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Evidence of Professional Knowledge of a 4th-grade Mathematics Teacher from the Perspective of a Research Project Involving Lesson Study

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ABSTRACT

Context: Lesson Study has been used in several countries in the West and East with adaptations to the possibilities of local education systems and cultures. **Objectives:** The article discusses and analyses the knowledge to teach mathematics evidenced by a teacher who taught in the fourth grade of elementary school in the municipal education network of São Paulo and who participated in a research project using the lesson study training methodology. **Design:** The study is situated in qualitative research, with an interpretive nature. **Setting and participants:** It involves teachers from the interdisciplinary cycle who teach mathematics in the municipal education network of São Paulo, especially a teacher who worked in the fourth grade of elementary school. The training with the teachers was carried out fortnightly at the Cruzeiro do Sul University. **Data collection and analysis:** The analysis was based on empirical material consisting of observational protocols, training audios, photographs of students' curriculum materials, and transcripts of video recordings during class observation. **Results:** The research results show that the teacher's gaps in her knowledge to teach mathematics were shown in the class observation stage, even without appearing in the planning, and that the discussion in the reflection stage allowed the teacher to progress in her practice. **Conclusions:** With the training actions, the teacher's didactic and curricular knowledge was produced from discussions with the group, based on theoretical contributions that dealt with the mathematics teaching and learning process.

Keywords: Mathematical knowledge for teaching; Lesson study; Continuing education of teachers who teach mathematics.

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Evidências de conhecimentos profissionais de uma professora que ensina matemática no 4º ano na perspectiva de um projeto de pesquisa envolvendo *lesson study*

RESUMO

Contexto: Os estudos de aula (*lesson study*) vêm sendo utilizados em vários países do ocidente e do oriente com adaptações às possibilidades dos sistemas de ensino e culturas locais. **Objetivos**: o artigo discute e analisa os conhecimentos para ensinar matemática evidenciados por uma professora que lecionava no quarto ano do ensino fundamental na rede municipal de São Paulo e que participou de um projeto de pesquisa usando a metodologia de formação *lesson study*. **Design**: O estudo está situado em uma pesquisa de natureza qualitativa, de cunho interpretativo. **Ambiente e participantes**: Envolve professores do ciclo interdisciplinar que ensinam matemática na rede municipal de São Paulo, em especial uma professora que atuava no quarto ano do ensino fundamental. As formações com os professores foram desenvolvidas quinzenalmente nas dependências da Universidade Cruzeiro do Sul. **Coleta e análise de dados**: A análise baseou-se no material empírico constituído de protocolos observacionais, áudios das formações, fotografias dos materiais curriculares dos estudantes, transcrições das gravações em vídeos durante a observação das aulas. **Resultados**: Os resultados da pesquisa mostram que as lacunas da professora em seus conhecimentos para ensinar matemática mostraram-se na etapa de observação das aulas, mesmo sem surgir no planejamento, e que a discussão na etapa de reflexão permitiu avanços da professora em relação à sua prática. **Conclusões**: Com as ações de formações, os conhecimentos didáticos e curriculares da professora foram sendo produzidos a partir de discussões com o grupo, baseados em aportes teóricos que versavam sobre o processo de ensino e de aprendizagem de matemática.

Palavras-chave: Conhecimentos Matemáticos para o Ensino; Estudos de Aula; Formação Continuada de Professores que ensinam Matemática.

INTRODUCTION

This study is anchored in a doctoral research (Martins, 2020) linked to the research project called “Discussões Curriculares: contribuições de um grupo colaborativo para a implementação de um novo currículo de Matemática e o uso de materiais curriculares na Rede Pública Municipal de São Paulo" [Curricular Discussions: contributions of a collaborative group for the implementation of a new Mathematics curriculum and the use of curricular materials in the Municipal Public Network of São Paulo] developed by the research group "Conhecimentos, Crenças e Práticas de Professores que ensinam Matemática” — CCPPM" [Knowledge, Beliefs and Practices of Teachers who teach Mathematics" - CCPPM, which, in recent years, has been carrying out investigations with teachers from the public education network using the “lesson study” training methodology and curriculum materials produced by education departments.

The research project in question was coordinated by the second author of this article, having as a partnership the triad: the Postgraduate Programme in Science and Mathematics Teaching at Cruzeiro do Sul University, São Paulo Municipal Department of Education, and the United Nations for Education, Science, and Culture (UNESCO). This project sought to promote discussions and reflections about the participants’ understanding of mathematics and its teaching and the knowledge to teach mathematics related to documents and curriculum materials implemented at the time of the research.

Given the above, the present text aims to discuss and analyse the knowledge to teach mathematics evidenced by a teacher who taught in the fourth grade of elementary school in the municipal education network of São Paulo, who participated in a research project with other teachers working in the interdisciplinary cycle in the context of a lesson study training methodology and in a curriculum implementation scenario.

To analyse teachers’ knowledge to teach mathematics, we use the data triangulation strategy and rely on authors who discuss teachers’ knowledge in general, such as Shulman (1986), and those who discuss the knowledge of teachers who teach mathematics, such as Ball et al. (2008). Before portraying the research scenario and the data collection procedures, we present a summary of the studies of the authors cited above.

THEORETICAL REFERENCES

Strands of Knowledge Teachers need to teach Mathematics

Ball, Thames, and Phelps (2008), based on studies by Shulman (1986), discuss the teacher’s knowledge to teach mathematics. According to Ball et al. (2008), understanding the mathematical content to teach is crucial, as this knowledge determines the teacher’s views on teaching. Only in this way can the teacher recognise the students’ strategies and understand their learning, which means that the teacher needs to be inserted into a school environment that embraces the teacher him/herself and the students.

The model proposed by Ball et al. (2008) is organised from the two domains of knowledge conveyed in studies by Shulman (1986) - content-specific knowledge and pedagogical content knowledge. The researchers subdivide the strands of knowledge proposed by Shulman (1986) into six other subdomains, as shown in Figure 1.

Figure 1

Domains of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008)

According to Ball, Tames, and Phelps (2008), the left side of Figure 1 refers to the "content-specific knowledge" domain, which was organised into three subdomains: (1) common content knowledge (2) horizon content knowledge and (3) specialised content knowledge. The right side deals with the domain "pedagogical content knowledge", which was organised into "knowledge of content and students", "knowledge of content and curriculum" and "knowledge of content and teaching".

Ball et al. (2008) discuss each of those domains, which will be briefly presented below. Common content knowledge is not exclusive to teachers, as other professionals also master this type of knowledge. In this sense, a professional who has the knowledge, for example, of the concept of proportionality and uses it to build a bridge does not have the specific content knowledge to teach it. They consider that the teacher must have this common content knowledge "for their use", but for them to teach this knowledge, they must go further. Every teacher must understand when the book presents an improper definition or even when their students give a wrong answer. The central idea is that the teacher understands the nature of mistakes and successes from a mathematical and teaching (pedagogical) perspective.

We agree with the authors because every teacher should master common content knowledge to solve an activity they propose to students. However, this is not enough. They should identify inappropriate responses and even recognise, from a conceptual point of view, inappropriate terminologies students use or that are displayed in curriculum materials, which leads them to have specialised knowledge to teach.

Specialised content knowledge refers to specific knowledge to teach, unique and fundamental for teachers, as it enables professionals to recognise patterns in students' errors and analyse procedures and strategies used by their students.

Ball, Thames, and Phelps (2008) draw attention to the difference between common knowledge and specialised content knowledge. They give an example by discussing the recognition of student errors as common knowledge, and the reflection on the nature of those errors and their meanings as the specialised content knowledge.

In her doctoral thesis, Curi (2004) highlighted that teachers need to "appropriate" some mathematical content for teaching. As one of the results, the researcher showed that when teachers have little knowledge of the mathematical content they must teach, they feel insecure before teaching situations that involve this content, often failing to carry out activities proposed in their planning.

We consider this type of knowledge essential in the teacher's practice, and believe it is related to the common content knowledge. Without common content knowledge, one cannot develop specialised content knowledge.

Horizon content knowledge refers to the knowledge that allows the teacher to recognise and place a concept throughout the mathematics curriculum. This knowledge is linked to the teacher's recognition of the correlations present between the mathematical contents over a year of schooling.

The knowledge of content and students allows a close relationship between the content to teach and the knowledge about students, what they master or not of the content to teach. The perspective is that teachers can anticipate what students think about a given content to teach, predicting what students find interesting, what they can or cannot do easily, and their possible errors, among other aspects.

For Ball, Thames, and Phelps (2008), knowledge of content and teaching allows a relationship between the contents to teach and how to teach them. Often, to teach a given content, teachers use teaching sequences, choose examples or counterexamples that allow students to make approximations with content they already know and define or choose questions or problems that deepen the theme, in addition to methodologies of teaching and appropriate representations.

For the authors, the knowledge of content and curriculum refers to the confluence of knowledge that allows the teacher to understand the organisation, the objectives, and the principles that underlie the curriculum, which enables analysis of curriculum materials to make decisions and use them or not in their teaching proposal. However, it is not enough to focus only on the teacher's knowledge to teach mathematics. Curi (2004) and other researchers argue that teachers' knowledge to teach mathematics is subjectively close to the beliefs and conceptions teachers have about mathematics and its teaching. Those discussions corroborate Ball's (1991, p.1) studies: "the knowledge teachers have of mathematical content interacts with their assumptions and beliefs about teaching and learning, students, and ways to teach them". According to Curi (2004), changes in beliefs can occur when teachers are challenged, when they verify the students' learning in their proposals or even when they broaden their understanding of mathematics and its teaching.

METHODOLOGY

This study is situated in qualitative research with an interpretive nature. According to Creswell (2010), qualitative research allows a different approach from academic research, because it employs philosophical concepts, investigation strategies, different methods of data collection, and data analysis and interpretation. Thus, we corroborate this idea presented by the author that qualitative research is interpretative, because, as researchers, we are mainly involved in a supported and intensive experience with research subjects (teachers).

When thinking about the structure of the data collection process, we resort to triangulation, which, according to Denzin and Lincoln (2006), means employing multiple methodological practices in a single research to guarantee rigour, richness, and complexity to the study. For the authors, triangulation is a safe path that reflects an in-depth understanding of the phenomenon studied.

We understand that there is an undeniable need for an articulation between different instruments and strategies for the production of data to enrich the validity of our research. In this way, we organised the data sources based on the resources explained below.

- **Logbook** - a resource the researcher elaborated during the participant observation of the classes imparted by a teacher of the school involved in the project;
- **Video** – a resource used to record the lessons observed, seeking to capture interactions between the students, the teacher, and the knowledge of the students and the teacher.
- **Audios –** a resource used in the training meetings at the university that hosted the project to analyse the data rigorously.
- **Photographs** — a resource used in significant moments of the training and students' protocols, in addition to teacher's and students' registers on the board.
- **Observational protocol** —a resource developed by the researcher to make careful and detailed notes of the multiple interactions and actions experienced in the context of training and the classroom, based on some elements of analysis arranged in grids — more open grids, giving the freedom to proceed to reflective notes and grids with pre-established questions to be observed in the field of investigation.

As we have already highlighted, this article involves teachers of the interdisciplinary cycle who teach mathematics in the municipal education network of São Paulo, especially a teacher who taught in the fourth grade of elementary school. The group was formed by three teacher educators, including the first author of this article and twenty-three teachers, six of whom taught in the 4th grade. We used the following documents and curriculum materials from the municipal education network of the city of São Paulo:

1. Currículo da Cidade: componente curricular Matemática [City Curriculum: Mathematics Curriculum Component] (São Paulo, 2017)
2. Orientações Didáticas do Currículo da Cidade- Matemática, volume 1 e 2 [Didactic guidelines of the city curriculum - mathematics, volumes 1 and 2] (São Paulo, 2018)

3. Caderno da Cidade Saberes e Aprendizagens- Mathematics, nas versões estudantes e professores [City Notebook Knowings and Learnings- Mathematics, in student and teacher versions] (São Paulo, 2019)

Having clarified the materials and documents used in the study, we must know the profile of the teacher - the focus of our study - we will call P4. Teacher P4 worked in the 4th grade of a municipal elementary school in the East Zone of São Paulo, belonging to the Regional Board of Education of Penha. She completed her teaching degree in 1991, has a degree in pedagogy and letters and, in 2013, attended a specialisation course in Portuguese for elementary and high school teachers. She worked as a teacher for twelve years, six of them as an effective teacher in the municipal education network of São Paulo. Regarding continuing education courses, the teacher mentioned her participation in two held at the Regional Board of Education (DRE-PENHA). The first aimed at the new city curriculum and the second about the revision of the Cadernos da Cidade Saberes e Aprendizagens.

We now move on to the next topic, to present a summary of the training methodology, called lesson study.

The Training Methodology Lesson Study.

Originated in Japan, the methodology began at the end of the 19th century as a collaborative and reflective teacher training process, mediated by experienced researchers, aimed at improving student learning and teachers' professional development. It unfolds into three main stages: the collective and individual class planning, the achievement of the planned classes, and, finally, the reflection on the developed classes, which can lead to a replanning of future classes. This methodology has been used in several countries in the West and East with adaptations to the possibilities of local education systems and cultures. In Brazil, especially in the research projects developed by the CCPPM Group, teachers linked to basic and higher educational levels have always participated spontaneously. Although with different academic and professional qualifications, every teacher becomes responsible for researching on mathematics teaching and learning and on practical teaching situations.

In the course of the training processes, those groups end up becoming collaborative since there is a plurality of characteristics such as voluntariness, identity, and spontaneity (Fiorentini, 2006); trust, dialogue, and negotiation

(Boavida & Ponte, 2002); intellectual and affective support, mutual respect, shared leadership, and co-responsibility (Fiorentini, 2006).

For Martins (2020), collaborative groups in research projects are essential for the success of the lesson study methodology, as it is centred on dialogue, negotiation of meanings, and trust between peers. Such methodology makes the participating teachers feel free to offer their classes for observation of the classes they planned as it is in the observation that the teachers' gaps regarding mathematical and didactic knowledge emerge for curriculum development in action.

Two aspects are characteristic of the lesson study in the CCPPM Group. One is the use of curriculum material in which teachers identify the objectives of activities and plan an activity; the second is the presence of the teacher educator in the schools involved in the research, encouraging the engagement of other teachers and school leaders; and the propagation of research in mathematics education and the use of the lesson study methodology for local training. In this article, we will prioritise three steps in this training methodology — planning, observation, and reflection.

We must note that depending on the project, the CCPPM Research Group focuses on other stages, such as the teacher educators' training and disseminating results.

Before moving on to the next topic, we must clarify that, in this article, we will only discuss one 4th-grade activity that the teachers selected, which was planned, observed, and reflected on collectively. They also discussed two other activities in the *Caderno da Cidade Saberes e Aprendizagens — Matemática*, one for the 5th and one for the 6th grade 6. The activity will be presented and commented on below for readers' further elucidation.

The selected activity

The activity selected was number 1, incorporated in Unit 3 of Sequence 2 and involves the structuring axis "numbers". This activity includes part of a more significant objective: "(EF04M10) To analyse, interpret, formulate, and solve problems with natural numbers, understanding different meanings of the additive field (composition, transformation, comparison, and composition of transformations) and the multiplicative field (proportionality, rectangular configuration, and combinatorics), and validate the adequacy of the results through estimates or digital technologies".

The activity was organised in two situations, with some questions in each of them. The first one is simpler. It involves counting by groups of numbers, in this case by threes, and the meaning of proportionality of the multiplicative field. The objective is to calculate the number of beads of a given number of bracelets based on the number of beads in a bracelet, which involves the meaning of proportionality of the multiplicative field in the simple quaternary relationship "one to many". Students were requested to calculate the number of large beads of 25 bracelets, knowing that each used 12 large beads. They should also calculate the number of small beads in the 25 bracelets based on the number of small beads in each. Then, students were asked to calculate the number of beads in the 25 bracelets.

One of the questions asked students to describe the procedures used to calculate the total of beads, which indicated concern with the explanation of their reasoning and communication in mathematics classes.

As it is possible to perceive, in this first proposal, the concept of proportionality[1] is evident in the two questions presented.

The second problem also focused on the meaning of the proportionality of multiplication. However, the questions were more complex and demanded an explanation of the reasoning. In this problem, the information was that each bracelet used 15 beads, and 120 bracelets were made weekly. The problem was whether, with 68 packages of 100 beads each, it was possible to make the weekly 120 bracelets. The students had to say whether the material was enough and explain their reasoning. They had to calculate and explain how they thought to find out whether the artisan's material was enough to make this type of bracelet and how many weeks she would spend making it. Finally, the students had to calculate how many bead packs were needed to produce this type of bracelet in four weeks. Below, we present the description of the selected activity.

After presenting the activity, we returned to the lesson study methodology, specifying aspects of training that drew our attention during the development of each stage.

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[1] The concept of proportionality refers to the declaration of equality between two relations, i.e., the relationships existing between two proportional variables (Van de Walle, 2009).

Lesson planning

The planning was based on an observational protocol intended for this stage and analysed by the authors of this article (Figure 2) to identify the knowledge that emerged from the teachers when filling out the form.

Figure 2

Planning Protocol.

Year
Structuring Axis
Unit, sequence, and activity
Learning and development objects
Fundamental ideas
Types of reasoning
What students need to know
Possible students' and teachers' doubts
What to do
Time
Class organisation
Evaluation indicators

Although all teachers of the interdisciplinary cycle participated in the planning stage, the analyses will be centred on 4^{th}-grade teachers, with emphasis on P4's planning form, because she allowed the other group participants and researchers to watch her class. In the planning meetings, the teachers were confronted with situations when they could put into play the experiential knowledge of their practice, the studies on the multiplicative conceptual field and the curriculum knowledge acquired in the first part of the course about the constitutive elements of the mathematics city curriculum (São Paulo, 2017), materialised in *Cadernos da Cidade Saberes e Aprendizagens* (São Paulo, 2019). The 4^{th}-grade teachers discussed the selected activity based on studies theoretical studies proposed and were instructed to fill out the planning form. Although the audio and video recordings were from the overall group's discussions, we analysed P4, whose knowledge will be presented in item 5 of this article.

The Class Observation

The second stage of the lesson study to be prioritised in this article is the observation of the classes planned by the teachers and teacher educators. For Reis (2011), observation plays an important role in improving the quality of teaching and learning, being a source of inspiration and an active catalyst for transformation in the educational context.

P4 developed the activity planned for the 4th grade, and the class was observed through some criteria established in the observation form. The class was observed on September 11, 2019, at 3:30 pm at EMEF, attended by the two coordinators of the educational unit and two teacher educators (including the first author of this article). The observations were registered in the first author's instruments (logbook and observational protocols) in her participant observation, besides videos and photos taken for data collection. Those instruments made it possible to evidence P4's knowledge to teach mathematics in the 4th grade, which will be highlighted below.

Before starting the activity, P4 introduced the observers to the students to establish an approach of trust and respect and to prevent students from behaving differently from their usual way. We understand that without creating a climate of trust with the students, the expressive number of observers could jeopardise the teaching and learning processes and the progress of the research we had set out to develop. After the introduction, P4 organised the students into pairs, as previously agreed with the class. After that, she chose a student and asked him to read the first problem in activity 1: "Dona Nina, Tainá's mother, needs to know how many beads she needs to buy to make three types of bracelets. Tainá separates the quantities for her and calculates the number of accounts to be purchased. Help Tainá estimate it". Afterwards, she asked students to tell what the problem was.

The teacher educators realised that the teacher did not follow what had been planned for the beginning of the class, did not follow the initial guidelines about the activity students should develop, did not discuss the mathematical contents incorporated in the activity, or surveyed the students' prior knowledge.

The teacher proposed to her students to make a bracelet. To produce the bracelet, each student received individually 12 large and 36 small beads and one thread.

A fact observed by the teacher educators that deserves to be highlighted is that when giving the parts to the students, the teacher asked them to count the pieces received one by one. With this, we found out that the teacher unconsciously motivated the students to solve the problems using a counting strategy.

In the students' statements, the educators noticed that most of them had already appropriated the multiplicative reasoning in the problems involving the proportional idea and that students understood what that calculation represented in the given situation. However, they also observed that some students found counting difficult. Students' eagerness to finish the production of the bracelet jeopardised the incorporation of the activity learning and development objective: (EF04M10) Analysing, interpreting, and solving problems with natural numbers, understanding different multiplicative field meanings (proportionality).

The educators concluded that the planned strategy of producing a bracelet was not a facilitator, which was reflected in the students' learning. This happened because, after finishing the bracelet, at no time the teacher resumed the purpose of grouping by threes, nor were the students encouraged to calculate how many large beads would be needed to produce 25 bracelets and, then, how many small beads would they would need for this production, according to the statement of items a and b of the activity.

After making the bracelet, the teacher allowed time for the students to solve all the items in activity 1. At that moment, she walked around the room, following the activity and asking questions. Then, the teacher educators pointed out that there was more knowledge exchange between the class and the teacher than among the students.

The Reflection About the Class

In the lesson study methodology, the phase of reflection about the classes involved the collective analysis of the data obtained from the video footage, written registers, student protocols, and photographs. Teachers and educators of the interdisciplinary cycles attended the meetings of this stage. At this stage, the group watched some episodes related to the activity already described in the planning stage. The episodes below, selected by the teacher educators, stemmed from the footage of P4's class (4th-grade) and were submitted to collective reflection. The video recordings of the classes allowed us to identify P4's knowledge, which will be analysed in the item intended for this purpose.

Episode 1: Making the bracelet: The teacher proposed to the students to make a bracelet with the beads.

Episode 2: Zero is worth nothing: During the activity correction, a student said to a colleague: Zero is worth nothing.

Episode 3: The idea of proportionality: When correcting a problem situation, the teacher assertively incorporated the meaning of proportionality involving a simple direct proportion of two variables (number of bracelets and beads).

Episode 4: Register of the conventional multiplication algorithm: to simplify the calculation for the students, the teacher registered some calculations on the blackboard incorrectly. For example, number 18 multiplied by number 3. The correct one would be 1,800 by 3. Therefore, the teacher removed the zeros from the operation, including them in the result.

The first episode, "The Making of the bracelet," allowed the teachers to reflect on the influences of manipulative materials while carrying out the proposed activities. It was possible to perceive a deep-rooted belief in some teachers' conceptions, as they genuinely believe that the bracelet, as a "playful" material, could overcome all the students' learning difficulties. The educators' interventions were aimed at making the group reflect on the intentionality of using this type of material without parting from the object of knowledge present in the activity, which, in this case, involved the meaning of proportionality of multiplication. After this reflection, the group realised that the additive principle was used in counting by ones during the production of the bracelet, which parted from the activity's object of knowledge and the learning objective.

As for the second episode, entitled "Zero is worth nothing", the teachers unanimously agreed that it was necessary to draw students' attention to the meaning of zero in the composition of numbers within the scope of the decimal number system (DNS). When replanning this activity, the teachers proposed revisiting with the students some of the characteristics and regularities of the DNS.

When watching episode 3, “The idea of proportionality”, the teachers disagreed with the educators about including the number 10 to understand the numerical relationships and analyse the relationship between the variables (number of bracelets and beads). However, after discussing approximations, multiplication, and division by 10, they understood the importance of going through 10 bracelets before reaching 20 (double 10).

As for the last episode, “Register of the conventional multiplication algorithm”, seeking to strengthen the relationship with P4 to preserve her from any uncomfortable feelings, the educators chose not to disclose this episode to the group. Instead, they performed specific interventions and gave specific feedback to clarify doubts so that they could contribute to the teacher’s professional development. This stance could help her overcome the gaps in her training to teach mathematics to her students.

Afterwards, the educators showed and discussed the video footage of this episode with P4. In this way, the teacher understood that both the registers made by her students and her register on the board gave indications as to why one of the students said that “zero is worth nothing” and showed a lack of understanding of the meaning of zero in the decimal number system (DNS).

During this conversation, we observed the professional posture of teacher P4, always flexible and concerned with her practice and, above all, with her students’ learning. As a result of the educators’ interventions, the teacher reflected on some attitudes that had major consequences on the teaching and learning processes. One point that caught her attention was her unsuitable register of multiplication on the blackboard, which impacted students’ conception of the meaning of zero in the decimal number system, prevailing the incongruous thought that zero has no value. In this way of thinking, the zero could be inserted and removed any time one deemed necessary.

RESULTS AND ANALYSES

Our analyses were based on our follow-up of teacher P4 and the data presented in the resources used in the research, i.e., the registers written in the logbook, the observational protocols used in the planning, the student protocols and the capture of audio, video recording, and photography. For the construction of this section, we took as a theoretical contribution the model of Ball, Thames, and Phelps (2008), who, based on Shulman’s (1986; 1987) studies, announced some domains and subdomains of mathematical knowledge for teaching. Thus, the analyses elaborated were organised by categories, which, in turn, emerged from the theoretical contributions of this study and that gave us subsidies for this analytical process.

We started with the domain "Specific Content Knowledge", subdivided by Ball, Thames, and Phelps (2008), into three subdomains: (1) "common content knowledge", (2) "horizon content knowledge ", and (3) "specialised content knowledge". In the "common content knowledge" subdomain, in the three stages of the lesson study, the data reveal that P4 showed to master the objects of knowledge of the proposed activity, i.e., the meaning of proportionality of the multiplicative field present in the two problems stated in the activity chosen. She thought that if a quantity increased – in our study, the number of bracelets - the number of beads for the new order should increase in the same proportion. However, as already stated in the theoretical part of the article, this type of knowledge is not enough for the teacher to teach it.

Regarding the "specialised content knowledge", in the first planning meetings, the registers in the logbook indicated that P4 often had some difficulty pointing out the students' prior knowledge to develop the activities. In the group meetings, the teachers commented that the students were undisciplined and, for that reason, did not understand mathematics. However, with the educators' support, the group could move forward and indicate what the students needed to know to develop the proposed activities. They also recognised the evaluation indicators. In the case of the selected activity, P4 identified that the students should have previous knowledge of the interpretation of the problem statement and the multiplication tables, but she did not relate the multiplication table with the counting by groups involved in one of the questions. There was an excessive concern with conventional calculations, and there are indications that teacher P4 did not realise that in solving the second problem, students needed to make relationships, an essential skill in problem-solving. Those notes may mean that the teacher has poor specialised content knowledge, which was reinforced in class observation.

The class observation revealed that the research data showed flaws, such as, for example, P4's concern with making the bracelet, offering the counting procedure by ones and not by threes, as the activity proposed, which impaired the development of the idea of proportionality. We understand that for the idea of proportionality to be developed, the teacher's approach should be centred on counting by threes, analysing regularities, and arriving at the multiplicative procedures.

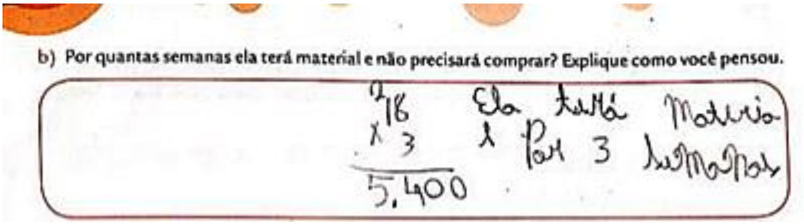
Regarding the "horizon content knowledge" subdomain, we analysed only one activity focused on proportionality. Thus, it was not possible to perceive this knowledge, which is commonly evidenced when one observes several sequences of activities developed throughout the school year. After analysing the content knowledge, we moved to the “pedagogical content knowledge” domain, which, according to Ball, Thames, and Phelps (2008), is subdivided into three categories: (1) “Knowledge of content and students”, (2) “Knowledge of content and teaching ” and (3) “Knowledge of content and curriculum”.

For Ball, Thames, and Phelps (2008), knowledge of content and students is about how students reflect, know, or learn a certain mathematical content. This type of knowledge is based not only on the cognitive aspects of students but also on relational knowledge. The activity favoured the development of this knowledge, as the statement requested students to explain and reflect on their calculation procedures or the resolution and relationships between questions and answers. However, this knowledge also evidences the teachers’ knowledge of their students. In the planning stage, teacher P4 showed knowledge of the content and her students when she planned the activities with her colleagues. She and her colleagues could anticipate possible doubts students might have regarding learning situations and what they should do about them. However, they could not say whether their students could explain their procedures and referrals to obtain the answers to the activities, showing evidence of gaps in the knowledge of the content and the students. Nevertheless, regarding class management, P4 showed to master it and suggested organising the class with productive groups, prioritising heterogeneity.

Another subdomain of teacher knowledge that caught our attention was the “knowledge of content and teaching”, which, for Ball, Thames, and Phelps (2008), is a conciliation between specific knowledge of mathematics with that of teaching. P4 noticed the focus on the multiplications involved in the problems. However, another aspect that we believe is convenient to discuss in this context concerns mathematical communication in the classroom. We consider that mathematical communication is part of the knowledge of content and teaching because, to communicate properly, the teacher needs to know well the content to teach and have knowledge about teaching the content through an appropriate mathematical language, more appropriate representations, explanations of the reasoning involved, and explanations on the procedures used, among other aspects. Also, the students’ interactions with each other and between the teachers and their students are part of the communication in mathematics classes. Regarding communication, in the observation of classes, data from the logbook and videos show that, when producing the bracelet, the students and the teacher communicated more than the students - organised in productive pairs - communicated with each other. The videos show that P4 did not explore orality in the questions of the problems that prioritised explanation of reasoning or procedures, besides relationships between data and the operations performed to arrive at the justifications of the answers. They also point to the little socialisation of students' resolutions, making discussions and socialisations between groups impossible. They also show little concern with mathematical symbology and the use of proper nomenclature. The video also showed the teacher's difficulties in relation to the mathematical knowledge that emerged during the multiplication algorithm involving a number with two zeros at the end (1,800). The data show that to simplify the multiplication calculation for the students, P4 recorded on the board the multiplication of 18 by 3 instead of 1,800 by 3, removing the two zeros from the number 1,800, and, after solving the operation, added those zeros to the result without explaining it. So, the students reproduced it in their notebooks, as we can see in the protocol (Figure 3).

Figure 3

Student's protocol for activities in item B.



We understand that the teacher may have intended to simplify the resolution process for the students. However, on the other hand, it may also reveal the teacher's lack of knowledge of the decimal number system and indicate a lack of knowledge within the knowledge of content and teaching subdomain, which can result in obstacles to students' learning and influence the understanding of the characteristics and regularities of the decimal number system.

We emphasise that the other episode selected from the videos that showed weaknesses in the knowledge about zero in the composition of a number by P4 was a speech by a student who stated that: Zero is worth nothing!

After reading this statement out loud, the teacher continued correcting the activity without worrying about clarifying to the student that in the number 900, the zero represents the absence of quantities in the units, and in the tens in the corresponding order, i.e., this number has 90 (ninety) tens, or 900 (nine hundred) units, or even 9 (nine) hundreds.

We understand that the idea that zero has no value for students can mean that they can either insert it or exclude it from a number whenever they want. The teacher's explanations would facilitate understanding the importance of zero in a number, and the use of overlapping cards could facilitate this understanding.

It should be noted that knowledge of content and teaching is often guided spontaneously by the teacher's beliefs and conceptions about how mathematics is learned and by mathematics-related myths. The data revealed in the videos show that P4's teaching practices were based on procedures and techniques to support students in their mathematical learning, without concerns about the relationships that should be highlighted in the students' answers and explanation of the procedures.

The "Content and curriculum knowledge" subdomain, according to Ball et al. (2008), refers to the confluence of knowledge that allows the teacher to understand the curricular documents, which enables a deeper analysis of the curriculum to make decisions and use them or not in their teaching proposal. Although the focus of our analysis is the second module of the course, it is worth noting that the data collected in the questionnaire, in the first module of the course, indicated that teachers needed to deepen their knowledge of the city curriculum and the city curriculum didactic guidelines.

In this first stage, the data from the project reports presented to UNESCO revealed that the teachers had weaknesses in the conceptions that subsidise the *Currículo da Cidade - Matemática*, such as the organisation of the curriculum in networks of meanings, the fundamental ideas of mathematics, the types of reasoning and the articulating axes. This indicated the need for a substantial investment in training, indicating that those concepts require much more time for their appropriation than a simple reading of documents.

Those discussions were resumed during the trainings. In the planning stage, the audios revealed that some teachers still had some knowledge gaps regarding the fundamental ideas of mathematics, as the linearity view predominated among the interdisciplinary cycle teachers. The activity of identifying the fundamental ideas present in the activities allowed the expansion of that knowledge. Another important point taken up during the planning was the organisation of the curriculum in a network of meanings, materialised in *Cadernos de Saberes* and discussed extensively by the teacher educators. This brought to teachers the idea that the learning and development goals can be organised and articulated in different ways, without losing their specificity, but seeking to understand that it is possible to establish relationships between them. This type of organisation broke with the idea of a prerequisite, very present in the initial instruments answered by the teachers when they referred to the teaching of mathematics.

When planning the selected activities, the teachers developed some perceptions that they did not have about their students, the curriculum, and the practices. Then, they began to give greater importance to collective planning.

In the second stage of training, we found that the planning phase allowed for a deeper discussion of the city curriculum and the didactic guidelines document, so that teachers could explore in more depth the constitutive elements of the curriculum and find the starting points and suggestions. Regarding the teachers' knowledge about the mathematics *Caderno da Cidade Saberes e Aprendizagens*, the planning stage allowed an analysis based on the knowledge developed in the first module of the course. The actions implemented in the first module contributed to the understanding and analysis of curricular materials, bringing a greater perception to teachers of the teaching practices they were planning, organising and proposing to their students. When planning the chosen activity, teachers could experience their understanding of the curriculum. In the planning stage, P4 expressed knowledge of the content and the curriculum by identifying the learning and development objectives in the selected activity and highlighting the important part. Furthermore, P4 identified the fundamental mathematics ideas present in the activity, indicating a bias opposite to the idea of linearity present in the conceptions of mathematics teaching of the interdisciplinary cycle group.

In the reflection phase of the classes, the data show that the 4th-grade teachers indicated improvements to be incorporated into the curriculum materials. In the teacher's version of the *Caderno*, the group pointed out the need to understand more deeply the didactic guidelines for the development of the activity, especially discussing mathematical communication and aspects that require clarification of students' reasoning about their procedures in solving the calculations and the type of intervention that could help them in their practice.

CONCLUSIONS

Although the focus of this article is not the lesson study methodology, we consider it essential to point out that it was crucial to expand P4's knowledge to teach mathematics. The research results show that P4's gaps in her knowledge to teach mathematics appeared in the class observation stage, although they did not appear in the planning. Another result also indicates that the discussion in the reflection stage allowed the teacher to progress in her practice. This finding may show that the teacher was not always aware of the limits of her knowledge, whether from the very mathematics or the pedagogical knowledge of the topic that would be addressed in class.

With the training actions, P4 began to mobilise knowledge from her experiences and group studies. Their didactic and curricular knowledge were produced from discussions with the group, based on theoretical contributions that dealt with the mathematics teaching and learning process. P4's discussions with her peers broadened and deepened her views on mathematics and its teaching, curriculum documents, and students and allowed the teacher to assume a leading role in decisions about her practice based on theoretical studies and collective reflections. It is worth noting that the training actions revealed some teachers' beliefs, especially regarding the use of manipulative materials in the classroom. During the training, many of these beliefs were problematised, and the reflection provided their resignification. However, beliefs are not easily overcome.

In P4's case, her belief about using manipulative materials as facilitators of her students' learning was firm. The contradiction between the belief that mathematics is abstract, complex, and for the understanding of few. The belief that manipulative material makes mathematics more attractive to children still needs to be an object of reflection in teacher training courses. In this article, we can perceive that although the subdomains of the teacher's knowledge are presented separately by Ball et al., they overlap in the teacher's practice. However, the separate analysis of each subdomain can provide essential indications for teacher education. To exemplify how the subdomains are connected in the teacher's knowledge, we emphasise, as already pointed out, that P4 found it challenging to teach some mathematical content, perhaps resulting from conceptual weaknesses.

On the other hand, she was empathetic and clear about her students' knowledge and had a more favourable interpersonal relationship. She also presented the necessary curriculum knowledge to understand the proposed activity, but she was aware of the need to improve more to teach mathematics, which must be emphasised. When the teacher realised her need to improve her education and was open to discussions, her evolution was consistent. We understand that the consolidation of knowledge of the teacher at stake resulted from training strategies that catalysed her professional development.

To conclude this article, it is worth clarifying that at no time do we intend to belittle the work of the teacher who opened her room for the teacher educators' observation and research. On the contrary, during the project development, the educators and the teacher built an affective bond. The teacher's availability stemmed from her closeness and admiration for them. In the reflections carried out in the third stage of the lesson study, the teacher always reported that she was learning much from the educators' comments about her classes. She also said that although she had good support from the planning and the theoretical studies, those analyses allowed her to realise some errors in her practice. We understand that the collaboration of basic education teachers is fundamental to advance research that discusses teacher continuing education.

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AUTHORS' CONTRIBUTIONS STATEMENTS

This work is the result of the doctoral research of the first author, PBM, under the guidance of the second author, EC.

DATA AVAILABILITY STATEMENT

Data from this study will be made available by PBM.

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