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Investigating Functional Thinking Processes that Impact on Function Composition Problems in Indonesia

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ABSTRACT

Background: Every student has the ability to think, especially the ability to think when solving mathematical problems. The teacher must explore this ability to determine student understanding of the material being taught. Functions are essential because they are the basis for understanding algebra. The way of thinking about function is called functional thinking. **Objective**: This study aims to investigate the functional thinking process of students in solving mathematical problems based on the APOS theory. **Design**: This type of research is qualitative through an exploratory, descriptive approach. **Setting and participants**: Two out of 44 university students that can communicate fluently when working on questions using the think-aloud and interview methods. **Data analysis**: Analysis of students' functional thinking processes using the triangulation method, namely comparing think-aloud data, student answer sheets, and interview results. **Results**: This study found two ways of student functional thinking processes, namely semi-compositional functional thinking processes and compositional functional thinking processes, where students can generalise the relationship between quantity variations in the form of a composition function. **Conclusion**: This study investigates the functional thinking process of students in exploring the understanding of the concept of function so that students are expected to be able to represent and generalise function forms.

Keywords: Functional thinking, APOS theory, Mathematics Education, Problem-solving.

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Investigando processos de pensamento funcional que afetam problemas de composição de funções na Indonésia

RESUMO

Antecedentes: Cada aluno tem a capacidade de pensar, especialmente a capacidade de pensar na resolução de problemas matemáticos. Essa habilidade precisa ser explorada pelo professor para descobrir até que ponto o aluno compreende o material que está sendo ensinado. Funções são materiais importantes porque são a base para a compreensão da álgebra. A maneira de pensar sobre a função é o pensamento funcional. **Objetivo**: Este estudo tem como objetivo investigar o Processo de Pensamento Funcional de alunos na resolução de problemas matemáticos com base na Teoria APOS. **Desenho**: Esta pesquisa é qualitativa, de abordagem descritiva exploratória. **Sujeitos da pesquisa**: Dois de 44 estudantes universitários, que são capazes de se comunicar fluentemente ao trabalhar em perguntas usando os métodos de pensar em voz alta e de entrevista. **Análise de dados**: Análise dos processos de raciocínio funcional dos alunos usando o método de triangulação, ou seja, comparando os dados do *think-aloud*, as folhas de respostas dos alunos e os resultados das entrevistas. **Resultados**: Este estudo encontrou duas formas de processos de pensamento funcional dos alunos, a saber, processos de pensamento funcional semicomposicional e processos de pensamento funcional composicional, onde os alunos podem generalizar a relação entre variações de quantidade na forma de uma função de composição. **Conclusão**: Este estudo investiga o processo de pensamento funcional dos alunos ao explorar a compreensão do conceito de função, de modo que se espera que os alunos sejam capazes de representar e generalizar formas funcionais.

Palavras-chave: Pensamento Funcional, Teoria APOS, Educação Matemática, Resolução de Problemas.

INTRODUCTION

A function is a mathematics topic taught at almost all levels of education to junior high school, senior high school, and university students. The comprehension of function can provide a basis for the students to succeed in more complex subjects in mathematics, such as calculus and algebra. Knowledge of the concept of function is essential to support students' achievement in studying calculus, advanced mathematics, or science (Subanji, 2011). According to Chazan (Warren et al., 2006), the concept of function constitutes a fundamental relationship and transformation associated with how particular quantities correlate. A function is represented or expressed in terms of the relationship between the first quantity and the second quantity. In other words, functions are mathematical statements that describe how two (or more) variant quantities are correlated with one another (Tanişli, 2011). For instance, there are a large number of square tables and many people sitting around the tables. The correspondence relationship between many square tables and many people is known as function (Blanton et al., 2015).

Chazan (Warren et al., 2006) states that function is not a concept students understand easily. Most students have difficulty representing and interpreting functions. Tanişli (2011) also says that many students experience misunderstandings about functions and difficulties representing the use of algebraic notation, where most students find it challenging to solve the general forms $y = 2x - a$ and $y = 3x - a$. Carlson et al. (2002) identified students' difficulties understanding functions, among others; 1) there is no emphasis on understanding functions as a form of input and output; 2) students view functions as two expressions separated by an equal sign (=); 3) students assume that all functions can be defined by a single algebraic formula; 4) students often find it hard to accept different forms of the same function; 5) students tend to think of functions only as linear and quadratic forms; and 6) students cannot easily distinguish between the visual attributes of the physical situation and the visual attributes of the graph of a function which is a model of the situation. With the problems mentioned above, an analysis is necessary to determine how students think when solving problems about function. A related way of thinking about function is functional thinking. Functional thinking should be introduced from an early age. This situation is supported by NCTM (2001), which states the importance of developing algebraic and functional thinking in earlier grades (pre-kindergarten). Several studies have also shown that early learners (kindergarten to elementary school) can understand functional relationships and begin to think functionally and use algebraic notation (Blanton and Kaput, 2004; Markworth et al., 2010; Warren, 2006; Warren and Cooper, 2005). For example, Blanton and Kaput (2004) stated that kindergarten students could determine covariational relationships and correspondences since grade 1. The results of the following study are that novice students can generalise and provide examples of relationships and functions. Students can also explain the inverse of the relationship and correctly explain how to determine the inverse relationship (Warren, 2005). Likewise, the results of Tanişli's (2011) research show that 5^{th}-grade elementary school students can determine covariation relationships and correspondence when working on linear function tables. Warren et al.'s (2007) research showed that elementary school students are not only able to think functionally but can communicate functional thinking verbally and symbolically.

Functional thinking and growth pattern

Functional thinking is an important aspect of learning mathematics in schools (Stephens et al., 2011; Tanişli, 2011; Warren et al., 2006). Functional thinking is defined as representational thinking focused on the relationship between two (or more) quantity variations (Markworth, 2010). This is in line with what was stated by Blanton et al. (2015), that functional thinking involves generalising the relationship between covariant quantities (covarying), reasoning and representing these relationships through natural language, algebraic notation (symbols), tables and graphs. Blanton and Kaput (2005) also define functional thinking as the relationship between specific quantities called "correspondence". Blanton et al. (2016) give an example of a functional thinking task, "rope cutting: the relationship between the number of pieces of rope and the number of pieces of rope produced", and the type of function that can be formed is $y = x + 1$, where x = number rope cut and y = number of strings produced. Thus, based on the example of the functional thinking task, it can be explained that there is a relationship between the two quantities which are then generalised into the appropriate form of function. Some of the benefits of functional thinking are: 1) it can facilitate students learning about algebra and understanding functions; 2) it can be used as an alternative way of thinking in generalising the relationship between quantity variations; 3) it can be used as the development of students' reasoning abilities; and 4) it can be used as basic competencies to support successful learning of calculus, advanced mathematics, or science (Tanişli, 2011).

Functional thinking processes are mental activities that are in accordance with a functional thinking framework, namely: 1) identifying problems, 2) organising data, 3) determining recursive patterns, 4) covariational relationships, 5) correspondence, and 6) checking generalisation results. This functional thinking framework was adopted from Blanton et al., 2015, Pinto and Cañadas 2012, and Tanişli 2011. In this case, identifying the problem is a mental activity of reading the test sheet, observing and understanding the sequential many tenths, many squares, and many triangles. Organising data is a mental activity in sorting and grouping data by registering or grouping them into tables. This agrees with Blanton et al. (2015), who state that in organising data, it can be described in tables and lists. Defining recursive patterns is a mental activity to determine patterns based on previous values, which follows the opinion of Pinto & Cañadas, 2012, Stephens et al., 2011, and Tanişli, 2011. The authors state that a recursive pattern is looking for patterns of variation in a series of values so that a particular value is obtained based on the previous value. Determining a covariational relationship is a mental activity in the coordination of two quantities related to the change in the value of one quantity against another (for example, when x increases by 1, y increases by 3). This is in accordance with the opinion of Carlson et al. (2004), Stephens et al. (2011), Subanji (2011), Subanji and Supratman (2015), and Tanişli (2011), who affirm that a covariational relationship is a mental activity in coordinating two quantities (independent variables and dependent variables) that are associated with changes in the value of one quantity to another. Determining correspondence is a mental activity that produces general conclusions by changing two quantities (e.g., y is 3 times x plus 2 or y = 3x + 2). Finally, checking the results of generalisations is a mental activity in retracing the entire completion process of the general conclusions obtained. Indicators of functional thinking processes in solving problems can be seen in Table 1.

One of the mental activities that can improve functional thinking is growth patterns. Repeating patterns are a tool that can be used to understand the concept of function (Blanton & Kaput, 2004; Warren, 2004; and Wilkie, 2014). Growth patterns can explore concepts related to functional thinking (Warren et al., 2006). In other words, the use of growth patterns can be used to find functional relationships so that students' functional thinking can be explored. According to Wilkie (2014) the experience of visualising and generalising geometric growth patterns provides students with a new context for developing a conceptual understanding of functional relationships and what they can look like in mathematics (e.g., word descriptions, symbolic equations by representing variables, value tables, and graphs). This provides a great foundation for primary school students to engage effectively in learning algebra.

Literature review

Almost every year, many conduct research on how students think related to the concept of function, including Blanton et al., (2015) who stated that students in the intervention group could identify a covariational relationship between two quantities and were able to use variable notation. Pinto and Cañadas (2012) show that students can distinguish two types of functional relationships: some students can generalise (correspondence), and some can determine covariational relationships. Tanişli (2011) investigated elementary school students' functional thinking through function tables, where students were able to determine recursive patterns on the dependent variable without looking at the independent variable, students were able to determine covariational relationships in linear function tables and work on linear function tables in the form of y = 2x -a and y = 3x-a. Other research findings include designing and developing learning tools in the learning process to improve functional thinking skills (Blanton & Dartmouth, 2005; Doorman et al., 2012; Stephens et al., 2017; Stephens et al., 2017; Warren et al., 2006; Wilkie 2004, 2015; Wilkie & Clarke 2016; and Yuniati et al., 2020). On the other hand, Mceldoon & Rittle-Johnson (2010) designed and developed an assessment of the ability of elementary school students in functional thinking, especially in determining correspondence in linear function tables. Based on the studies mentioned above, no research examines the functional thinking process in solving mathematical problems based on APOS theory.

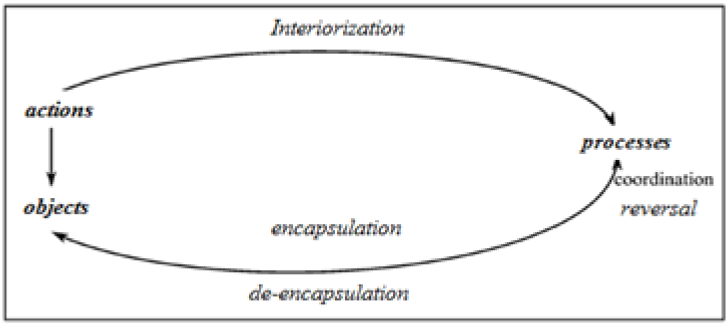
APOS theory

APOS theory emerged to understand the abstraction reflection introduced by Piaget, which explains the development of logical thinking for children. These ideas were then developed for broader mathematical concepts (Dubinsky, 2002). This theory is based on the hypothesis that one's mathematical knowledge is a tendency to overcome situations that are mathematical problems by constructing actions, processes, objects and schemes and organising them in schemes to understand and solve problems (Dubinsky & Michael, 2008). According to Arnon et al., 2014, APOS theory is principally a model to describe how mathematical concepts can be learned; the model is a framework used to explain how individuals build their mental understanding of mathematical concepts. From a cognitive perspective, certain mathematical concepts are framed in genetic decomposition, which describes how concepts can be constructed in an individual's mind. According to Dubinsky (2001) APOS theory is a constructivist theory about how learning mathematical concepts is possible. APOS theory is a theory that can be used as an analytical tool to describe the development of a person's schema on a mathematical topic which is the totality of knowledge related (consciously or unconsciously) to that topic (Dubinsky & McDonald, 2001). Regarding analytical tools, Tall (1999) analysed the role of APOS theory in the reality of learning and mathematical thinking, in particular comparing its role in various contexts of basic mathematical thinking and higher-order mathematical thinking. Based on this description, APOS theory can be used as a tool to analyse mental activities carried out by someone in building knowledge.

APOS theory inspires that all mathematical entities can be represented in mental structures of actions, processes, objects, and schemas, and mental mechanisms consisting of interiorisation, coordination, reversal, encapsulation, de-encapsulation and thematization. The mental structures and mental mechanisms are described in detail in Figure 1. Furthermore, Arnon et al. (2014) explain that an individual's ability to make connections between mental structures and their constituent elements can determine the depth and complexity of their understanding.

Figure 1

Mental Structures and Mental Mechanisms for the Construction of Mathematics Knowledge (Arnon et al., 2014)



METHODS

Research Design

This research is qualitative descriptive exploratory research. This is in accordance with the characteristics of qualitative research proposed by Creswell (2012) as follows: 1) Scientific environment (natural setting). The researcher collects data in the class where the research subjects are solving the problem under study. Researchers do not bring individuals into situations that have been set, 2) Researchers as key instruments. Researchers themselves collect data through audio-visual recordings/documentation, observations, or interviews with subjects, 3) Various sources of data (multiple sources of data). Researchers collect data from various sources, such as recording/documentation, observation, or interviews, then review all the data, giving it meaning, and processing it into categories or topics that cross all data sources, 4) Inductive data analysis. Researchers build categories or topics inductively by processing data into more abstract information units, 5) Emergent design. The research process is always evolving dynamically, which means that the initial research plan cannot be strictly adhered to. All stages in the research process may change after the researcher enters the research location and begins to collect data, and 6) holistic view.

Participants

The participants in this study were 44 students in semester 4 and semester 6, Department of Mathematics Education, Universitas Islam Negeri Suska Riau. The selection of subjects was carried out at the university because, based on preliminary studies conducted by researchers, there were indications that students could carry out functional thinking processes when solving mathematical problems. Details of the subject can be seen in Table 1.

Table 1

Research subject details

Students	Number of students
Semester 4	20
Semester 6	24

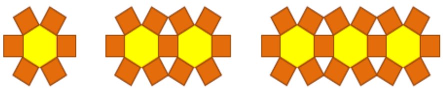
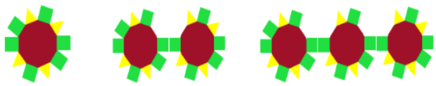
The criteria for research subjects are 1) subjects who have fluent and clear communication skills when solving problems with think aloud and interviews, 2) subjects who can solve problems and meet functional thinking indicators, and 3) subjects who are willing to be involved in the data collection process to obtain accurate data. Thus, students who meet these criteria are two out of 44 students. The two students are distinguished by the symbol S, namely S1 = the first subject and S2 = the second subject.

Data collection

Data collection was carried out in several stages, namely first, holding a written test, where, when doing the written test, students were asked to express their thoughts aloud, i.e., think aloud. The written test aims to overview students' functional thinking processes when solving mathematical problems. Experts then validate the test sheets that have been prepared until the draft is valid for use in research. Second, check the results of student answer sheets. In this case, looking for the correct answer sheet and obtained two different groups of answers. Furthermore, to explore the students' functional thinking processes, interviews were conducted with one student from each group. Interviews were conducted to explore and clarify students' functional thinking processes that have not been revealed in think aloud (this activity is documented with an audio-visual recorder). The questions in the interview guide used are still very likely to develop according to the conditions or characteristics of the respondents. Thus, the interview used is an unstructured interview. Interview guidelines that have been prepared are then validated by experts. Based on the results of the expert validation test, the interview guide is valid to use. In addition, field notes were made on interesting and unique important events related to students' functional thinking processes in solving mathematical problems. The test sheet instrument in this study is the development of a growth pattern task sheet from Wilkie (2014). The differences are presented in Table 2.

Table 2

Development of Research Instruments

Task Sheet Instrument (Wilkie, 2014)	Test Sheet Instruments in this Research
Use two quantities  1st flower 2st flower 3st flower	Use three quantities  1st image 2st image 3st image

Data analysis

The data analysis in this study was modified from Creswell (2012), namely: first, preparing the data for analysis. At this stage, the activities carried out are transcribing think-aloud data and interviews, scanning student answers, and compiling the data into certain types based on the characteristics of the data, and reducing data, namely explaining, choosing the main things, focusing on things what is important, discarding unnecessary and organising raw data obtained from the field. Data reduction is intended to select, focus, abstract and formulate raw data. Second, read the entire data. Build a general sense of the information obtained and reflect on its overall meaning. At this stage, the activities carried out are writing special notes or general ideas about the data obtained. Third, analyse the data in more detail by coding or categorising the data. Coding the data is done to facilitate the interpretation of the data, simplify the problem, and simplify the process of analysing the subject's thinking. The activities carried out at this stage are taking the written data or pictures that have been collected, segmenting the sentences or pictures into categories, and then labelling these categories with special terms. Fourth, draw the structure of students' functional thinking in solving mathematical problems based on data categorisation. Fifth, drawing conclusions is based on the results of data analysis, both those obtained by using test sheets with think aloud and those obtained from interviews.

RESULTS

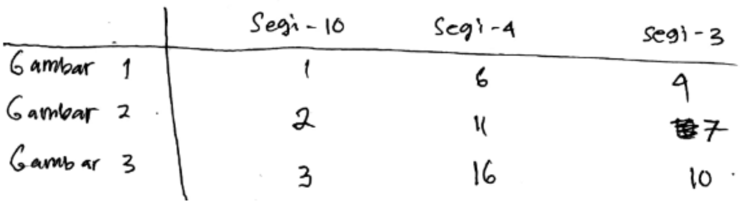
Of the two groups with different answers, one student was in the first group, and four students were in the second group. To find out how these students explore functional thinking processes, here are the results of the descriptions of the two students, called here S1 and S2.

Functional thinking process in solving problems (S1)

The initial activity carried out by the S1 subject identifies the problem, i.e., observing and understanding Figure 1, Figure 2, and Figure 3, aiming to observe many flat shapes from each figure. Then, S1 organised the data into a table and grouped them based on the number of two-dimensional figures found in Figure 1, Figure 2, and Figure 3. The participant's data organisation can be seen in Figure 2.

Figure 2

Data Organisation by S1



	Segi-10	Segi-4	Segi-3
Gambar 1	1	6	9
Gambar 2	2	11	7
Gambar 3	3	16	10

Figure 3

Recursive Patterns Predicted by S1

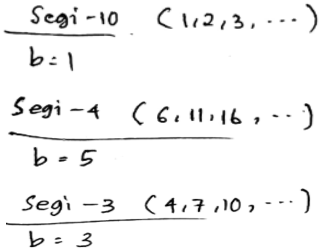
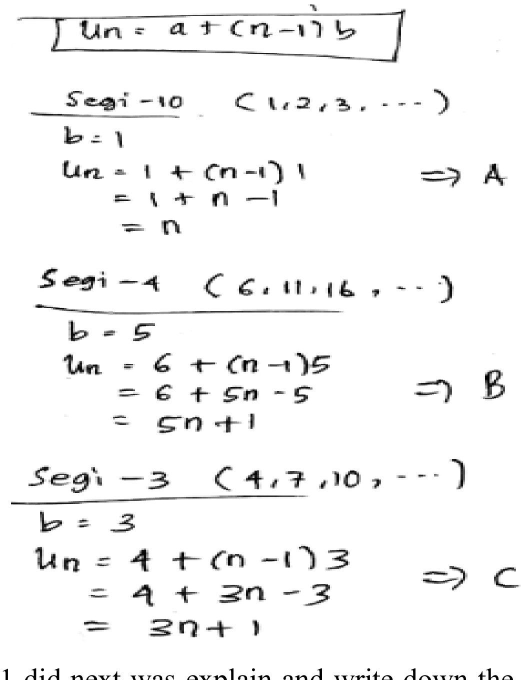


Figure 4

Generalisations by S1



What S1 did next was explain and write down the number pattern of the decagons $(1, 2, 3, \ldots)$, the quadrilaterals $(6, 11, 16, \ldots)$, and the triangles $(4, 7, 10, \ldots)$. The participant then searched for the common difference using the following formula $U_2 - U_1$ (the next term is subtracted by the previous term). S1 found that the decagons had the common difference of $b = 1$, the triangles had the common difference of $b = 3$, while the quadrilaterals had the common difference of $b = 5$. S1 termed "pattern" as "difference", symbolised by "b". By doing so, it is possible to observe that S1 had recursive patterns in mind. It was also supported by the recursive patterns predicted by S1, such as presented in Figure 3.

In the next step, S1 indirectly determines the change in value between the location of an item and the item itself. After that, S1 applied the arithmetic formula $U_n = a + (n-1)b$ to determine the n^{th} term; $U_n = 3n + 1$ for triangles; $U_n = 5n + 1$ for quadrilaterals; and $U_n = n$ for decagons. Therefore, we can say that S1 could make generalisations about the relationship between quantity variations (correspondence). S1's generalisations to show the relationship between quantity variations can be seen in Figure 4.

Figure 5

The Relationship between Two Quantities According to S1

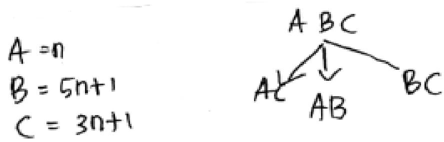
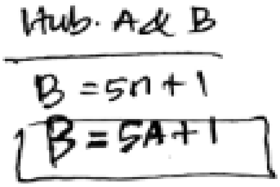


Figure 6

The Relationship between A and B According to S1



Then, S1 generalised the relationship between two quantities by writing pentagons, quadrilaterals and triangles as A, B and C and generating the following formulas: $A = n$; $B = 5x + 1$; $C = 3n + 1$ for pentagons, quadrilaterals and triangles, respectively. He further looked for the relationship between A and C, A and B, and B and C. This finding was confirmed by Figure 5, which shows S1's attempt to determine the relationship between two quantities.

The participant obtained the general formula of AB relationship by substitution. He thus generated a new formula $B = 5A + 1$. Furthermore, the relationship between A and B is depicted in Figure 6.

To find the relationship between A and C, S1 substituted A into the formula $C = 3n + 1$ and generated $C = 3A + 1$. Figure 7 below shows the relationship between A and C generated by S1.

Figure 7

The Relationship between A and C According to S1

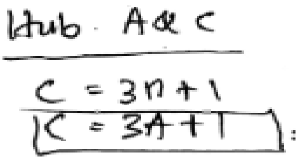
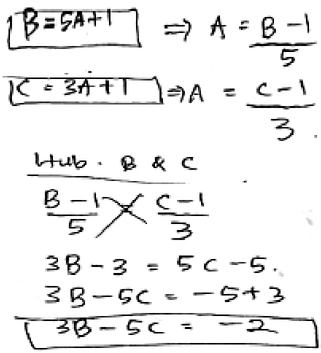


Figure 8

The Relationship between B and C According to S1

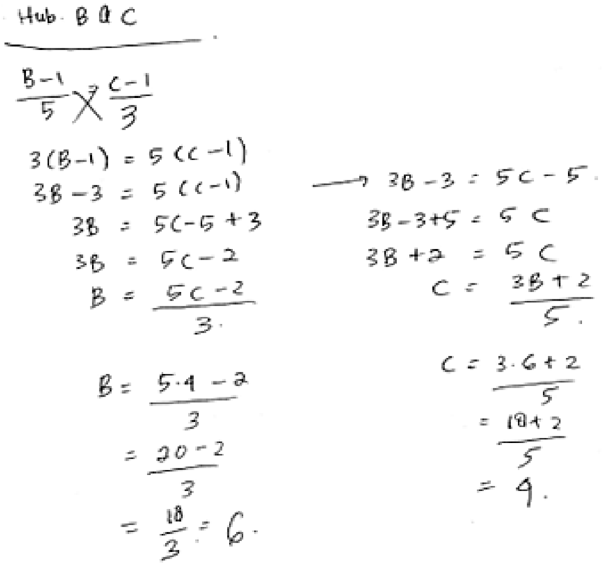


Finally, S1 explained the relationship between B and C by substituting $B = 5A + 1$ with $A = \frac{B-1}{5}$ and $C = 3A + 1$ to generate$= \frac{C-1}{5}$. The participant demonstrated an effort to figure out the general formula of BC relationship but failed to do so. Figure 8 presents the relationship between B and C suggested by S1.

After that, S1 examined the generalisation he made regarding the relationship between B and C by working on the existing formulas to generate $B = \frac{5C-2}{3}$ and $C = \frac{3B+2}{5}$. This finding was confirmed by Figure 9, depicting S1's effort to re-examine the relationship between B and C.

Figure 9

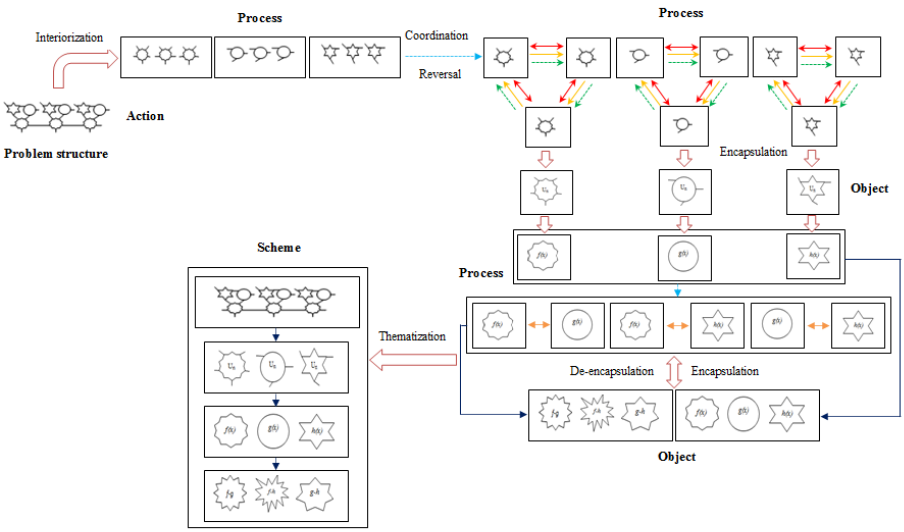
The S1's Effort to Re-examine the Relationship between B and C



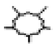
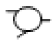
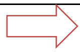
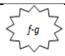
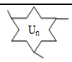
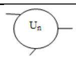
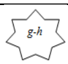
The following is the S1 functional thinking process presented in Figure 10.

Figure 10

Functional Thinking Process S1



Explanation of Symbols

	Decagon		$g(x) = 3x + 1$
	Triangle		$h(x) = 5x + 1$
	Quadrilateral	↔	relationship between quantities (applies otherwise)
↔	Recursive Pattern		Move to another mental structure
→	Covariational relationship		Move to another mental structure (applies otherwise)
⇢	$U_n = a + (n-1)b$		$g(x) = 3f(x) + 1$ or $f(x) = \frac{g(x)-1}{3}$
	$U_n = n$		$h(x) = 5f(x) + 1$ or $f(x) = \frac{h(x)-1}{5}$
	$U_n = 3n + 1$		$g(x) = \frac{3h(x)+2}{5}$ or $h(x) = \frac{5g(x)-2}{3}$

U_n	$U_n = 5n + 1$	→	The resulting process
$f(x)$	$f(x) = x$	⇢	The next step

Functional thinking process in solving problems (S2)

The initial activity carried out by the S2 identifies the problem, namely observing and understanding Figure 1, Figure 2, and Figure 3, aiming to observe many flat shapes from each image. Furthermore, S2 assumes a tenth as x, a triangle as y, and a quadrilateral as z. Based on the observations made, S2 organised the data by making lists and grouping many flat shapes in Figure 1, Figure 2, and Figure 3. S2's work organising the data is presented in Figure 11.

Figure 11

Work Results of S2 Organising Data

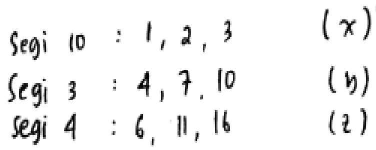
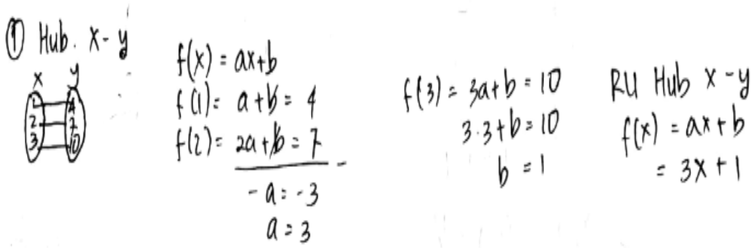


Figure 12

Generalisation of the Relationship between x and y by S2

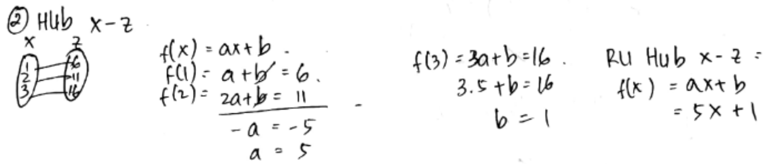


S2 used a Venn diagram to describe the relationship between variant quantities. S2 associated x with y and used the formula $f(x) = ax + b$ to generalize the relationship between the two variables (*x* and *y);* hence $f(1) = a + b = 4$, $f(2) = 2a + b = 7$, by subtracting $f(2)$ by $f(1)$, $a = 3$ was obtained. The value of $a = 3$ was substituted to$f(3) = 3a + b = 10$, hence $f(3) = 3.3 + b = 10$, and $b = 1$. Therefore, the relationship between *x* and *y* is $f(x) = 3x + 1$. This finding corroborated S2's answer in generalising the relationship between *x* and *y*, as presented in Figure 12.

S2 used a Venn Diagram to describe the relationship between *x* and *z*. The standard formula of $f(x) = ax + b$ was used to obtain $f(1) = a + b = 6$, $f(2) = 2a + b = 11$. The value of $f(2)$ was subtracted by $f(1)$; hence, $a = 5$ was obtained and substituted into $f(3) = 3a + b = 16$, Since $f(3) = 3.5 + b = 16$, $b = 1$. The relationship between *x* and *z* thus can be written as $f(x) = 5x + 1$. This finding corroborated S2's answer in generalising the relationship between *x* and z, as presented in Figure 13.

Figure 13

Generalisation of the Relationship between x and z by S2



Again, S2 drew a Venn Diagram to explain the relationship between *y* and *x*. They firstly mentioned the general formula of $f(y) = ay + b$, and concluded that $f(4) = 4a + b = 1$, $f(7) = 7a + b = 2$. They added that if $f(7)$ was subtracted by $f(4)$, then $a = \frac{1}{3}$, if $a = \frac{1}{3}$ was substituted into $f(10) = 10a + b = 3$, then $f(10) = 10.\frac{1}{3} + b = 3$, and $b = -\frac{1}{3}$. In conclusion, the relationship between *y* and *x* can be written as $f(y) = \frac{1}{3}y - \frac{1}{3}$.

This finding was confirmed by S2's answer in generalising the relationship between *y* and *x*, as presented in Figure 14.

Figure 14

Generalisation of the Relationship between y and x by S2

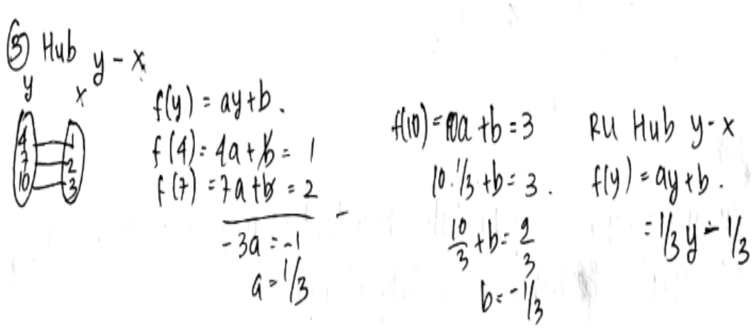
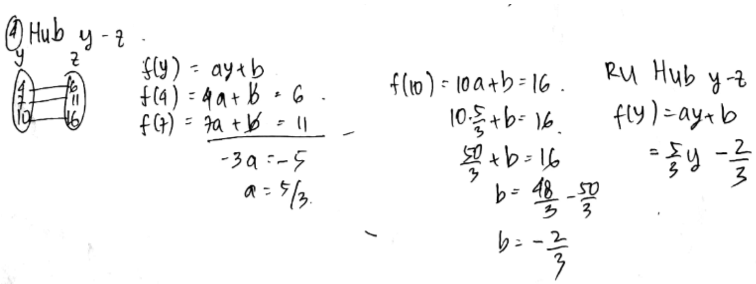


Figure 15

Generalisation of the Relationship between y and z by S2



S2 also drew a Diagram Venn to describe the relationship between *y* and *z*. They used the standard formula of $f(y) = ay + b$, $f(4) = 4a + b = 6$, $f(7) = 7a + b = 11$, and subtracted $f(7)$ by $f(4)$ to obtain $a = \frac{5}{3}$. Thus, the relationship between *y* and *z* can be written as $f(y) = \frac{5}{3}y - \frac{2}{3}$. This finding was strengthened by S2's answer in generalising the relationship between *y* and z*,* as presented in Figure 15.

The relationship between *z* and *x* was described by the subjects using a Diagram Venn. The standard formula used for this relationship was $f(z) = az + b$, so that $f(6) = 6a + b = 1$, $f(11) = 11a + b = 2$. The value of $f(11)$ was subtracted by $f(6)$; thus, $a = \frac{1}{5}$. If $a = \frac{1}{5}$ was substituted into

$f(16) = 16a + b = 3$, $f(16) = 16.\frac{1}{5} + b = 3$, $b = -\frac{1}{5}$. Therefore, the relationship between *z* and *x* can be written as $f(z) = \frac{1}{5}z - \frac{1}{5}$. This finding was strengthened by S2's answer in generalising the relationship between *z* and *x,* as presented in Figure 16.

Figure 16

Generalisation of the Relationship between z and x by S2

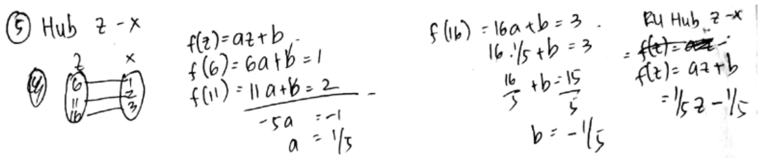
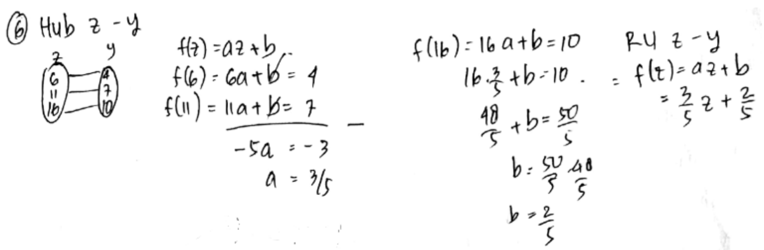


Figure 17

Generalisation of the Relationship between z and y by S2



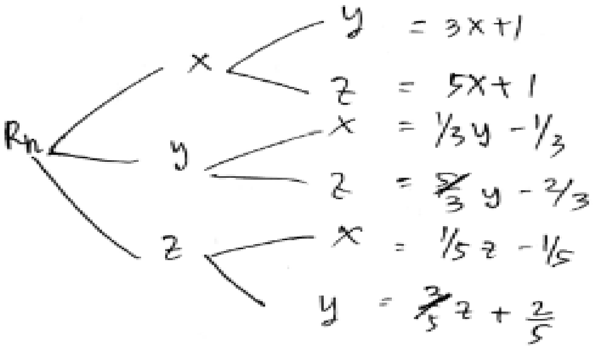
To explain the relationship between *z* and *y,* S1 and S2 used a Venn Diagram and the standard formula $f(z) = az + b$, $f(6) = 6a + b = 4$, $f(11) = 11a + b = 7$; $f(11)$ was subtracted by $f(6)$ to obtain $a = \frac{3}{5}$, that was substituted into the formula so that $f(16) = 16a + b = 10$, $f(16) = 16.\frac{3}{5} + b = 10$, and $b = \frac{2}{5}$. In short, the *z* and *y* relationship was explained as $f(z) = \frac{3}{5}z + \frac{2}{5}$. This finding corroborated S2's answer in generalising the relationship between *z* and *y,* as presented in Figure 17.

Furthermore, based on the relationships between the variant quantities, S2 concluded six standard formulas for: 1) the *x* and *y* relationship, that is $y =$

$3x+1$, 2) the x and z relationship, that is $z = 5x + 1$, 3) the y and x relationship, that is $x = \frac{1}{3}y - \frac{1}{3}$, 4) the y and z relationship, that is $z = \frac{5}{3}y - \frac{2}{3}$, 5) the z and x relationship, that is $x = \frac{1}{5}z - \frac{1}{5}$, and 6) the z and y relationship, that is $y = \frac{3}{5}z + \frac{2}{5}$. Following is the result of S2's work in drawing a conclusion on the relationship between *x, y* and *z* (Figure 18).

Figure 18

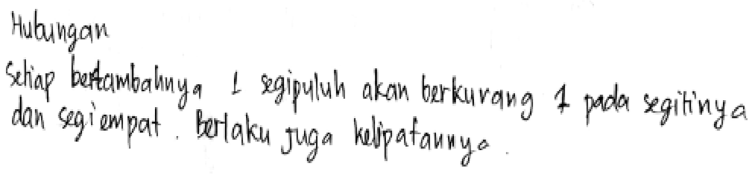
The Relationship Between x, y and z by S2



Finally, the relationship between *x, y* and *z* was verbally expressed as follows: the number of triangles and the number of quadrilaterals will decrease by 1 when the number of decagons increases by 1. The same rule also applies to the multiplication of the two-dimensional figures. The generalisation of the relationship between *x, y* and *z* can be seen in Figure 19.

Figure 19

Generalisation of the Relationship between x, y and z by S2

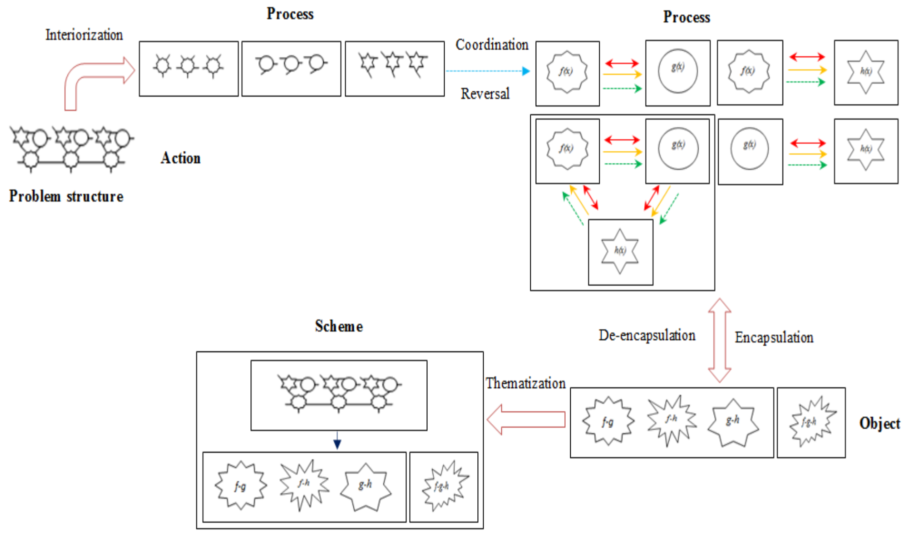


(The relationship is every increase in 1 tenth, it will decrease by 1 triangle and quadrilateral, also applies to multiples)

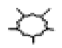
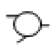
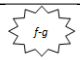
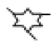
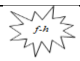
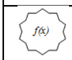
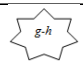
The following is S2's functional thinking process presented in Figure 20.

Figure 20

Functional Thinking Process S2



Explanation of Symbols

	Decagon	→	Generated process
	Triangle		$g(x) = 3f(x) + 1$ or $f(x) = \frac{g(x)-1}{3}$
	Quadrilateral		$h(x) = 5f(x) + 1$ or $f(x) = \frac{h(x)-1}{5}$
	$f(x) = x$		$g(x) = \frac{3h(x)+2}{5}$ or $h(x) = \frac{5g(x)-2}{3}$

h(x)	$g(x) = 3x + 1$	f-g-h	Generalisation of the relationship between $f(x)$, $g(x)$, and $h(x)$
g(x)	$h(x) = 5x + 1$	⇨	Switch to another mental structure
↔	Recursive patterning	⇢	$f(x) = ax + b$
→	Covariational relationship between two or more variant quantities	⇕	Switch to another mental structure and vice versa
⇢	Next step		

The functional thinking process is a student's mental activity that is in accordance with the functional thinking framework. The functional thinking framework in this study is 1) identifying the problem, 2) organising the data, 3) determining the recursive pattern, 4) determining the covariational relationship, 5) generalising the relationship between quantity variations (correspondence), and 6) checking the generalisation results again. This study found two functional thinking processes of students in solving mathematical problems based on APOS theory. The two functional thinking processes are called semi-compositional functional thinking processes and compositional functional thinking processes. The semi-compositional functional thinking process is a process where mental activity in generalising the relationship between quantity variations in the form of compositional functions is carried out partially on a given quantity variation. The compositional functional thinking process is a functional thinking process in which mental activity is generalising the relationship between quantity variations in the form of compositional functions. The following functional thinking processes are analysed based on the APOS theory:

Functional Thinking Process at the Action Stage

All subjects in the semi-compositional and compositional functional thinking process categories took the same initial step, reading all the information on the test sheet. Next, the subject identified the problem by observing and understanding the case. Observing particular cases is one of the activities of the inductive reasoning process in solving problems. This is supported by Cañadas et al. (2007), Ikram et al. (2020), Canadas and Castro (2007), Pinto and Cañadas (2005), Polya (1973), Reid (2002), Yuniati (2018), and Yuniati (2020), who states that observing cases is an activity of an inductive reasoning process that is carried out on certain cases of the proposed problem. Thus, the functional thinking process is an inductive reasoning process.

The subject's activity in identifying the problem, the mental structure that emerges, is called action. This agrees with the opinion of Dubinsky and McDonald (2008), who say that action occurs through physical or mental manipulation involving the transformation of objects that are influenced by external stimuli, which is in the form of cognitive objects that have been previously constructed in the individual's mind through learning experiences. The mental mechanism that arises in this activity is interiorisation. This is in accordance with Dubinsky's (2001). The researcher states that individuals interiorise actions by repeating and reflecting on the action in their mind, so that they can imagine and explain the transformation without having to do it explicitly.

Functional Thinking Process at the Process Stage

In the next step, all subjects counted objects according to the same shape and colour. This agrees with one of the Gestalt laws, the law of similarity, which describes the tendency to perceive the same group of objects as a single unit, whether the objects are the same in terms of shape, colour, or texture. From the grouping, data 1, data 2, and data 3. Then, the subject in the category of compositional functional thinking processes in organizing data by making lists. This is in accordance with Sutarto et al. (2016), who indicate that the strategy used in organising particular cases is by making lists. Meanwhile, in the semi-compositional category, organising data is done through tables. This is in accordance with Blanton et al. (2015), who state that the method used in organising data is described in the table.

In the next activity, subjects in the semi-compositional and compositional functional thinking process categories wrote down data 1, data 2, and data 3 in sequence and formed a number pattern. The number pattern is a recursive pattern obtained inductively using the formula $b = U_n - U_{(n-1)}$, where b= different, U_n= nth term, and $U_{(n-1)}$= the term before n. This agrees with Pinto and Cañadas (2012), Stephens et al. (2011), and Tanişli (2011), who point out that the recursive pattern is looking for variations or patterns of variation in a series of values for variables so that specific values can be obtained based on previous values.

The recursive pattern of data 1, data 2, and data 3 is used as a benchmark by the subject in the semi-compositional functional thinking process category to determine changes in the value of the relationship between variations in quantity (covariational relationship), i.e., changes in value occur between the location of an item and the item itself. This is in accordance with Wilkie (2014), who affirms that a covariational relationship in a number sequence occurs between the location of an item and the item itself. Whereas in the subject of compositional functional thought processes category, the recursive pattern is used as a benchmark to determine changes in the value of the relationship between quantity variations, i.e., changes in the value of two (or more) quantity variations (independent variable and dependent variable). This is in accordance with the opinion of Carlson et al. (2004), Blanton and Kaput (2005), Subanji (2011), Subanji and Supratman (2015), and Tanişli (2011), who indicate that the covariational relationship is a mental activity in coordinating two quantities (independent variable and dependent variable) related to changes in the value of one quantity to another quantity.

The mental structure in this activity is the process, while the mental mechanisms that arise are coordination and reversal. According to Dubinsky et al. (2005), coordination is a mental mechanism for coordinating actions that have been interiorised. Coordination is used to construct new processes. Two or more processes can be coordinated to form a new process. Reversal is an activity to trace back knowledge that has been previously owned to construct a new concept.

Functional Thinking Process at the Object Stage

In the next activity, subjects in the categories of semi-compositional and compositional functional thinking usually generalise the relationship between quantity variations (correspondence) by using algebraic representations. This is in accordance with the research findings of Yuniati et al. (2019) and Cabral et al. (2021), who state that students, when generalising relationships between quantities (correspondence), mostly use algebraic representations. Algebraic representation is the most dominant representation used by students because the learning experience in the teacher's class uses algebraic formulas. This is different from the results of research from Goldin (2002), Lannin et al. (2006), MacGregor and Stacey (1995), Blanton et al. (2015), Swafford and Langrall (2000), and Tanişli (2011), who found the results of generalising the relationship between variations in quantity using verbal representations. On the other hand, the results of research from Tanişli (2011) and Blanton et al. (2015) also found the results of generalising the relationship between variations in quantity using symbolic representations. The difference is that the subjects used in the previous study were elementary school students, while the subjects used in this study were university students.

Subjects in the semi-compositional functional thought process category generalise the relationship between quantity variations separately, i.e., generalising data 1, generalising data 2, and generalising data 3 using the formula from the arithmetic sequence, i.e., $U_n = a + (n-1)b$. However, the subject realised that the generalisation of data 1, data 2, and data 3 was in the form of a function, then connected to obtain a new form of function, namely a composition function. Thus, the semi-compositional functional thinking process is a mental activity in generalising the relationship between quantity variations in the form of compositional functions that are carried out partially on a given quantity variation. Meanwhile, in the category of compositional functional thinking processes, the subject realised that data 1, data 2, and data 3 were a function. Then these functions are connected to produce a new function, namely the composition function. Thus, the process of functional compositional thinking is a mental activity in generalising the relationship between quantity variations in the form of a compositional function.

The mental structure that appears in this activity is the object, while the mental mechanism that arises is encapsulation and de-encapsulation. According to Dubinsky and McDonald (2001), an individual is said to have encapsulated the mental structure of the process into an object if he/she is aware of the process as a totality, realising that actions can be taken on the process. Arnon et al. (2004) explained that not only one object can be de-encapsulated, but two objects can be de-encapsulated into their constituent processes. The two processes are coordinated and re-encapsulated as a new object.

Functional Thinking Process at the Schematic Stage

In the last activity, subjects in the semi-compositional and compositional functional thinking process categories re-checked the generalisation results of the relationship between quantities and believed that the resulting formula was correct. The mental structure that appears in this activity is called a schema. A schema is a collection of mental structures of actions, processes, objects, and other schemas combined to form the totality of students in understanding a concept being studied (Dubinsky & McDonald, 2008; Dubinsky & McDonald, 2001). Characteristics of students' functional thinking processes in solving mathematical problems based on APOS theory can be seen in Table 4.

CONCLUSION

Based on the research questions, analysis results, and discussion, we can conclude that the functional thinking processes of students in solving mathematical problems based on the APOS theory are as follows: in the first step, students identify problems by observing and understanding objects separately according to the shapes and colours shown. In this activity, the mental structure that appears is action, while the mental mechanism that appears is interiorisation. In the second step, students organise data 1, data 2, and data 3 by making lists or tables. In the third step, students determine the recursive pattern inductively by using the formula $b = U_n - U_{(n-1)}$, where b= different, $U_n =$ nth term, and $U_{(n-1)} =$ the term before n. In the fourth step, students determine covariational relationships, i.e., students look for changes in the value between the location of an item and the item itself and changes in the value of two (or more) variations in quantity (independent variable and dependent variable). The mental structure in these activities is the process, while the mental mechanisms that arise are coordination and reversal. In the fifth step, students generalise the relationship between variations in quantity (correspondence). In generalising the relationship between quantity variations, the students do two different things, 1) generalising the relationship between quantity variations in the form of a composition function which is carried out partially on a given quantity variation and 2) generalising the relationship between quantity variations in the form of a composition function. In this activity, the mental structure that appears is the object, while the mental mechanisms that arise are reversal, encapsulation, and de-encapsulation. In the sixth step, students re-check the results of the generalisation of the relationship between variations in quantity and believe that the resulting formula is correct. In this activity, the mental structure that appears is a schema, while the mental mechanism that appears is thematization.

AUTHORS' CONTRIBUTIONS STATEMENTS

SY, TN, and IMS developed research ideas, research theories, and instruments. SS was in charge of collecting data while SY was also in charge of processing and analysing the data. All authors were active in discussing the results, reviewing, and approving the final version of the work.

DATA AVAILABILITY STATEMENT

All data of this study are available from the first author, SY.

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APENDIX

Table 1

Functional thinking process indicators in solving problems based on APOS theory

Functional Thinking Framework	Indicator	Alleged Answer	Mental Mechanism	Mental Structure	Information
Identify the problem	Identify all the information on the test sheet, such as: ➢ Read the test sheet given. ➢ Observe and understand the many tenths in Figure 1, Figure 2, and Figure 3. ➢ Observe and understand the many triangles in Figure 1, Figure 2, and Figure 3. ➢ Observe and understand the many quadrilaterals in Figure 1, Figure 2, and Figure 3.	➢ Many tenths in the 1st picture, 2nd picture and 3rd picture. ➢ Many triangles in the 1st picture, 2nd picture and 3rd picture. ➢ Many quadrilaterals in the 1st picture, the 2nd picture and the 3rd picture.	Interiorisation	Action	P_n = natural number X_n = many Tenth Y_n = many triangle Z_n = many quadrilaterals
Organising data	Organising submitted data such as: ➢ Make a list or table to organise data from the 1st figure, 2nd figure, and 3rd figure. ➢ Organising the many tenths of Figure 1, Figure 2, and Figure 3. ➢ Organising many triangles from Figure 1, Figure 2, and Figure 3. ➢ Organising many quadrilaterals from Figure 1, Figure 2, and Figure 3.	➢ Many tenths (X_n) is 1, 2, 3 ➢ Many triangles (Y_n) is 4, 7, 10 ➢ Many quadrilaterals (Z_n) is 6, 11, 16	Coordination	Process	
Define a recursive pattern	Observing particular objects in the form of a list/table and thinking about the next unknown object, such as: ➢ Determine the number pattern of the tenth, triangular, and quadrilateral objects and think about the number pattern up to the nth.	➢ $X_n = 1, 2, 3, \dots$ ➢ $Y_n = 4, 7, 10, \dots$ ➢ $Z_n = 6, 11, 16, \dots$	Coordination reversal	Process	
	➢ Determine the difference between the object of a tenth, triangle, and quadrilateral.	➢ Different from X_n is 1 ➢ Different from Y_n is 3 ➢ Different from Z_n is 5	Coordination reversal	Process	

Determine the covariational relationship	Determine the change in the value of the relationship between quantity variations in a given problem, such as: ➢ Sequences X_n ➢ Sequences Y_n ➢ Sequences Z_n ➢ Connecting between X_n and Y_n ➢ Connecting between X_n and Z_n ➢ Connecting between Y_n and X_n ➢ Connecting between Y_n and Z_n ➢ Connecting between Z_n and X_n ➢ Connecting between Z_n and Y_n ➢ Connecting between X_n, Y_n and Z_n	➢ When X_n increase 1, P_n increase 1 ➢ When Y_n increase 3, P_n increase 1 ➢ When Z_n increase 5, P_n increase 1 ➢ When X_n increase 1, Y_n increase 3 ➢ When X_n increase 1, Z_n increase 5 ➢ When Y_n increase 3, X_n increase 1 ➢ When Y_n increase 3, Z_n increase 5 ➢ When Z_n increase 5, X_n increase 1 ➢ When Z_n increase 5, Y_n increase 3 ➢ When X_n increase 1, Y_n increase 3, Z_n increase 5	Coordination	Process
Determine correspondence	Generalise the relationship between quantity variations on a given problem, such as: ➢ Generalising sequences X_n ➢ Generalising sequences Y_n ➢ Generalising sequences Z_n ➢ Generalise the relationship between X_n and Y_n ➢ Generalise the relationship between X_n and Z_n ➢ Generalise the relationship between Y_n and X_n ➢ Generalise the relationship between Y_n and Z_n ➢ Generalise the relationship between Z_n and X_n ➢ Generalise the relationship between Z_n and Y_n ➢ Generalise the relationship between X_n, Y_n and Z_n	➢ $X_n = P_n$ ➢ $Y_n = 3P_n + 1$ ➢ $Z_n = 5P_n + 1$ ➢ $Y_n = 3X_n + 1$ ➢ $Z_n = 5X_n + 1$ ➢ $X_n = \frac{Y_n - 1}{3}$ ➢ $Z_n = \frac{5Y_n - 2}{3}$ ➢ $X_n = \frac{Z_n - 1}{5}$ ➢ $Y_n = \frac{3Z_n + 2}{5}$ ➢ $Y_n + Z_n = (3X_n + 1) + (5X_n + 1)$	Reversal Encapsulation De-encapsulation	Object
Re-check the generalisation results	Check the truth of the generalisation results based on certain cases.	➢ Verbal representation	Thematisation	Scheme

Table 4

Characteristics of students' functional thinking processes in solving mathematical problems based on APOS Theory

Functional Thinking Framework	Functional Thinking Process		Mental Mechanism	Mental Structure
	Semi Composition	Compositional		
Identify the problem	➢ Read the given test sheet ➢ Observe and understand Figure 1, Figure 2, and Figure 3	➢ Read the given test sheet ➢ Observe and understand Figure 1, Figure 2, and Figure 3	Interiorization	Action
Organising data	Create tables and group multiple objects on each image	List and group multiple objects on each image		
Define a recursive pattern	➢ Determine the number sequence of each object ➢ Using formulas $b = U_n - U_{(n-1)}$ ➢ Representing algebraically	➢ Determine the number sequence of each object ➢ Using formulas $b = U_n - U_{(n-1)}$ ➢ Representing verbally	Coordination	Process
Determine the covariational relationship	➢ Determining the change in value from the relationship between variations in quantity in a number series, i.e., determining the change in value based on the location of an item with the item itself ➢ Verbal representation	➢ Determine the change in value between 2 quantities (independent variable and dependent variable) ➢ Determine the change in value between 3 quantities (independent variable and dependent variable) ➢ Representing verbally	reversal	
Determine correspondence	➢ Using formulas $U_n = a + (n-1)b$ ➢ **Generalise the relationship between quantity variations in the form of a composition function that is carried out partially on a given quantity variation** ➢ Representing algebraically	➢ Using function formulas $f(x) = ax + b$ and Venn Diagram (connecting 2 quantities) ➢ **Generalise the relationship between quantity variations in terms of compositional functions** ➢ Representing algebraically (connecting 2 quantities) ➢ Representing verbally (connecting 3 quantities)	Reversal Encapsulation De-encapsulation	Object
Re-check the generalisation results	Representing verbally	Representing verbally	Thematisation	Scheme