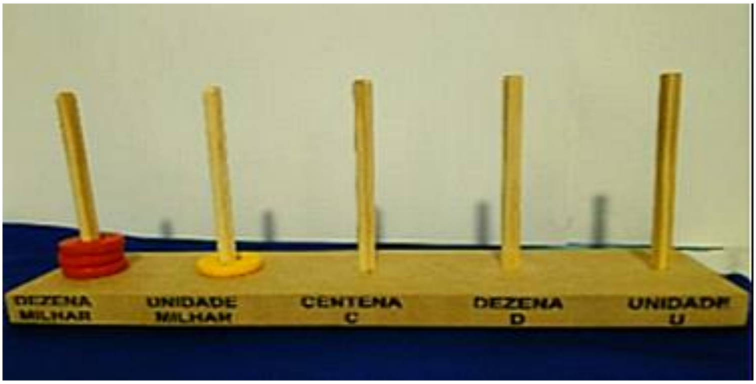
This fragment reinforces the clear difficulty found by teachers in the use of the physical abacus as a didactic instrument introduced as one of the conditions to help demystify the situation of quantification of physical quantities, above all, when ***FI₂*** admits, "*If I get confused... imagine the child! The child will be confused too!"*. However, this use seems to find limitations evidenced by teachers for the task of quantifying physical quantities using the physical abacus.

In addition, we observed the teachers' practice with the use of the physical abacus, the representation of a numeral in an abacus and vice versa, and their interpretations, which were focused on the positions of the abacus with the potencies and with the formula researched and socialised by $\boldsymbol{FI_2}$ during their training and, vice versa, i.e., they made relations of the potencies of the formula $\boldsymbol{\tau_2}$ with the abacus positions, in particular, as the following dialogues seem to evidence.

Figure 6 shows the representation of the numeral on the abacus of group 2 of teacher educators

Figure 6

$\boldsymbol{FI_2}$ representation of the quinary numeral 31000 at the decimal numeral 2000 (Ferreira, 2020)



- $\boldsymbol{FI_5}$ **-** Observing the quinary system, I did it with the power for the decimal system, and it works great, it resets, really!! It's a cool tip to get back to zero!

- $\boldsymbol{FI_5}$ **-** Next tie $5^4 = 625$ and so on, until each one's needs.

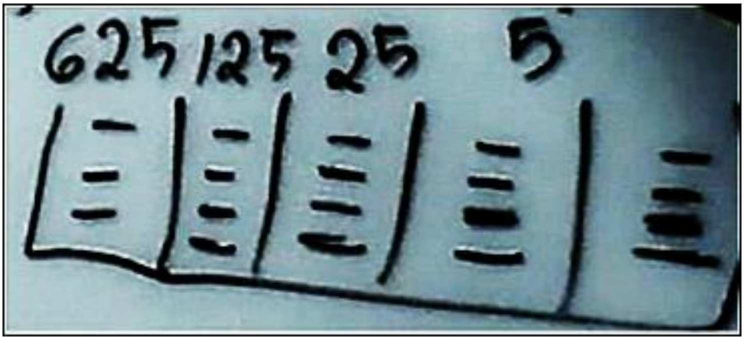
Ultimately, group ***FI₃*** defended their praxeological answers supported by the written abacus discourse from the exemple of a numeral cluster in fives, as highlighted in figure 7.

- ***FI_3*** **-** So let's go, I'll use the chart to show, so it's 1 US, 1 UOS1, 1 UOS2. Do you agree? ... Let's remember, in the case of the quinary, it only supports 4 of 4, because when it forms 5, it passes to another unit of a higher order, in this case, the following one, because it grouped, so I will register here in the table, it will be 5, 25, 125, and 625.

Figure 7 shows the register on the written abacus on the teacher's board.

Figure 7

Use of the written abacus in base five (Ferreira, 2020)



Before the manifestation of ***FI_3***, group ***FI_2*** refuted the answer:

- ***FI_2*** **-** No, I don't understand, did you just explain the numbers on the board; what about in practice?
- ***D*** **-** They did it by the algorithm...
- ***FI_2*** **-** I understand that this question is clear in our minds, but when it comes to teaching children... using concrete material... they couldn't understand... and I think the teacher's purpose is to relate to the material.

Faced with the difficulties teachers ***FI***$_k$ found when using the abacus to quantify physical quantities, the director of studies **D** forwarded the following condition:

- ***D*** – Now, we need to build the practice because this is at our level. Now we need to do it as if we were going to teach the children. We need to understand how this can work with children.
- ***FI***$_4$ **-** Now let's work on the quineral system, where the base is not 10. So, it's: 1 cap, 2 caps, 3 caps, 4 caps and 5 caps and a tie...[...] remember the tie?

Figure 8 shows a ***FI***$_4$ member handling the soft drink caps

Figure 8

Formation of groups in fives (Ferreira, 2020)

Even resorting to the use of concrete materials for structuring the numeral through physical quantities, as directed by the director of studies ***D***, it is necessary to highlight that group ***FI***$_4$ also abandoned the concrete practice, making use of the whiteboard to justify or clarify the argument used in defense of the praxeological answer presented.

Thus, another attempt to perform the task of quantification of physical quantities was not satisfactory to the class [**FI, D**] in finding a technique τ_3 that would allow us to tackle perhaps the most interesting task: ***T***$_3$ **-** To represent a numeral from the quantification of a cluster of physical units, considering different types of clusters of those units.

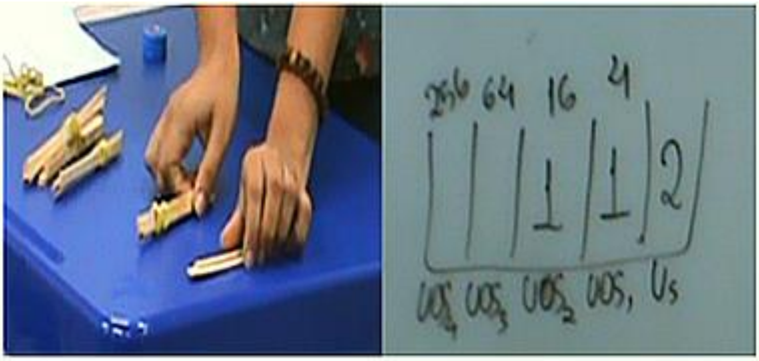
After some guidance from the director of studies ***D***, group ***FI$_5$*** made the following defense:

- ***FI$_5$*** **-** There are 7 caps, I'll add them together, or rather group them in twos because we're doing it on base 2.
- ***FI$_5$*** - The one that's left, is it the Single Unit?!
- ***D*** – Yes
- ***FI$_5$*** - Okay, now, what? Do we regroup, teacher?
- ***FI$_5$*** - So, will it stay like this? [doubt], do I join again? Two out of two, 1 is left over again, so there is one cluster of 4 and one cluster of 2, so there is 1 in the single unit (SU), 1 in the unit of higher-order one (UOS1), and 1 in the unit of higher-order two (UOS2).

Figure 9 shows the clustering process from popsicle sticks.

Figure 9

Clusters 4 of 4 sticks and their representation on the written abacus (Ferreira, 2020)



- ***FI$_1$*** **-** I got it!!!!! I want to do it again!
- ***FI$_4$*** **-** Pal, change the base to see if it improves, do it on base 4.
- ***FI$_1$*** **-** Let's separate and tie, so there are five groups of 4, remaining 2, which is the single unit...clustering them, we have one group of

4 of 4, remaining one group of 4. This leaves us one single unit, one unit of higher-order 1, and one unit of higher-order 2.

Ultimately, the group's reaction "***FI₁*** *- I got it!!!!! I want to do it again!"*It seems to us that these teachers are clearly admitting that the attempts made until then on the task of quantifying physical quantities seemed inconsistent or unsuccessful, even with the use of the physical and written abacus mobilised by them in their praxeologies.

Furthermore, it is worth noting that the answer to question $\mathbf{Q_1}$ found by class **[FI, D]** can be described through four tasks divided into two steps:

- **1st Stage**: Definition:
- t_1 – Set the β limit, or base, of the unit count;
- **2nd Step: Iterative process -** Starting with the single units:
- t_2 – If possible, build clusters of a higher order than the existing ones, forming new clusters of β units of clusters of immediately lower order;
- t_3 – Register the number of cluster units that were being clustered in the previous step.
- t_4 – If the number of new higher-order clusters formed is less than the count limit β, register this amount in the corresponding position and finish by highlighting the written numeral. Otherwise, return to task t_2.

Despite the complexity of the praxeologies expressed by the teachers during the SRP, the director of studies ***D*** played an indispensable role in leading class **[FI, D]**, especially, the *situational-problematising*attitude , which "consists of problematising the situations experienced or observed, i.e., raising questions about them. This is, of course, an essential attitude from which both the question Q of the Herbartian schema and the engendering of the questions Q_j arise"[9](Chevallard, 2013, p. 4, our translation).

[9]Text snippet: *consiste à reconnaître la « problématicité » des situations vécues ou observées, c'est-àdire à soulever des questions à leur propos. C'est évidemment une attitude essentielle, d'où naît tant la question Q du schéma herbartien que les questions engendrées* Q_j.

The set of conditions, both initial and introduced by the class [**FI, D**], showed the teachers' difficulties using the physical and written abacus to carry out the social practice of quantifying physical quantities of numeral construction.

Teachers revealed qualities of relationships (Chevallard, 2005) with praxeologies of representing numerals on the abacus and vice versa and interpreting the powers of the formula $\boldsymbol{r_N \beta^N + \cdots + r_1 \beta^1 + r_0 \beta^0 = \sum_{k=0}^{N} r_k \beta^k}$ through the abacus and vice versa. However, the task of quantifying physical quantities for the construction of the numeral using the abacus was gradually built in the training course, i.e., it did not integrate the teachers' praxeological equipment.

FUTURE RESEARCH OUTCOMES

This investigation, from the study of a problem in an unusual context about non-decimal numbers, confirmed the hypothesis that there is a problem regarding the use and handling of concrete materials, specifically, the use of the abacus as a facilitator in the structuring of non-decimal numbers through the practice of quantification of physical quantities. It also revealed that its handling does not ensure the accomplishment of the practice of quantification of physical quantities.

Our focus was on the diffusion and interactions of groups of teachers who resort to the use of the physical abacus as a possible instrument for quantifying physical quantities. Moreover, the type of problem in an unusual context mobilised teachers to meet various praxeologies revealed by them, for instance, the technique of one-to-one correspondence between decimal numerals, from the name of quantities, and quinary numbers, in that they follow the same intuitive rule for writing decimal numbers, i.e., they are made up of positions occupied by digits, in this case, O (zero), I (one), V (two), A (three) and Z (four) and each position, when occupied by the digit that corresponds to the maximum value, in this case, Z, must be restarted from O (zero), taking the successor of the digit of the next left position.

However, some questions in the training process revealed the limitation of the specific technique of correspondence and, not least, the strategy used with concrete material, which, for the moment, was not successful. Using concrete materials clearly unveiled part of the teachers' difficulties in quantifying quantities for structuring the numeral.

Although the formula $\boldsymbol{\tau}_2 = \boldsymbol{r_N}\boldsymbol{\beta}^{\boldsymbol{N}} + \cdots + \boldsymbol{r_1}\boldsymbol{\beta}^{\boldsymbol{1}} + \boldsymbol{r_0}\boldsymbol{\beta}^{\boldsymbol{0}} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{N}} \boldsymbol{r_k}\boldsymbol{\beta}^{\boldsymbol{k}}$ (Ripoll, Rangel, & Giraldo, 2016) evidenced by one of the groups of teachers may have ensured the legitimacy and institutionalisation of the most far-reaching mathematical technique in dealing with the type of problem considered the quality of relationship (Chevallard, 2005) of teachers with it, was not enough to allow constructing an answer to the problem of converting a decimal reality to a non-decimal one.

In this way, the teachers decided, in common agreement with the director of studies, to use the physical abacus to find answers that would allow confronting the problems posed, especially the quantification of physical quantities. However, the teachers' dialogues when dealing with task H**5** of the SRP revealed difficulties with its use as a didactic instrument.

From the proposal for using the physical abacus in the teachers' practice regarding the representation of a number in an abacus and vice-versa, those teachers' interpretations focused on the positions of the abacus with the potencies and with the formula $\boldsymbol{\tau}_2 = \boldsymbol{r_N}\boldsymbol{\beta}^{\boldsymbol{N}} + \cdots + \boldsymbol{r_1}\boldsymbol{\beta}^{\boldsymbol{1}} + \boldsymbol{r_0}\boldsymbol{\beta}^{\boldsymbol{0}} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{N}} \boldsymbol{r_k}\boldsymbol{\beta}^{\boldsymbol{k}}$, i.e., they made relations between the powers of the formula $\boldsymbol{\tau_2}$ with the positions of the abacus, which was not sufficiently capable of assuring success in structuring the physical quantities for the construction of the number.

Hence, it somehow ratifies what Chevallard (2005) highlights about the distance between knowledge and social practice, in this case, the knowledge of the technique $\boldsymbol{\tau_2}$ (the practice of quantification of discrete units), since knowledge about practices can constitute the specific knowledge of the social practice of quantification, but such an assertion cannot be guaranteed, since implicit or non-mathematical knowings give sense and meaning to social practice, as Sodré (2021) notes.

Thus, considering the empirical results found, we motivate other researchers to carry out future research, including facing some problems that emerged, especially in mathematical modelling (Sodré & Guerra, 2018; Sodré, 2021) for the study of normative mathematical models, such as a numeral as a mathematical model that "governs" real-world situations.

AUTHORSHIP CONTRIBUTION STATEMENT

GJMS delimited the introduction of the article, the problematisation of the investigation articulated to the didactic-methodological device. RSRF referred to the type of unusual problem and that of empirical data. GJMS, RSRF, and MLSG, together, participated in the analysis of the empirical results found and the future developments of the investigation.

DATA AVAILABILITY STATEMENT

The data that support and integrate this study and investigation will be made available by the corresponding author, GJMS, upon prior request.

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