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Solving Double and Multiple Proportion Problems in the Final Years of Elementary School

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ABSTRACT

Background: Many studies analysed solutions to simple proportionality problems, but few focused on double and multiple proportionality. **Objectives**: Analysis of students' solutions to double and multiple proportionality problems. **Design**: Quantitative and qualitative analyses of empirical data. **Setting and participants**: Ninety students in grades 7 to 9 in Recife, Brazil. **Data collection and analysis**: Participants solved, in writing, double and multiple proportionality problems and orally explained their solutions. The analysis considers performance and solution strategies. **Results**: Percentage of correct answers per problem and school grade ranged from 67 to 97%, with no significant differences between types of problems or grades. Partipants used scalar, functional, and mixed strategies, with higher frequency of scalar strategies for double and of functional strategies for multiple proportionality problems. **Conclusions**: Double and multiple proportionality problems were equally accessible. Correct answers associated with the mixed strategy suggests that students considered multiple relationships in problems' statements.

Keywords: Proportionality; Mathematical problems; Scalar and functional relations; Double proportionality; Multiple proportionaly.

Resolução de problemas de proporção dupla e múltipla nos anos finais do Ensino Fundamental

RESUMO

Contexto: Muitos estudos analisaram soluções para problemas de proporcionalidade simples, mas poucos consideraram proporcionalidade dupla e múltipla. **Objetivo**: Analisar como estudantes resolvem problemas de proporção dupla

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e múltipla. **Design**: Análise quantitativa e qualitativa de dados empíricos. **Ambiente e participantes**: Noventa estudantes do 7º ao 9º ano em Recife. **Coleta e análise de dados**: Participantes resolveram por escrito problemas de proporção dupla e múltipla e explicaram oralmente suas estratégias. A análise considera performance e estratégias. **Resultados**: O percentual de respostas corretas por problema e ano variou de 67 a 97%, sem diferenças significativas entre tipos de problemas ou anos. Participantes usaram estratégias escalares, funcionais e mistas, sendo estratégias escalares mais frequente nos problemas de proporção dupla e estratégias funcionais nos de proporção múltipla. **Conclusões**: Problemas de proporção dupla e múltipla foram igualmente acessíveis. Respostas corretas associada à estratégia mista sugere que os estudantes consideram as múltiplas relações no enunciado dos problemas.

Palavras-chave: Proporcionalidade; Problemas Matemáticos; Relações escalares e funcionais; Proporção dupla; Proporção múltipla.

INTRODUCTION

This investigation is part of an interdisciplinary field called psychology of mathematics education, which seeks to understand the epistemological, psychological, and didactic obstacles involved in the mathematics teaching and learning processes (Carraher et al., 1988; Carrião et al., 2018; Da Rocha Falcão, 2003; Lautert et al., 2020; Spinillo et al., 2021; Utsumi, 2020). Among the theoretical contributions used in the studies of this field of knowledge, attention is drawn to the theory of conceptual fields developed by Gérard Vergnaud, which seeks to understand the dynamics involved in the process of construction of knowledge that inexorably leads to the study of a variety of *situations* in close relationship with the *operational invariants* (as fundamental properties that characterise a given concept) and the linguistic and symbolic representations that allow the *representation* of concepts and their relationships. According to Vergnaud (2017, p. 39), to "analyse the development of competencies and conceptualisations of the subject in the different registers of his/her activity, it is essential to fragment the object" to "take as the object of study a set of situations and concepts, i.e., a conceptual field."

In this investigation, we explore the multiplicative conceptual field, which requires multiplication and division operations and includes a variety of concepts such as fraction, ratio, proportion, probability, rational numbers, and other ideas (Vergnaud, 1983, 1988, 1994). Proportionality, the object of this investigation, involves the sense of covariance and multiple comparisons and refers to the ability to mentally gather and process different sets of information. It is also characterised by the ability to understand the multiplicative relationship inherent in situations of comparison (Lesh et al. 1988). Therefore,"understanding concepts and procedures within multiplicative structures requires attention to how the measured quantities are interrelated" (Lautert & Schlieamann, 2021, p. 1421).

Some studies (Hart, 1984; Nunes et. al., 1993; Ricco, 1982; Schliemann & Carraher, 1992; Schliemann & Nunes, 1990; Tournaire & Pulos, 1985; Vergnaud, 1983, 1988, 1994) describe the use of scalar and functional strategies in proportionality problems, which may show a conceptual appropriation and understanding of more advanced representations and procedures regarding proportionality (Levain & Vergnaud, 1994-1995).

The concept of proportion

In mathematics, the concept of proportionality refers to the equality between two ratios A/B=C/D, for example, 2/3=4/6. Proportional situations vary according to the relationship between the quantities, which can be (i) *directly proportional*, when the change in one variable is observed in the same direction as the other variable, or (ii) *inversely proportional,* when the change in one variable is observed in the opposite direction in the other variable. Lesh et al. (1988) emphasise that the concept of proportionality involves understanding covariance and multiple comparisons and the ability to mentally gather and process different sets of information, identifying the multiplicative relationship inherent in the situation. The relationship described in a proportionality problem may involve the idea of a one-to-many correspondence or a many-to-many correspondence. In both cases, the relationship between the two sets of values constitutes a linear function, the basis for understanding the concept of proportion. As emphasised by Schliemann and Carraher (1992, 1993), situations involving proportions can be seen not as pairs of lone numbers, but as part of a linear function where a variable y is directly proportional to another variable x, which is expressed as $y = f(x) = ax$.

Proportional reasoning requires going beyond verifying equivalence between different situations, thinking in relative terms rather than in absolute terms, and establishing relationships between relationships, i.e., second-order relationships that link two or more first-order relationships (Inhelder & Piaget, 1976). From a psychological point of view, proportional reasoning is considered to promote the development of specific knowledge, which favours the evolution of higher-order reasoning, because it is related to inferences and predictions that involve both qualitative and quantitative thinking, often based on real-life situations (Lamon, 2007; Lesh et al. 1988; Spinillo, 2002; Sousa et al. Oliveira, 2015).

Situations involving proportionality can be presented in three different ways: (i) simple proportion, defined by the existence of a constant relationship between the two numbers or two quantities; (ii) double proportion, situations that involve two or more independent proportions linked together by a common variable; and (iii) multiple proportion, characterised by situations that involve two or more concatenated simple proportions. Each one has specific characteristics and forms of resolution, which mobilise different reasoning, depending on its configuration. (Gitirana et al., 2014; Levain, 1992; Lautert et al., 2017; Levain & Vergnaud, 1994/1995; Vergnaud, 1983; 1988; 2011).

When solving simple proportion problems from a pair of data relating two variables, *x* and *y*, we determine the value of the variable *y*, which corresponds to another value of the variable *x*, keeping the relationship between the two values constant. For example, if a car has four wheels, three cars will have 12 wheels; if in a cake recipe, for every glass of milk, two eggs are added, for three glasses of milk, six eggs are added.

Vergnaud (1983) describes three main strategies for solving simple proportion problems: the scalar strategy, the functional strategy, and the use of the algorithm known as the rule of three or calculating the fourth proportional. Consider, for example, the following problem:

> *"Three candies cost nine reais. How much should I pay for 12 candies?"*

Using the scalar strategy, for every three candies added to the number of candies, nine reais are added to the price, obtaining the answer through successive additions of the value 3 to the number of candies and the value 9 to the price. This same strategy can use multiplication, considering that if 12 candies are equivalent to four times three candies, the cost of three candies (9 reais) must be multiplied by four, yielding 36 as a result. The solution to the same problem, using the functional strategy, establishes the relationship between the two initial quantities, i.e., the 1:3 ratio between the number of candies and the price, or that each candy costs 3 reais, and we must multiply 12 by 3 to find the price of 12 candies.

The rule of three algorithm states that 3/12=9/x (or, 3 is to 12 as 9 is to x) and the value of x is calculated by cross-multiplication, *9*12=3x* and by solving the equation through the following steps: *x= (9*12)/3, x=108/3, x=36*.

The scalar strategy has been considered as more accessible to young students since ratios between quantities of the same nature appeared earlier than ratios between different quantities in the History of Mathematics (Freudenthal, 1983) and that, as suggested by Vergnaud (1983), the functional strategy is more abstract than the scalar strategy. However, empirical studies have shown that children and street vendors (Carraher et al., 1988; Ricco, 1982; Schliemann & Carraher, 1992; Schliemann & Nunes, 1990) solve many-to-many proportionality problems by calculating the price of a unit before operating on the values of each variable using the scalar strategy. When calculating the price of an item, the child or adult is implicitly considering the functional relationship between the two variables. Thus, a possible strategy for solving proportionality problems may involve both aspects of the scalar relationship and typical aspects of the functional relationship. Schliemann and Carraher (1992, 1993) found that, in the case of many-to-many simple ratio problems, scalar, functional, or mixed strategies among educated children depend on the relationships between the quantities described in the problem.

The double proportion and multiple proportion problems, extensively described by Vergnaud (1983, 1988, 2011) and Gitirana et al. (2014), have three or more pairs of relationships between quantities. In double proportion problems (also called bilinear function), the pairs of quantities involved establish independent relationships with each other, for example:

> *"At FabriCar plant, production is monitored by counting the cars produced. Two (2) workers working 8 days in a row can assemble 4 cars. How many cars will be produced by 6 workers who work 16 days?"* (Leite, 2016, p. 36)

In this case, the number of cars produced is proportional to the number of workers and the number of days worked; these two quantities (workers and days) do not maintain a relationship of dependence, i.e., the number of workers does not influence the number of days worked, or vice versa.

Multiple proportion problems require, as the term implies, the concatenation of several proportions, as we see in the example:

> *"To prepare a sidewalk, a bricklayer Seu José uses 6 buckets of sand for every 3 buckets of cement and 4 buckets of water for every 2 buckets of sand. When he received the material, he noticed that there were 6 buckets of cement and 12 buckets of sand. So how many buckets of water will Seu José need to make the dough in the same way?"* (Leite, 2016, p. 36)

In this case, cement, sand, and water maintain dependency relationships with each other, so that any change in any value, even if proportionality rates are kept constant, will produce a change in the final situation of the problem. In other words, if the amount of cement changes, the amount of sand and the amount of water also change.

In summary, in double proportion problems, the relationships between the quantities appear in pairs separately (two by two) while, in multiple proportion problems, all pairs of quantities are related to each other. Levain (1992) identified several characteristics that affect students' performance in simple proportion, multiple proportion, and double proportion problems. Their results show that, for French students in the school year that corresponds to the fifth grade of the elementary school in Brazil, the double proportion problems are the most difficult. Differently from these results, in a study on the resolution of multiple proportion and double proportion problems, among students from the 2nd to the 9th grade of elementary school in public schools in São Paulo, Gitirana et al. (2014) found that, in the case of double proportion, 4th graders present around 30% of correct answers, reaching almost 70% in the last year of elementary school (9th grade). On the other hand, the multiple proportion problems proved to be more difficult, with the percentage of correct answers in the 9th grade lower than 40%. It should be noted that these investigations did not include analyses of the types of strategies used by students, as is the case of the present study.

Considering the complexity involved in double proportion and multiple proportion problems, the divergent results of previous studies (Levain 1992; Gitirana, et al. 2014), it becomes relevant to investigate the performance and strategies mobilised by students when solving problems involving double proportion and multiple proportion. Do students in the final years of elementary school have an adequate understanding of the multiplicative relationships involved in double and multiple proportion problems? Do these two types of problems present the same degree of difficulty? What strategies are mobilised by students in solving these problems? This study seeks to answer these questions.

METHOD

Participants

The study included 90 7th-, 8th-, and 9th-grade students of both sexes, from a public school in the city of Recife, considered to be a reference school in the state of Pernambuco.These students were randomly placed in three groups of 30 students each, enrolled in the 7th grade (M = 12 years and 5 months, SD 5.13 months), in the 8th grade (M= 13 years and 3 months, SD 4.83 months) and in the 9th grade (M= 14 years and 6 months, SD 4.03 months). The choice of the schoolgrades was guided by the National Curriculum Parameters of Mathematics, which assume that at the end of elementary school, students will be able to solve problems of proportionality (Brasil, 1998).[1]

Procedures and materials

Figure 1

Double proportion and multiple proportion problems presented in the investigation (Leite, 2016, p.36)

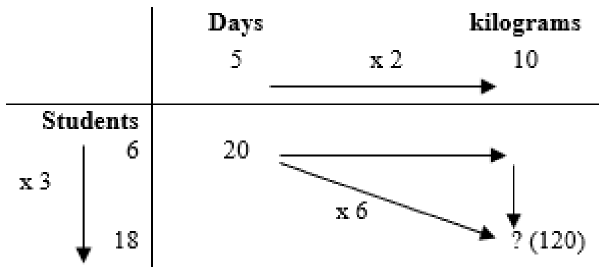
Double proportion (DP)	**Multiple proportion** (MP)
DP1: A school competition is being held at Escola Rui Barbosa and this year one of the tasks proposed to students is that they get mobilised to collect food for donation. In the 5th grade class, a group of 6 students managed to collect 20 kilos of food in 5 days. How many kilos of food would be collected if the group consisted of 18 students working for 10 days?	**MP3:** To prepare a sidewalk, a bricklayer sr. José uses 6 buckets of sand for every 3 buckets of cement and 4 buckets of water for every 2 buckets of sand. When he received the material, he noticed that there were 6 buckets of cement and 12 buckets of sand. So how many buckets of water will sr. José need to make the dough in the same way?
DP2: At FabriCar plant, production is monitored by counting the cars produced. Two workers working eight days in a row can assemble four cars. How many cars will be produced by 6 workers who work 16 days?	**MP4:** Marina is making a chocolate cake. The recipe says that for each glass of milk you need to use 2 eggs. For each egg, we must use 3 cups of flour. How many cups of flour will Marina need to make the same cake with 3 glasses of milk?

[1]In accordance with Resolution 466/12 National Health Council (CNS), after approval by the Research Ethics Committee of the Federal University of Pernambuco 03/04/2015, Review N. 1541330, the signature of the parents or guardians and the students of the Free and Informed Consent Term (FIC).

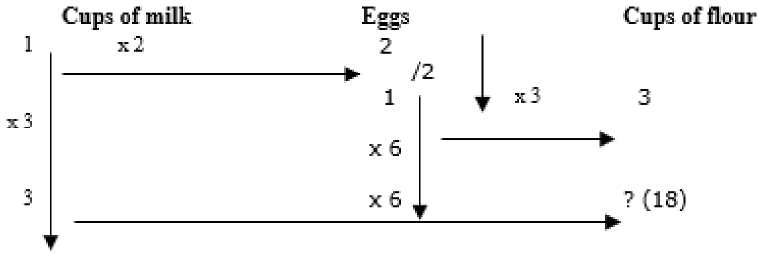
Figure 2

Problems explained by students and their representations. Representation scheme proposed/adapted from Vergnaud's (1988, 2011) work (Lautert & Schliemann, 2021, p. 1427).

DP1 (double proportion): A school competition is being held at Escola Rui Barbosa, and this year, one of the tasks proposed to students is that they get mobilised to collect food for donation. In the 5th grade class, 6 students managed to collect 20 kilos of food in 5 days. How many kilos of food would be collected if the group consisted of 18 students working for 10 days?



MP4 (multiple proportion): Marina is making a chocolate cake. The recipe says that for each glass of milk, you need to use 2 eggs. For each egg, we must use 3 cups of flour. How many cups of flour will Marina need to make the same cake with 3 glasses of milk?



Note: School relationships are represented by vertical arrows and functional relationships are represented by horizontal arrows.

Data were collected in two sessions[2]. In the first session, in a collective application during math class time, all students were asked to solve, in writing, four problems, two of double proportion and two of multiple proportions. Half of the students solved the double proportion problems first and then the multiple proportion problems. The other half solved the problems in reverse order, being given the instruction: *"We would like you to solve the problem individually, in this form, using only a pencil, eraser or pen. Below each question, there is a space for the resolution, and you can use other blank spaces of this material"*

Presented in Figure 1, the problems were constructed based on Vergnaud (1983, 1988, 1994, 2011) and Gitirana et al. (2014).

In the second session, held five to 15 days after the first session, in an individual application during math class time, each student was asked to explain how they solved two of the problems proposed in the first session. They began to explain the MP4 problem and later explained the DP1 problem, with their answers being recorded and fully transcribed into individual protocols. Figure 2 illustrates the relationships between the quantities in each of the problems for which the students explained their solving strategies. As we can see, in problem DP1 (school competition), to find the number of kilos collected, it is necessary to determine the product of the number of students by the number of days in which there was collection. However, those two quantities do not maintain a dependency relationship with each other, i.e., the number of students participating in the collection does not change the number of working days. In problem MP4 (Marina's recipe), to find the number of cups of flour to make the same cake with three glasses of milk, the student must coordinate the relationship between all the quantities involved, glasses of milk, eggs, and cups of flour. This is because, in the multiple proportion, all pairs of quantities maintain coordination with each other and any change in one of the pairs produces changes in the whole set.

RESULTS

Initially, we will present the results of the quantitative analysis regarding the number of hits by type of problem (double proportion *vs*. multiple proportion) and for each of the four problems. Then we will discuss the resolution strategies implemented by the students, as revealed by the analysis

[2]This research was carried out before the publication of the National Curricular Base (BNCC). In the BNCC, proportionality must be present in the study of: operations with natural numbers, fractional representation of rational numbers; areas; functions and probability (Brasil, 2017).

of the written material and the verbal explanations during the individual interview.

Performance Analysis

Table 1 shows the average proportion of correct answers in double and multiple proportion problems as a function of education. As can be seen, students in all school grades present similar results (except 7th graders on double proportion problems) and slightly better averages of hits on multiple proportion problems. An analysis of variance (ANOVA) for repeated measures on one factor did not reveal significant differences between the results across school grades (7th, 8th and 9th grades) [$F_{(2.84)} = 0.36$; $p = 0.69$ nor between types of problem (double and multiple proportion) [$F_{(1.84)} = 2.77$; $p = 0.099$]. The interaction between these two factors was also not significant [$F_{(2.84)} = 0.44$; $p = 0.644$]. Such results suggest that, as of the 7th school grade, most students at this school are able to solve double proportion and multiple proportion problems. The effect of order of presentation of the types of problem (double-multiple vs. multiple-double) did not reveal significant differences.

Table 1

Mean of correct answers and standard deviation (in parentheses) by type of problem and school year

School year	Double Proportion	Multiple Proportion
7th grade	1.47 (0.78)	1.73 (0.52)
8th grade	1.63 (0.67)	1.77 (0.57)
9th grade	1.63 (0.67)	1.70 (0.60)
Total	1.58 (0.70)	1.73 (0.56)

Note: Maximum score on each problem equal to 2.

Table 2 presents the mean of students' correct answers in each of the two problems of each type. In general, students show better results in the Recipe problem, (mean 0.90) and lower results in the Car problem (mean 0.77), with small variations across school years. The ANOVA results for repeated measures on the problem factor did not reveal significant differences between the four types of problems [$F_{(1.87)} = 3{,}290$; $= 0.73$] nor between school years [$F_{(2.87)} = 0.921$; $p=0.402$].

Table. 2

Mean of correct answers and standard deviation (in parentheses) by type of problem and school year.

School year	Double Proportion		Multiple Proportion	
	Competition	Car	Sidewalk	Recipe
7th grade	0.80 (0.41)	0.67 (0.48)	0.77 (0.43)	0.97 (0.18)
8th grade	0.77 (0.43)	0.87 (0.35)	0.87 (0.35)	0.90 (0.31)
9th grade	0.87 (0.35)	0.77 (0.43)	0.87 (0.35)	0.83 (0.38)
Total	0.81 (0.40)	0.77 (0.43)	0.83 (0.38)	0.90 (0.30)

Analysis of resolution strategies

The written material and the answers to the interviews in which the students explained how they solved the problems about the "Competition" (double proportion) and about the "Recipe" (multiple proportion) were analysed by a mediator, with 18 cases for which the classification was problematic, referred to a second independent mediator. Four cases of discordant classification were detected. Those cases were discussed by the two adjucators and assigned a final classification by consensus. In the few cases where the verbal explanation was unclear, what the written work suggested was considered. Only four students said they did not remember how they arrived at a written answer and two presented an explanation that was not clear.

In the 172 problems solved in writing and for which the students explained their solving strategies during the interview, three strategies were identified: (i) the scalar strategy, when students demonstrate or explain the solution using the scalar factor for each pair of values of the same magnitude; (ii) the functional strategy, when students demonstrate or explain the solution using the functional factor that relates the values of two different quantities and, (iii) the mixed strategy, when students use or explain the solution using both scalar and functional relationships. Although teaching the rule of three is part of the textbook adopted for the 7th grade, only one 9th grader used the rule of three to calculate his answers. This student explained his work using the scalar strategy. Examples of those strategies, based on written material and verbal explanations for both types of problems will be presented after quantitative analysis of student responses. Table 3 shows the distribution of participants in each school grade according to the type of strategy used in solving the double proportion problem (Competition) and the multiple proportion problem (Recipe).

Table 3

Frequency and percentage of the types of strategies adopted by students to solve each type of problem

School year	Double Proportion (Competition)				Multiple Proportion (Recipe)			
	Absent	Scalar	Functional	Mixed	Absent	Scalar	Functional	Mixed
7th grade	3	18	1	8	0	6	19	5
	(10,0)	(60,0)	(3,3)	(26,7)	(0)	(20,1)	(63,3)	(16,6)
8th grade	2	24	0	4	0	11	11	8
	(6,7)	(80,0)	(0)	(13,3)	(0)	(36,6)	(36,6)	(6,8)
9th grade	1	22	1	6	0	10	19	1
	(3,3)	(73,3)	(3,3)	(20,1)	(0)	(33,3)	(63,4)	(3,3)
Total	6	64	2	18	0	27	49	14
	(6,7)	(71,1)	(2,2)	(20,0)	(0)	(30,0)	(54,4)	(15,6)

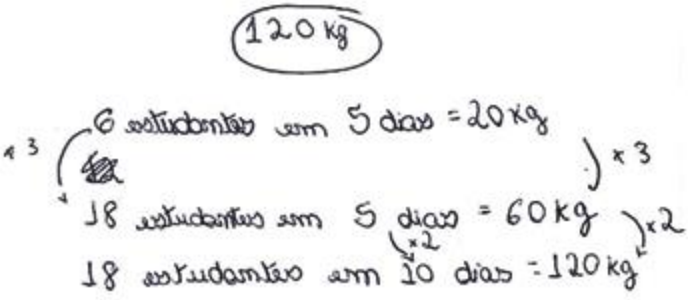
In double proportion problems, most students adopted scalar strategies, with little use of functional and mixed strategies. In multiple proportion problems, more than half of the students adopted functional strategies, followed by the use of scalar strategies and some mixed strategies. These trends appear in the three school grades, except for the answers of the 8th graders to the multiple proportion problem: in this case, equal percentages of students adopted the scalar and the functional strategy. The association between the types of problem (multiple proportion and double proportion) and resolution strategies (scalar relationships, functional and mixed relationships) was significant ($\chi^2_{(2)} = 58.72$; $p <. 0001$).

Examples of solutions to the double ratio problem

As shown above, the *scalar strategy* is used by 71.1% of students to solve the double proportion problem. Figure 3 shows the written work of a student who used scalar relationships to solve and explain how she solved the problem.

Figure 3

Protocol 13, 8th-grade student, female.



Examiner: [reading of the School Competition problem] *Tell me how you solved this problem.*
Participant: *First I saw how many kilos 18 students were going to get in the same amount of days; so I saw that in five days they would get 60 kg because that's three times 20 kg. Then, in ten days, which is twice as much, I would have gotten one hundred and twenty kilos.*

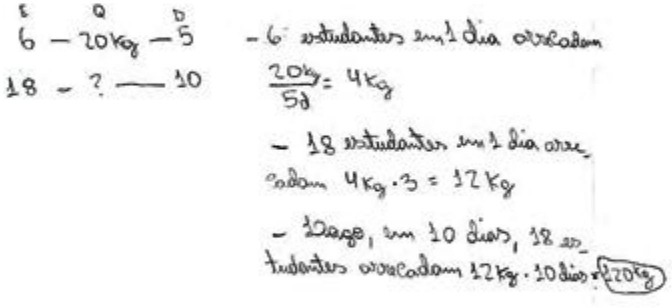
As the written work shows, the student uses the scalar relationship (times 3) between the initial number of students (6) and the final number (18), as well as the application of this scalar factor to the number of kilograms (20), leading to the first result of 60 kg produced by 18 students. For the second part of the problem, the student emphasises the scalar relationship between numbers of days (times 2) and applies this scalar factor over the previously determined number of kilograms (60), obtaining 120 kg as a result.

In only two double proportion problems the students tried to use the functional relationships, in both cases showing wrong results.

The mixed strategy was used by 20% of the students to solve the double proportion problem, considering scalar and functional relationships in the written work and the explanation, as shown in the example in Figure 4.

Figure 4

Protocol 17, 8th-grade student, male



Examiner: [reading of the School Competition problem]. *Tell me how you solved this problem.*
Participant: *I wrote down the details of the problem. Oh, first I saw how many kilos six students collected in one day. So, they collected four kilos. Then, how much do eighteen students collect in a day? Then, [you] multiply the initial number of students by three and, according to the principle of constancy, it is also necessary to multiply the number of kilos collected in a day by three, which equals twelve kilos in a day. But the question is in ten days. So [you] just multiply the twelve kilos per ten days, which gives one hundred and twenty kilos of food collected.*

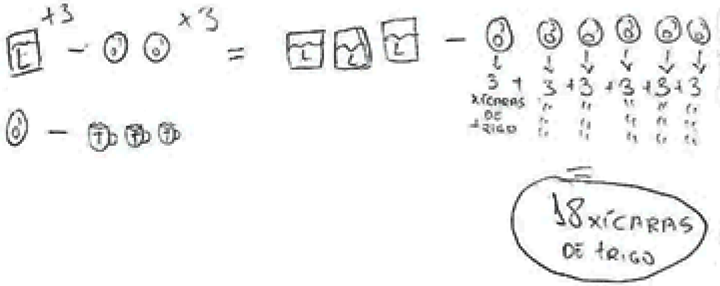
In this example of using the mixed strategy, the student registers the problem data in the first line of the written work and concludes that the students collected 4 kg per day (the result of dividing 20 kg per five days). In his explanation, he describes that "*first I saw how many kilos six students collect in one day, so, they collect four kilos" (sic)* considering, therefore, the functional relationship between kilos of food per day of work. Subsequently, he multiplies the kilos collected in 1 day by 3, the scalar factor relating 6 students to 18 students, emphasising that, "*according to the principle of constancy, it is also necessary to multiply by three the amount of kilos collected in a day, which equals twelve kilos in a day"* (sic). At the end, he returns to the question of the problem, "but the question is in ten days, so we only have to multiply the twelve kilos by ten days, which gives one hundred and twenty kilos of food collected", applying the functional factor, 10 days, to the amount of kilos collected, to find the correct answer, 120 kg.

Examples of solutions to the multiple proportion problem

As shown in Table 3, the scalar strategy was used by 30% of the students to solve the multiple proportion problem. Figure 5 shows how an 8th grader solved the problem and explained her solving strategy based on scalar relationships.

Figure 5

Protocol 15, 8th-grade student, female.



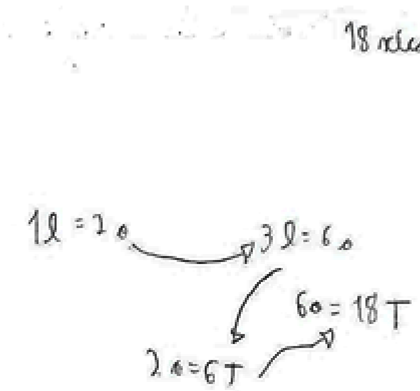
Examiner: [reading Marina's Recipe problem]. *Tell me how you solved this problem.*
Participant: *Let me see how I can get started. I saw first that to make a cake she would need... I first tried to reproduce the question in a drawing to make it easier, and here it said that in the recipe it is written for each glass of milk I need to use two eggs. So I put a glass of milk, I need two eggs. And she said that for each egg two cups of flour were needed. So I put here that for each egg three cups of flour... when he said that Marina would need to make the same cake with three glasses of milk, I multiplied by three, multiplied by three, which gave three glasses of milk and six eggs. So, if three cups of flour are needed for each egg, I put three cups of flour here, so I added each one and it came to eighteen.*

Based on the representation and the transcript of the interview, we identified that the student presents the scalar factor (times 3) applied to the number of glasses of milk and the number of eggs, leading to the result for the first part of the problem as drawings of three glasses of milk corresponding to six eggs. Underneath the drawing of each egg, she writes 3 six times, with the plus sign between them. So, for the second part, she uses the scalar relationship by successive additions, adding 3 cups of flour, six times. Consistent with the written solution, in the interview, the participant explains that *"the same cake, with three glasses of milk, I multiplied by three, which gave three glasses of* *milk and six eggs, and, "I put three cups of flour here so I added each one and it came to eighteen"*. (sic)

Figure 6 presents an example of the functional strategy, which was used by 54.4% of students in solving multiple proportion problems. In this figure, the written work shows that the student establishes functional relationships between a glass of milk and eggs (1 to 2) and multiplies 3 glasses of milk by 2 to get 6 eggs. He later indicates that 2 eggs correspond to 6 cups of flour, without explaining the calculation, and that 6 eggs correspond to 18 cups. When explaining the written work, he mentions "I multiplied three glasses of milk by two eggs, it gives six eggs", therefore using the functional factor, 2. It then continues: "*then I multiplied six by three which are the number of cups used by each egg*" (sic) also explaining the functional factor, 3.

Figure 6

Protocol 8, 7th-grade student, male

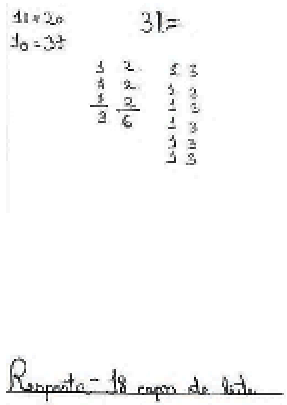


Examiner: [reading Marina's Recipe problem]. *Tell me how you solved this problem.*
Participant: *I saw that for every glass of milk you need two eggs and three cups of flour. Then, first, I multiplied three glasses of milk by two eggs, that's six eggs. Then, I multiplied six by three, the number of cups used by each egg.*

The mixed strategy (see example in Figure 7) was used by 15.6% of students to solve the multiple proportion problem.

Figure 7

Protocol 4, 7th-grade student, female



> **Examiner:** [reading Marina's Recipe problem]. *Tell me how you solved this problem.*
> **Participant:** *I thought like this: as there are three glasses of milk, for every glass of milk there are two eggs. So, three times two makes six eggs, and for each egg, you need three cups of flour. So three times six is eighteen cups of flour. Jeez, here [in the written answer] I put glasses of milk, but they were cups of flour.*

In this case of using the mixed strategy, the student records the functional relationships described in the problem statement and then presents successive additions of the values in the functional relationships 1 to 2 (1 glass of milk for 2 eggs) and 1 to 3 (1 egg for 3 cups of flour). She considers the scalar relationships by adding, in the first case, 3 times (scalar factor) the value 1 and the value 2, obtaining three glasses of milk and six eggs. In the second case, she adds 6 times (scalar factor) the value 1 and the value 3, getting 6 eggs and 18 cups of flour. In the interview, the explanation begins by describing the functional relationship and multiplying 3 glasses by 2 eggs: "*as there are three glasses of milk, for each glass of milk there are two eggs, so three times two gives six eggs*". In the second part, she explains that she multiplied 3 cups of flour by 6 (the scalar factor), to get 18 cups: "*so three times six is eighteen cups of flour*" (sic). She concludes by calling attention to the fact that she was wrong to write the answer to the problem: "*Jeez, here I put glasses of milk, but they were cups of flour*" (sic).

Relationship between correct and incorrect answers and strategies

Considering that previous studies suggest that scalar strategies would be more accessible than functional ones, it is interesting to examine whether there is a relationship between the frequency of correct answers and the type of resolution strategy adopted by the student (see Table 4).

Table 4

Frequency and percentage (in parentheses) of correct and incorrect answers by resolution strategy.

Types	Double Proportion (Competition)		Multiple Proportion (Recipe)	
	Correct	Incorrect	Correct	Incorrect
Absent	1 (0.5)	5 (2.8)	0 (0)	0 (0)
Scalar	58 (32.2)	6 (3.3)	23 (12.8)	4 (2.2)
Functional	0 (0)	2(1)	44 (24.4)	5 (2.8)
Mixed	18 (10)	0 (0)	14 (8)	0 (0)

As shown in Table 4, incorrect answers are rare in both types of problems for both the scalar and the functional strategy. It is interesting to note that mixed strategies always led to correct answers in both types of problems. The association between the mixed strategy and the correct solution suggests that students, when making use of this type of strategy, would have greater flexibility when considering the proportional relationships between the quantities described in the problems.

DISCUSSIONS AND CONCLUSIONS

The results of this study show that students in the 7th, 8th, and 9th grades present 67 to 97% of correct answers for double proportion and multiple proportion problems, with no significant differences in the average of correct answers between the two types of problems. Unlike results from previous studies (see Levain, 1992; Gitirana, et al., 2014), in the present study, double and multiple proportion problems were equally accessible to the investigated sample. A possible explanation for these results may be related to the fact that in this school, the teaching of algorithms for solving proportionality problems suggested by textbooks started with the discussion of relations between quantities before the student was introduced to the use of the rule of three.

However, this issue needs to be further investigated, with studies where the introduction of the rule of three algorithm is not preceded by instructions on the relationships between quantities.

From an educational point of view, the results of this investigation point to the use of strategies to solve proportionality problems related to the type of previous experiences of the individual (Schliemann & Carraher, 1992, 1993). There was a higher frequency of use of scalar relations in the double proportion problems and the use of functional relations for the multiple proportion ones, in the three school years. Furthermore, consistent with results from other studies (Levain, 1992; Schliemann; Carraher, 1992, 1993) these results suggest an influence of the type of numerical relationships presented in the problem statement: in the double proportion problem analysed in the present study, the functional relationship between the number of students (6) and the number of kilos of food (20) was more difficult to compute than the scalar relationships.

The totality of correct answers associated with the mixed strategy (solving procedures and explanations that emphasise both the use of scalar relationships and functional relationships) is worth mentioning because it suggests that when using this strategy, students consider the multiple relationships present in the problem statement. These students make use of action schemes that involve both scalar operators and functional operators, considering the various relationships in double and multiple proportion problems (Lautert & Schliemann, 2021; Nunes & Bryant, 1997; Santos, 2015; Vergnaud, 2009).

Finally, the use of different forms of written representation produced by the students, drawings, successive additions, or arrows accompanied by the multiplication sign, show that even being instructed in the school context in terms of steps for solving algorithms (for example, the rule of three and algebraic procedures), students adopt unconventional notations and strategies that reveal an understanding of scalar and functional relationships for solving double and multiple proportion problems. Although this algorithm is part of the 7th grade textbook, the students may not have used the rule of three because the numerical relationships in problems are easy to be mentally determined. Future studies with larger numbers may better illustrate the importance and possibility of using the rule of three to solve double and multiple proportion problems in the school context. In any case, our results suggest that most students who participated in this study are prepared to use and understand the rule of three algorithm.

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AUTHORSHIP CONTRIBUTION STATEMENT

A.B. B. L. A. contributed to conceiving the research, constructing the instrument, collecting and tabulating data, analysing the data, reviewing the literature, and writing the article. S. L. L. contributed to conceiving the research, constructing the instrument, analysing data, writing the article, and formalising the article for submission. A. D. S. contributed to data analysis, literature review, and article writing.

DATA AVAILABILITY STATEMENT

The data produced in the investigation will be made available from the first author upon request considered reasonable by the authors.

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