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Indirect Distance Measurement and Geometric Work in the Construction of Trigonometric Notions

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ABSTRACT

Background: Among the multiple phenomena reported around the teaching and learning of trigonometry is the exclusive use as technical tools that trigonometric ratios receive. This use allows and promotes the construction of limited meanings and the dissociation of school trigonometry from geometric notions and procedures. **Objective:** Given this problem, we studied the germinal construction of trigonometric notions in a historical setting to identify meaningful uses of mathematical knowledge to enrich those that inhabit the school and its associated meanings. **Design:** We conducted this study from socio-epistemological theory and through a particular configuration of content analysis. **Setting and participants:** Being a documentary cut study, we did not have participants *stricto sensu*. **Data collection and analysis:** The historical study focused on the mathematical preliminaries of Ptolemy's *Almagest*, a work that the literature points out as the oldest evidence of the birth of trigonometry. **Results:** Among the results, we highlight the indirect measurement of distances as the use that allowed the initial construction of trigonometric notions and the synergy of three uses of geometric knowledge - which we call geometric work - as fundamental to this process. **Conclusions:** We conclude by stressing the importance of creating concrete didactic situations that explore the operation of these results in teaching environments and the need for historical studies around other points in the construction of trigonometric notions.

Keywords: Linear and arithmetic meaning; Trigonometric meaning; Use of knowledge; Historical-epistemological study; Geometric work.

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Medición Indirecta de Distancias y el Trabajo Geométrico en la Construcción de las Nociones Trigonométricas

RESUMEN

Contexto: Entre los múltiples fenómenos reportados alrededor de la enseñanza y aprendizaje de la trigonometría, se encuentra el uso exclusivo como herramientas técnicas que reciben las razones trigonométricas. Este uso permite y promueve la construcción de significados limitados y la disociación de la trigonometría escolar de las nociones y procedimientos geométricos. **Objetivo:** Ante esta problemática, realizamos un estudio de la construcción germinal de las nociones trigonométricas, en un escenario histórico, con el objetivo de identificar usos del conocimiento matemático que sean útiles para enriquecer los que habitan en la escuela y sus significados asociados. **Diseño:** Llevamos a cabo este estudio desde la teoría socioepistemológica y a través de una configuración particular del análisis de contenido. **Entorno y participantes:** Al ser un estudio de corte documental no contamos con participantes *stricto sensu*. **Recopilación y análisis de datos:** El estudio histórico se centró en los preliminares matemáticos del *Almagesto* de Ptolomeo, obra que la literatura señala como la evidencia más antigua del nacimiento de la trigonometría. **Resultados:** Dentro de los resultados destacamos a la medición indirecta de distancias como el uso que permitió la construcción inicial de las nociones trigonométricas y a la sinergia de tres usos del conocimiento geométrico - que denominamos trabajo geométrico - como fundamental para este proceso. **Conclusiones:** Concluimos subrayando la importancia de crear situaciones didácticas concretas que exploren el funcionamiento de estos resultados en entornos de enseñanza y la necesidad de estudios históricos alrededor de otros puntos de la construcción de las nociones trigonométricas.

Palabras clave: significado lineal y aritmético; significado trigonométrico; uso del conocimiento; estudio histórico-epistemológico; trabajo geométrico.

Medição Indireta de Distância e Trabalho Geométrico na Construção de Noções Trigonométricas

RESUMO

Contexto: Entre os múltiplos fenómenos relatados em torno do ensino e aprendizagem da trigonometria está o uso exclusivo como ferramentas técnicas que as razões trigonométricas recebem. Esta utilização permite e promove a construção de significados limitados e a dissociação da trigonometria escolar das noções e procedimentos geométricos. **Objetivo:** Face a este problema, realizamos um estudo da construção germinal das noções trigonométricas, num cenário histórico, com o objetivo de identificar utilizações dos conhecimentos matemáticos úteis para o enriquecimento daqueles que existem na escola e os seus significados associados. **Design:** Realizamos este estudo a partir da teoria sócio-epistemológica e através de uma configuração

particular de análise de conteúdos. **Ambiente e participantes:** Como se trata de um estudo documental, não tivemos participantes *stricto sensu*. **Coleta e análise de dados:** O estudo histórico centrou-se nas preliminares matemáticas do *Almagest* de Ptolomeu, uma obra que a literatura aponta como a mais antiga evidência do nascimento da trigonometria. **Resultados:** Dentro dos resultados, destacamos a medição indireta de distâncias como a utilização que permitiu a construção inicial de noções trigonométricas e a sinergia de três utilizações do conhecimento geométrico - a que chamamos trabalho geométrico - como fundamentais para este processo. **Conclusões:** Concluímos salientando a importância da criação de situações didáticas concretas que explorem o funcionamento destes resultados em ambientes de ensino e a necessidade de estudos históricos em torno de outros pontos na construção de noções trigonométricas.

Palavras-chave: significado linear e aritmético; significado trigonométrico; uso do conhecimento; estudo histórico-epistemológico; trabalho geométrico.

INTRODUCTION

In mid-high school education, students - about 15 years old - begin studying a set of different mathematical notions. Some entities with strange looks do not respond to the numerical-algebraic rules built throughout their eight or nine years of schooling trajectory: the trigonometric notions - ratio, function, and series (Weber, 2008).

Given this scenario, the number of phenomena reported around the teaching and learning processes of those notions is not surprising. Among them: the undisputed introduction of the unit circle (v.g. Brito & Barbosa, 2004) and the transition from the degree to the radian as a unit of measure for the angle (v.g. Díaz, Salgado & Díaz, 2010), the non-identification of the angle as an argument (v.g. Thompson, 2008), the indistinction of trigonometric notions - ratio, function, and series - (v.g. Montiel & Buendía, 2013) and their *exclusive use as technical tools* for calculating a missing value.

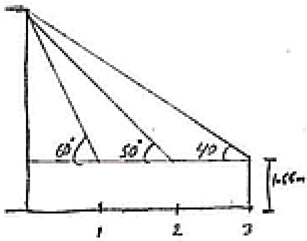
Concerning this last phenomenon focused in this study, Maldonado (2005) mentions that “trigonometric ratios are defined […] only to use them as means of a solution” (p. 15). Similarly, regarding the trigonometric function, the author considers that “it is used only without the need to be understood” (p. 68). Weber (2005), Araya, Monge, and Morales (2007), and Mesa and Herbst (2011) share those statements.

In this regard, Montiel and Jácome (2014) conclude that this restricted use limits the mathematical activity to dividing the sides of the right triangle, dissociating school trigonometry from geometric notions and procedures while allowing the construction of a linear meaning, and promoting an arithmetical meaning for trigonometrical notions.

The *linear meaning* refers to the conception and linear treatment that school trigonometry admits for the angle-length relationship in the triangle, the product of not explicitly analysing the nature of that relationship. For example, when Montiel and Jácome (2014) built a scale model of a problem situation to measure inaccessible distances, more than half of the participating teachers presented sketches in which steady decreases of the elevation angle coexist with constant growths on its adjacent side (Figure 1).

Figure 1

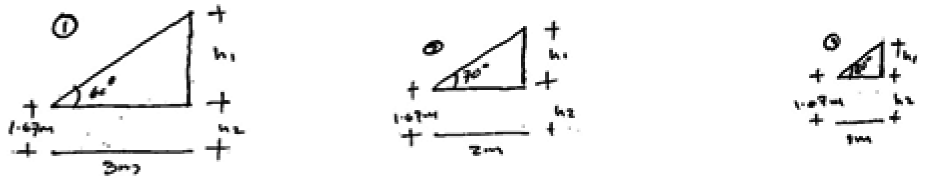
A proposed model that reflects a linear relationship between the elevation angle and its adjacent side. (Adapted from Montiel and Jácome (2014))



In turn, the *arithmetical meaning* alludes to the trigonometric notions - especially the ratio - as an 'ordinary' arithmetic process of dividing the lengths of the sides of the triangle, resulting from the focus of school trigonometry on the arithmetic domain of those notions. Thus, although Montiel and Jácome (2014) explicitly requested participants to build a scale model of the problem situation, none of their representations constituted a geometric scale model *stricto sensu* (Figure 2), as they are not proportional to reality. Instead, they only illustrate from where one should take the data to be replaced in the 'trigonometric formula' that solves the problem.

Figure 2

Proposed sketch where h1 - the height to be calculated - changes ‘proportionally’ from one figure to another. (Adapted from Montiel and Jácome (2014))



Previous research in the discipline warns that, although the use of the trigonometric notions as technical tools makes them essential instruments for many fields of application and school spaces (Montiel, 2011), it produces limited and conflicting meanings. Furthermore, it does not ensure a robust understanding of them (Weber, 2005) nor the development of trigonometric thinking when handling the triangle, its elements, and the relationships between them (Montiel & Jácome, 2014).

In other words, a student/teacher can achieve correct mathematical performance in typical school tasks related to trigonometric ratios and still develop the linear and arithmetic meaning associated with them, i.e., they can solve trigonometric exercises but not develop trigonometric thinking. Also, in terms of curricular transversality, we could ask ourselves what this ‘correct performance’ on the trigonometric ratios contributes to the understanding and treatment of the trigonometric function, considering that in this last notion, the study of its transcendent nature is essential. Are we not, from school mathematics itself, causing the disarticulation of both notions – trigonometric ratio and function – by endowing them with meanings of a different nature?

Faced with this problem, we propose a research project whose starting hypothesis is that, by expanding the uses of trigonometric notions - beyond its use as a technical tool - and reducing the gap between geometry and school trigonometry, we can confront and enrich the meanings that current school trigonometry allows and promotes.

Addressing this hypothesis requires, first, answering at least three questions: 1) What other uses of trigonometric notions are there? 2) What role do geometric notions and procedures play in these uses? 3) What geometric notions and procedures are involved?

History is a feasible scenario to answer these questions. As Gómez (2003) mentions, historical-epistemological studies make us aware that the same mathematical notion has had different meanings throughout the times. Moreover, it allows us to establish the constituent elements of its meanings, the different meanings, and their adaptation to the resolution of different situations and problems.

In this paper, we give an account of a study on the germinal construction of trigonometric notions in a historical setting, aimed at answering the three questions above from the socioepistemological theory perspective and directing our results to epistemological contributions.

THEORETICAL-METHODOLOGICAL ASPECTS

The socioepistemological theory and the problematization of mathematical knowledge

The socioepistemological theory - or socioepistemology -, developed within the sociocultural paradigm, departs from recognising that the construction and meaning of mathematical notions and procedures depend on their culturally, historically, and institutionally *situated use* (Cantoral, 2013).

For this, the theory understands use as the ways of employing a mathematical notion consciously or unconsciously, implicitly - in the individuals' actions - or explicitly - through school or contextual representations - in a specific context (Cruz-Amaya, 2019; Cabañas-Sánchez & Cantoral, 2012; Rotaeche, 2012).

This stance on the construction and significance of mathematical notions implies, among other things, that the context determines the type of rationality with which an individual, collective, or historical subject builds knowledge (Cantoral, Montiel, & Reyes-Gasperini, 2015). This view is contrary to the formalist or traditional view of rationality in which "being rational resides solely in thinking and acting according to abstract and universally applicable rules, such as logical, probabilistic, mathematical rules, etc." (Xiang, 2008, p. 103).

With this in mind, socioepistemology proposes a model to theoretically and empirically explain the construction of a specific mathematical notion through its situated use, expressed through the coordination of practices of various kinds called *nested model of practices*.

In the first three levels of that model - ascending reading - (Figure 3), convenient for empirically explaining the construction of a specific mathematical notion, we start from the subject's direct *actions* on the environment, organised as socioculturally situated human *activities* to outline a *socially shared practice* as the subject's deliberate iteration, and determined by the context (Cantoral, 2013).

Figure 3

Practice with nested model. (Adapted from Cantoral (2013)).

Socioepistemology has adopted the *problematization of knowledge* as a research route for studying the practices that accompany the construction of a specific mathematical notion. This theory requires two moments: historization and dialectization.

The *historization of mathematical knowledge* refers to the study and identification of those uses and meanings that are specific to the mathematical notion in question and that have been diluted, transformed, or lost in its introduction to school mathematics (Montiel & Buendía, 2012). We must explain that this study is not limited to the analysis of the mathematical object *per se*, but it includes - without ignoring the above - the detailed analysis of the sociocultural circumstances that allowed the construction of the piece of knowledge in question, "of the contextualised rationality with which it was conceived in its time and space" (Reyes-Gasperini, 2017, p. 54).

The *dialectization of mathematical knowledge*, on the other hand, alludes to the confrontation of the uses and meanings recognised in historization with those who live in the school, the technical-professional environments, and people's daily lives, among other spaces.

In this sense, the problematization of knowledge, from the theoretical perspective adopted, is a useful theoretical-methodological tool to identify uses, meanings, and other elements that allows the construction of a specific mathematical notion, and, based on the former, to confront and enrich the uses and meanings that inhabit the school, professional spaces, and others. A tool that allows "having the different positions and explanations in dialogue, signifying them based on their contexts, and understanding and studying them based on those contexts" (Reyes-Gasperini, 2017, p. 55).

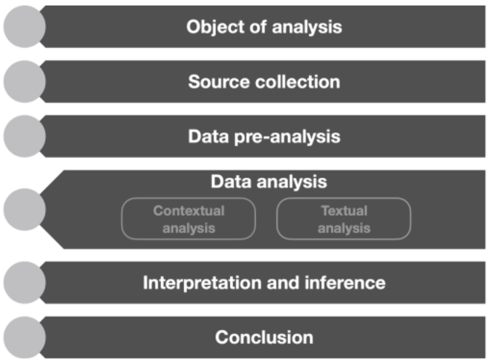
A historicization of the trigonometric notions

To address the three starting questions, we carried out a historization of the trigonometric notions, i.e., we studied the sociocultural conditions in which trigonometric notions were initially constructed and spread, and the use - in terms of actions and activities - that their 'producer' makes of mathematical notions and procedures.

This historization was done through a particular configuration of the *content analysis,*[1] as a sociological tool for studying communications related to their conditions of production/reception (Bardin, 1996 in Cáceres, 2003), composed of six stages (Figure 4).

Figure 4

General scheme of the methodological elements of the study



[1] The complete description of the method can be consulted in Cruz-Márquez (2018, pp. 54-64).

The *object of analysis* is the selected work in which we recognised the birth of trigonometric notions. During this stage, we chose chapter IX of book I of Ptolemy's *Almagest* as our object of analysis, given that literature usually agrees that the 'trigonometric table' constructed in that section is the oldest human communication that gives evidence of the birth of trigonometry. We understand trigonometry here as the systematic and quantitative study of the relationships established between an angle and the distances it subtends.

The *source collection*, in turn, refers to the search and organisation of works associated with the selected object of analysis. We classified the sources gathered during this stage into four types: 1) studies on the history of science and mathematics in general, where we placed, for example, *Science Awakening II* (van der Waerden, 1974); 2) modern versions of the *Almagest*, for example, *Ptolemy's Almagest* (Toomer, 1984); 3) analysis of the *Almagest*, e.g., *A Survey of the Almagest* (Pedersen, 2010); and 4) annotated translations of Ptolemy's masterpiece, among which the *Chapter IX of Book I of the Almagest by Claudius Ptolemy* (Saiz, 2003).

The *data pre-analysis*, the third stage of our content analysis, refers to the initial study of the sources collected to discriminate those that will constitute our data. During this stage, we made comprehensive readings of the sources, selected documents or sections of documents that we considered could help answer our initial questions, and translated and transcribed some of them, intending to make their analysis easier and more efficient.

The *data analysis* refers to the study - based on the theoretical elements - of the previously selected and processed data and to the establishment of causalities, correspondences, and links between them. In correspondence with our theoretical position, we carried out this stage in two phases: contextual analysis and textual analysis.

For the contextual analysis, we took as a starting point the methodological proposal by Espinoza-Ramírez and Cantoral (2010), which maintains that to approach the sociocultural meaning of a work, it must be seen from at least three perspectives: as a production with history, as an object of diffusion, and as part of a global intellectual expression. In this sense, the following questions became paramount in our study: Who was Claudius Ptolemy? What social, political and/or economic events determined the publication and dissemination of the *Almagest*? What relationship does the *Almagest* keep with other relevant mathematical or didactic works at the time? (Cruz-Márquez & Montiel, 2017).

Regarding the textual analysis, thanks to what is reported in the literature and a previous study carried out on Euclid's *Elements*, in chapter IX of book I of Ptolemy's *Almagest* we could identify discursive units as portions of text with a similar grammar and function (Figure 5), and units of analysis or propositions as sections of the text with a similar composition and with a specific objective (Cruz-Márquez & Montiel, 2017) (Figure 6).

Figure 5

Discursive structure of the first proposition (PP1)

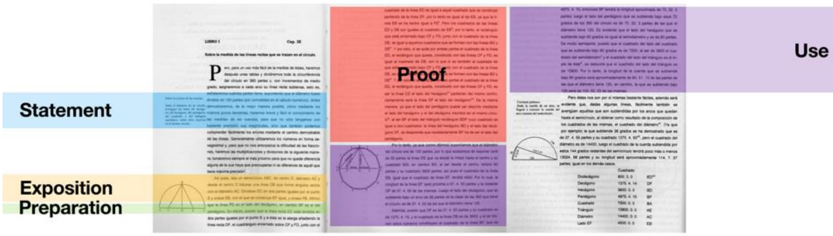
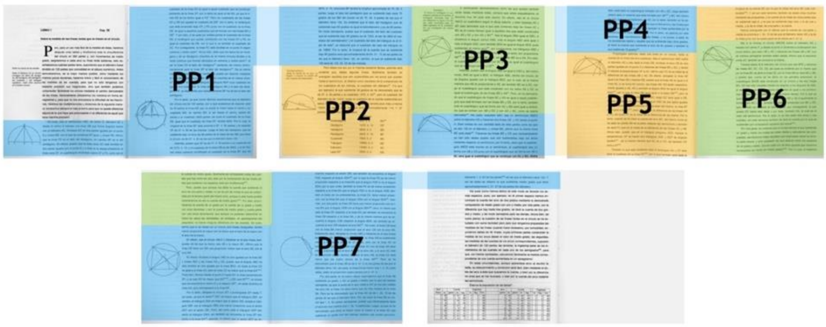


Figure 6

Propositions identified in Chapter IX of Ptolemy's Almagest. (Adapted from Cruz-Márquez (2018, p. 112)).



With this in mind, we began the textual analysis of the *Almagest* by studying in detail the discursive structure of the recognised propositions and carrying out each of the geometric constructions and mathematical proofs involved in them - a stage we called microanalysis - to identify the actions - in the sense of the practice nested model - that the author carried out directly on the objects and the useful tools. As the core of this first level of textual analysis, we ask: What does he [Ptolemy] do? How does he do it?

In the second level of analysis - which we called mesoanalysis -, we studied the objective of each of the identified propositions. By this, we derived relationships between propositions and possible activities - in the sense of the practice nested model. As the core of this second level of analysis, we ask: Why does he do it?

Finally, in the third level of analysis - called macroanalysis -, we articulated the specific objectives of each proposal and the links between them, to approach the objective of the document and answer our initial questions.

The *interpretation and inference*, the fifth stage of our content analysis, refers to the approach of answers to our starting questions based on the data collected, selected and analysed in the previous stages.

The *conclusion*, on the other hand, refers to the process of synthesis, writing, and final presentation of the results.

RESULTS

In correspondence with the implemented methodological tool, we present the results of the historization carried out in three large sections: contextual analysis, textual analysis, and a synthesis of the main results.

Contextual analysis of the *Almagest*

As a result of this first phase of analysis, we identified the problems faced by Ptolemy's contemporary astronomers and the conditions in which they did so. The results are described below.

About the context

We recognised the conditions in which Ptolemy's masterpiece was produced and disseminated as a plot of three components: the sociocultural context, the scientific context, and the tacit context.

The *sociocultural context* refers to events of a social and cultural nature that played a transcendental role in Ptolemy's life, education, and work. Among them, we highlight the development of ancient civilisations -Egyptian and

Mesopotamian -, the advantageous geographical location (Boyer, 1986) and ingenuity of the Greek people (Mateu & Orts, 2006), the foundation and economic and cultural power of Alexandria (Kline, 1972), the creation and rise of its Museum and Library (Melogno, Rodríguez & Fernández, 2011), and the concurrence of Ptolemy and his direct scientific predecessors - Euclid, Apollonius and Hipparchus - in the city (Boyer, 1986).

On the other hand, the *scientific context* alludes to the advances and events that shape the mathematical and astronomical environment in which Ptolemy created and spread his work. For example, the use of astronomy to anticipate terrestrial phenomena (van der Waerden, 1974), the astronomical observations by ancient civilisations, the Mesopotamian number system and calculation methods (Aaboe, 1964), the identification of incommensurability and the consequent rise of Greek deductive geometry (Sánchez, 2012), the composition of Euclid's *Elements* (Euclides, 1991; Boyer, 1986), the emergence of the angle as a quantification of the amplitude (Matos, 1990), the epicycle and eccentric theories of Apollonius (Boyer, 1986), and all the scientific production of Hipparchus (Maor, 1998).

Given the above, we considered that three tools were essential for Ptolemy's work: the Greek deductive axiomatic geometry - gathered in Euclid's *Elements* -, the arithmetic-algebraic advances of the Mesopotamian civilisations, and a quantitative notion of the angle.

Greek deductive axiomatic geometry, specifically Euclid's outstanding work, provided Ptolemy with a fairly rich accumulation of tools for geometric construction and mathematical proof. In addition, the *Elements* endow all scientific works after its publication with a particular language and rationality, and Ptolemy's *Almagest* is a clear example of that. For example, in the studied chapter alone, we identified the recurrent use of at least 20 propositions of Euclid's work, coming from books I, II, III, IV, VI and XIII.

Within the Mesopotamian arithmetic-algebraic contributions, it is worth highlighting its number system, which, sexagesimal and positional, had vast advantages over contemporary Greek and Egyptian systems, especially when working with large numbers and fractions. Moreover, the calculation methods of the Mesopotamian civilisations, such as linear interpolation and root approximation, were decisive for the astonishing degree of precision observed in the trigonometric table built by Ptolemy, up to seven decimal places, compared to the current calculation (Bressoud, 2010).

The quantitative notion of the angle, specified in the division of the circumference of the circle into 360 parts, each one subsequently divided into 60 smaller parts, which Greek mathematics built based on Mesopotamian astronomy and mathematics and which, in Ptolemy's times, was already in standard use, was an essential element for the astronomer's purposes, as it served as the reference unit based on which he expressed and operated the central angles and arcs of circumference.

Lastly, the *tacit context* refers to the articulated set of beliefs about the structure, composition, and functioning of the universe that prevailed in Ptolemy's time, currently called *Aristotelian worldview of the universe* (DeWitt, 2010) (Figure 7).

Figure 7

Scheme of the universe according to the Aristotelian worldview. (Adapted from Apian, Bellere and Gemma (1545, p. 6.))

Under that view, the universe is *finite* and *spherical*, and the Earth is a *static* and *floating* body in the void - beliefs from the Milesians' models (Asimov, 1975). In turn, the heavenly bodies are supposed to be *spherical* with *uniform circular* movements around the centre of the universe, where the Earth is located - beliefs suggested by the Pythagoreans (Boyer, 1986) and supported by the Platonic school (Saiz, 2003).

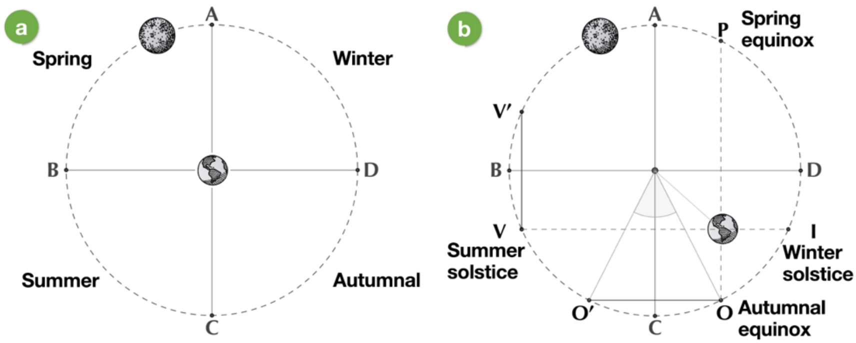
An important observation about those contexts is that, although separately exposing them helped give clarity and linearity to history, they are not independent; on the contrary, they intersect - oppose and favour - constantly and, as a whole, they shaped the conditions that made Ptolemy's work possible and fostered the emergence of trigonometric notions in his work.

About the problem

In the middle of the 2nd century BC, under the sociocultural, scientific, and tacit conditions mentioned, Hipparchus of Nicaea - Ptolemy's contemporary astronomer - observed that the celestial models built under the Aristotelian worldview of the universe could not explain the seasons of the year. In other words, if we consider the Sun as a body that moves in a uniform circular manner around the centre of the universe, where the Earth is located as a static entity (Figure 8a), the distance, size, and apparent brightness of the Sun should be similar throughout the year, which is not what we observe.

Figure 8

Uneven seasons problem. (Figure 8b adapted from Bressoud (2010)).



Consequently, Hipparchus proposed that, to match the model with the empirical facts, the Earth should not be located precisely in the centre of the ecliptic (Figure 8b). The consequent question is: How far from the centre is our planet located?

To address this issue, Hipparchus divided the ecliptic using empirical measurements of the length of the seasons of the year. Thus, if the distance between the spring equinox (P) and the autumn equinox (O) is 187 days out of the 365¼ of the year, then the arc PVO corresponds to 184.31º of the 360º into which the circle is divided. Consequently, the arc OO' equals 4.31º. Similarly, Hipparchus calculated that the arc VV' corresponds to 1.98º.

At this point, to calculate the distance between the Earth and the centre of the ecliptic through what is now called the Pythagorean theorem, Hipparchus only had to know the length of the chords subtended by the arcs OO' and VV'. However, he did not have a useful mathematical instrument to *measure distances indirectly in the context of the circle*, i.e., a tool that would quantitatively and systematically associate the arcs - or central angles - with their respective chords and vice versa. So, Hipparchus undertook the task of building it. This instrument became the first 'trigonometric table' that we know about, which made him creditor of the title of the father of trigonometry.

This table, like most of Hipparchus's works, has not come down to us, but we know about it thanks to the works of his successors, especially Claudius Ptolemy, who, in chapter IX of book I of the *Almagest*, our object of analysis, built a homologous table.

Textual analysis of *Almagest*

This second phase of analysis allowed us to identify three blocks of propositions or moments of work and three uses that Ptolemy gave to geometric notions in the course of his work. Next, we expand on those results and show examples of the micro[2] and mesoanalysis done around some propositions and blocks.

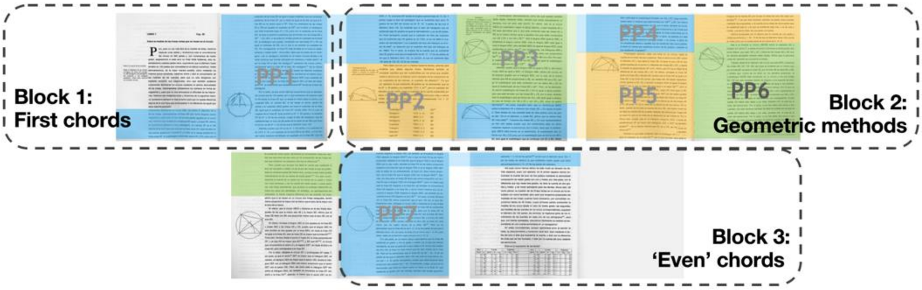
On the blocks of proposals or moments of work

To fulfil his objective of building a table that associated the angles between 0.5º and 180º, with an intermediate of half a degree, with their respective chords, Ptolemy required three blocks of propositions or moments of work, as a set of propositions with similar objectives: the first chords, the geometric methods, and the 'even' chords (Figure 9).

[2] The complete analysis of the geometric constructions and the mathematical proofs of each of the propositions can be consulted in Cruz-Márquez (2018, pp. 107-152).

Figure 9

Blocks of identified propositions



In the *first chords*, Ptolemy identified the first six angle-chord pairs in his table: the chords subtended by arcs - central angles - of 180º, 120º, 90º, 72º, 60º, and 36º. As an example, in the first proposition (PP1), to [what does he do?][3] calculate the chord subtended by a central angle of 90º, the author [how does he do it?] declared some basic geometric elements: a circumference divided into 360 parts and its diameter divided into 120 (Figure 10), each of them, then, divided into 60 smaller parts – a division that he used throughout the entire chapter. After that, he built the radius of the circle perpendicular to its diameter (BD) and drew the chord AB. Finally, with the help of what is now called the Pythagorean theorem, he calculated the length of the segment AB.

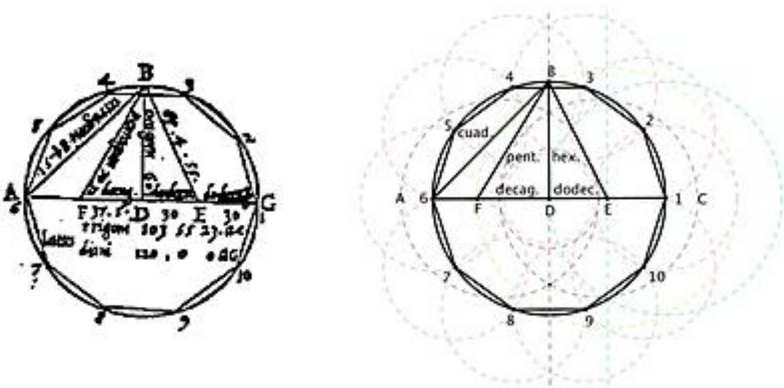
Although the angle-chord pairs built by Ptolemy in this first moment of work were quite well-known at the time - some of them were even part of Euclid's *Elements* (Bressoud, 2010) -, the objective [why does he do it?] was that it served as raw material for later moments.

In the *geometric methods*, second block or moment of work, Ptolemy took on the task of expanding the previously constructed angle-chord pairs. To do this, he proposed, tested, and made use of four geometric methods that constitute a bridge between the six pairs he already knew and those he required to complete his table.

[3] In this section we use square brackets to indicate "who does it, how he does it, and why he does it for" in the different propositions and blocks, with the intention of reflecting its role in their analysis.

Figure 10

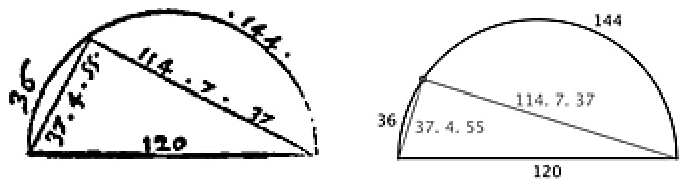
First chords. (Left: adapted from Saiz (2003, Appendix). Right: digitally designed according to the first, with dashed lines added, as necessary to build the solid figure, according to the proposals of Euclid's Elements).



For example, in the second proposition (PP2), Ptolemy [what does he do?] constructed and used his first geometric method, which allowed him to calculate the chord subtended by the central angle that is the supplement of a central angle whose chord is known. For that, the author [how does he do it?] started from a semicircle and its diameter, divided as explained before, to which he added a chord whose central angle he knew - 36º in the case he illustrated (Figure 11).

Figure 11

First geometric method. (Left: adapted from Saiz (2003, Appendix). Right: digitally designed according to the first).



Subsequently, Ptolemy constructed the chord that joins the remaining end of the diameter and the end of the first chord, i.e., the chord subtended by the central angle that is the supplement of the first angle - 144º in the illustrated case. Finally, the author used what today we call Thales' theorem and Pythagoras' theorem to establish a relationship between the lengths he knew - the diameter of the semicircle and the first constructed chord - and the length he intended to find out –the chord subtended by the central angle of 144º.

The importance [why does he do it?] of the methods constructed by Ptolemy during this second period of work lies not only in the number of angle-chord pairs that can be added to his table - somewhat over 120 -, but also in the fact that they represent the first systematic means to quantitatively describe the existing relation between the central angles and corresponding chords of which we have evidence, which constitutes the birth of trigonometry - in the sense we mentioned before.

In the *'even' chords* and the third and last block or identified moment of work, Ptolemy took the task of [what does he do?] approximating the chord subtended by a central angle of 1º and 0.5º. For that, the author [how does he do it?] proved that 'the ratio between the major chord and the minor chord in a circle is less than the ratio of their respective arcs'. In addition to [what does he do it for?] serving as a geometric foundation to approximate the desired chords, it is clear evidence of the author's full awareness of the non-proportional nature of the relationship between the arcs - central angles - and the chords they subtend.

On the uses of geometric notions

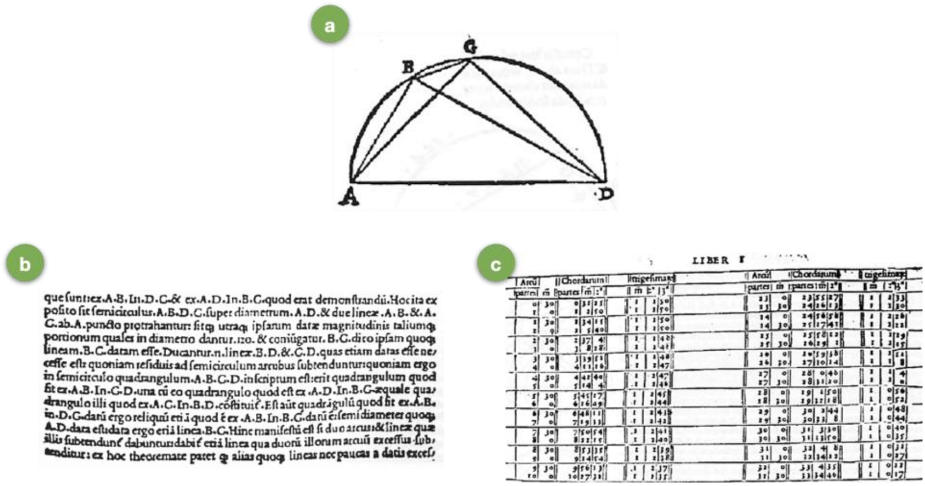
Finally, transversally to the propositions and moments of work cited above, Ptolemy used geometric notions and procedures, especially proportionality, the circle and the right triangle - their elements, relations and properties -, in at least three ways: as tools of construction, as theoretical tools, and as arithmetic-algebraic tools.

The use of geometric notions and procedures *as construction tools* refers to when Ptolemy declared and/or added the geometric objects that were going to intervene in a proposition; the *use as theoretical tools*, to when the astronomer formulated and proved the geometric relations between the objects declared and constructed above; and the *use as arithmetic-algebraic tools* refers to when the author used arithmetical and/or algebraic implications of the geometric relationships constructed to add new angle-chord pairs to his table.

As an example, in the fourth proposition (PP4), Ptolemy focused on [what does he do?] constructing a geometric method that allowed him to calculate the chord subtended by the angle that was the difference between two angles whose chords he knew. For that, the author [how does he do it?] introduced some geometric elements: a semicircle - divided as explained before - and two chords whose associated central angles are known (AB and AC), both with an endpoint in the diameter (Figure 12a). Up to this point, the author used geometric notions as construction tools.

Figure 12

Uses of geometric notions. (Adapted Saiz (2003, Appendix)).



Subsequently, Ptolemy argued that the chords subtended by the supplement angles of the first two (BD and CD) were also known, by the second proposition (PP2) described above. In addition, he used the fact that the quadrilateral ABCD was inscribed in the semicircle and that its two diagonals (BD and AC) and three of its sides (AB, AD, and DC) were known to ensure, by means of what is now called Ptolemy's theorem, that 'it is evident that if two arcs and the lines they subtend are given, the line subtended by the difference between those two arcs will also be given' (Figure 12b). In this section, we locate the use of geometric notions as theoretical tools.

Finally, the author used the arithmetic-algebraic interpretation of the relationship he built to calculate, among many others, the length of the chord subtended by a central angle of 12º - based on the chords of 60º and 72º -, hitherto unknown (Figure 12c). Here we place the use of geometric notions as arithmetic-algebraic tools.

The synergy of those three uses that Ptolemy gave to geometric notions in his work is what - for study purposes - we call *geometric work*.

A necessary summing-up

As a result of establishing a relationship between natural terrestrial and astronomical phenomena, the ancient Western civilisations saw the need to observe and register celestial phenomena and - later on - compose systems that would explain and anticipate them. A consequence of the desire to build those systems, within the framework of the Aristotelian worldview of the universe, is that the Greek astronomers had to indirectly measure distances in the context of the circle. This was a task that - as we have seen - made necessary the construction of a systematic and quantitative explanation of the relationship between a central angle and the lengths it subtended, i.e., the construction of the first relationship of a trigonometric nature.

To address this problem, Ptolemy inherited at least three fundamental tools: the Greek deductive axiomatic geometry, the arithmetic-algebraic advances of the Mesopotamian civilisations, and a quantitative notion of the angle. The first helped compose and justify geometric models that embodied the beliefs articulated under the Aristotelian worldview of the universe. The second and third were essential for adjusting those models to the more than four thousand years of empirical data.

Thus, when constructing his trigonometric table, Ptolemy took notions such as proportionality, the circle and the right triangle - their elements, properties and relationships -, and used them to introduce and build geometric objects, to establish some properties and relationships between them, and to calculate the desired angle-chord pairs. In other words, to fulfil his objective, Ptolemy used geometric work as a synergy of uses as construction tools, theoretical tools, and arithmetic-algebraic tools on the mathematical notions we mentioned.

Finally, it is essential to underline the role played by the sociocultural conditions for establishing an at least favourable atmosphere for the mathematical and astronomical production of Ptolemy's times, for the structure and composition of the *Almagest*, and – ultimately - for the construction of trigonometric notions.

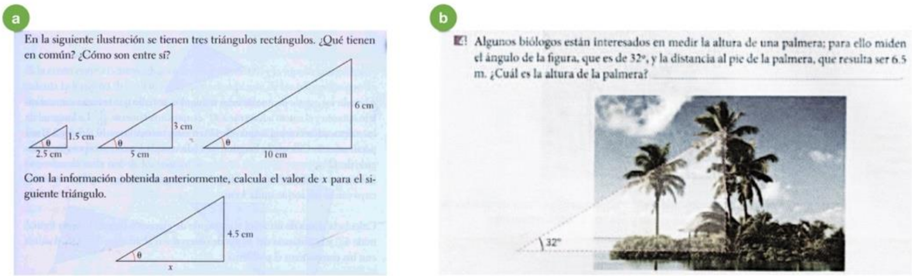
DISCUSSION AND CONCLUSIONS

The most significant results and contributions of this study – a consequence of the questions posed - are related to the initial use of trigonometric notions and the role that geometric notions and procedures play in it. Consequently, the first point of discussion is about what our historical analysis points out as the germinal use of trigonometric notions: the indirect measurement of distances in the context of a circle.

Reviewing study plans and programmes, textbooks, and trigonometry classes would make us think that the indirect measurement of distances occupies an important place in school trigonometry. However, research such as Montiel's (2014) shows that in the usual trigonometric tasks (Figure 13a) and the so-called 'application problems' (Figure 13b), the students are not required to carry out or analyse geometric constructions - given that they, as well as their proportionality, constitute starting conditions–, that obtain measurements or data or obtain the solution by another means.

Figure 13

Usual trigonometric homework and application problem. (Adapted from Montiel (2014)).



This type of mathematical activity restricts the students' work to choosing the appropriate 'trigonometric formula', substituting the given values, and performing the relevant arithmetic procedures to calculate the data that solves the problem (Brito & Barbosa, 2004; Weber, 2005; Díaz, Salgado & Díaz, 2010; Mesa & Herbst, 2011; and Montiel, 2014).

Consequently, we consider that, usually, in current school trigonometry, the indirect measurement of distances constitutes a fictitious scenario of application of definitions and formulas, instead of a context of use and meaning of trigonometric notions that favours their construction and the study of the proper nature of the angle-length relationship.

The second point of discussion is the role of geometry in the construction and significance of trigonometric notions. Although several studies (e.g. Patricio, García, & Arrieta, 2005; Navarro & Villalva, 2009; Jácome, 2011; and Montiel, 2014) agree on the usefulness and relevance of bringing geometric notions and procedures closer to the introduction and development of the trigonometric notions, questions about how to do it and what geometric notions and procedures are relevant to this process are areas of real debate. As an example, we briefly comment on the proposals by Bressoud (2010) and Weber (2005, 2008).

In the first, the author proposes introducing trigonometry from the 'circle model', based on historical considerations related to the trigonometric problems solved in the works of Euclid, Hipparchus, and Ptolemy, given that they precede the trigonometric ratios in the right triangle. However, his contribution does not include a didactic reflection on how those problems would live in class, given their complexity. Moreover, and regarding the recent contribution of Mandsfield and Wildberger (2017), today we accept that the Plimpton 322 tablet –dated a millennium before Hipparchus– contains exact trigonometric calculations based on ratios in the right triangle. Although such calculations come from a different epistemology than the one that gives rise to the current calculation of trigonometric ratios, and a deep analysis is required to recognise the relevance of any of those models in a didactic scenario, it is obvious that the chronological argument is not enough to decide which should be brought to the classroom.

In the second proposal, the author presents and puts into practice an experimental method of introducing trigonometric functions to study their understanding by higher education students in the USA school system. This method is based on the idea that trigonometric operations can be understood as 'geometric processes'.

Specifically, the geometrical process he proposes to calculate the sine of a given angle is to construct a unit circle in a Cartesian plane and use a protractor to draw a ray from the origin of that plane - so that the angle between the positive part of the x-axis and the ray is the desired angle -, locate the point of intersection between the ray and the unit circle, and determine the ordinate or height of that intersection (Weber, 2008).

Although this proposal brings into play some geometric elements, we believe they constitute mainly a method of introducing the unit circle and a means to 'geometrically justify' a metric or numerical procedure for the calculus of basic trigonometric ratios. Moreover, although it promotes the analysis of the angle-trigonometric function relationship and its properties, it does not explicitly delve into the transcendental –trigonometric– nature of that relationship.

Concluding, in light of the background and results presented, we affirm that when introduced into the school environment, the trigonometric notions lost their germinal functionality and their intrinsic nature, *the trigonometric element* that characterises them, and became a field for memorising and applying definitions and formulas, and space for exercising proportionality.

As an alternative - and answer to the starting questions -, we propose that the indirect measurement of distances in the context of the circle constitutes a favourable scenario for the analysis of the nature of the angle-length relationship and, consequently, to confront and expand the linear and arithmetic meaning associated with the trigonometric notions. We also postulate that the geometric work - the synergy of uses as construction tools, theoretical tools, and arithmetic-algebraic tools - on geometric notions such as proportionality, the circle and the right triangle - their elements, properties, and relationships - is essential in this process.

For future studies, we expect to create a specific didactic situation under those epistemological assumptions, subjected to empirical tests, besides studies around other historical points regarding the construction of trigonometric notions and their introduction and evolution within educational systems... but that will be a new story.

CONTRIBUTION STATEMENT

GCM conducted the investigation process. GME supervised the entire development of the investigation. GCM did the initial writing of the document. Both authors actively participated in the review and editing of this manuscript.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, GCM, upon reasonable request.

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