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Some Theoretical References Related to Advanced Mathematical Thinking

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ABSTRACT

Background: In Mathematics Education, the theoretical reference of Advanced Mathematical Thinking has contributed to some research. However, others opt for a different but somehow related theoretical reference. Identifying the similarities and differences between those references can elucidate the researcher's motivations when adopting one or another reference. **Objective**: to synthesise three studies involving Mathematical Creativity, Advancing Mathematical Activity and Advanced Mathematical Knowledge, respectively, and to illustrate possible contributions of those theoretical references in the analysis of a written production. **Design**: the research is qualitative and theoretical and speculative so that theoretical relationships were carried out and used later in the analysis of the written production. **Setting and Participants**: the research involves a participant with a degree in mathematics with whom the researchers had a virtual interaction to present the questions. **Data collection and analysis**: the procedures performed in the resolution of questions prepared based on the OBMEP question bank are analysed and, with the support of this example of analysis, possible contributions of each reference in the analysis of a written production are discussed. **Results**: none of the references covers all the thinking mobilised in the activity, but each one favours the focus on specific aspects that were verified in the written production of the research participant, in line with the similarities and divergences emphasised in the theoretical appreciation of each reference. **Conclusions**: the knowledge of different theoretical frameworks provides the teacher with foundations to exercise their teaching practice and the researcher with options for an adequate choice for their research.

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Keywords: Mathematics Education; Advanced Mathematical Thinking; Mathematical Creativity; Advanced Mathematical Activity; Advanced Mathematical Knowledge.

Alguns Referenciais Teóricos Relacionados ao Pensamento Matemático Avançado

RESUMO

Contexto: Na Educação Matemática, o referencial teórico do Pensamento Matemático Avançado tem sido um aporte para algumas pesquisas. Contudo, outras optam por um referencial teórico diferente, mas de alguma forma relacionado. Identificar as similaridades e divergências entre esses referenciais pode elucidar as motivações do pesquisador ao adotar um ou outro referencial. **Objetivo:** sintetizar três pesquisas envolvendo, respectivamente, a Criatividade Matemática, a Atividade Matemática Avançando e o Conhecimento Matemático Avançado e ilustrar possíveis contribuições desses referenciais teóricos na análise de uma produção escrita. **Design:** a pesquisa é de natureza qualitativa e teórica e especulativa, de modo que relações teóricas foram realizadas e utilizadas posteriormente na análise da produção escrita. **Ambiente e participantes:** a pesquisa envolve uma participante licenciada em Matemática com a qual os pesquisadores tiveram uma interação virtual para apresentação das questões. **Coleta e análise de dados:** analisam-se os procedimentos realizados na resolução de questões elaboradas com base no banco de questões da OBMEP e discute-se, com o apoio desse exemplo de análise, possíveis contribuições de cada referencial na análise de uma produção escrita. **Resultados:** nenhum dos referenciais ilumina todo o pensamento mobilizado na atividade, mas cada um favorece o foco em determinados aspectos que foram verificados na produção escrita da participante da pesquisa, em conformidade às similaridades e divergências enfatizadas na apreciação teórica de cada referencial. **Conclusões:** o conhecimento de diferentes quadros teóricos possibilita ao professor fundamentações para exercer sua prática docente e ao pesquisador opções para uma escolha adequada à sua pesquisa.

Palavras-chave: Educação Matemática; Pensamento Matemático Avançado; Criatividade Matemática; Atividade Matemática Avançando; Conhecimento Matemático Avançado.

INTRODUCTION

Studies have adopted the theoretical framework of Advanced Mathematical Thinking (AMT) developed by Tall (2002) and Dreyfus (2002). Other authors were based on different but somehow related references.

The book *Advanced Mathematical Thinking*, organised by David Tall in 1991 and republished in 2002, has chapters written by Tall (2002), Dreyfus

(2002), and Ervynck (2002), among others, still frequently adopted by researchers. In addition, some previous concepts of mathematical thinking were incorporated into this framework, such as concept image and concept definition (Tall & Vinner, 1981), resumed by Tall (2002).

Although Tall (2002) considers that AMT differs from the elementary because of the possibility of definition and formal deduction, proof in a logical way based on those definitions, formal abstraction, and consequence of advanced mathematics, from a perspective that aims to overcome the students' difficulties in the transition to higher education, other researchers consider that the AMT occurs since the most basic school levels. This perspective is consistent with that of Dreyfus (2002), who considers the AMT as a complex process that involves a large number of processes that interact in intricate ways. According to Dreyfus (2002, p. 26), "There is no sharp distinction between many of the processes of elementary and advanced mathematical thinking, even though advanced mathematics is more focused on the abstractions of definition and deduction", but "One distinctive feature between advanced and elementary thinking is complexity and how it is dealt with". Processes that allow managing the complexity of a mathematical situation, such as generalisation, synthesis, and translation between representations, can occur at different levels of schooling.

Bianchini and Machado (2015), Sousa and Almeida (2017), Vidotti and Kato (2018), and Mateus-Nieves and Jimenez (2020) are examples of research that used the AMT framework. The first worked on AMT processes as per Dreyfus (2002), with teachers in continuing education, who reviewed their reflections on their resolutions after studying the processes. The second analysed the AMT processes of a licentiate, according to Tall (2002) and Dreyfus (2002), while he was developing modelling activities. The third analysed the difficulties of mathematics degree students in learning the limit of functions of several variables, based on the concepts proposed by Tall and Vinner (1981). And the fourth articulated the Knot Theory with the AMT, according to research based on Tall, Dreyfus, and others, creating a seminar for undergraduate students in mathematics to delve into the process of mathematical generalisation.

As we have highlighted, those studies that used the AMT chose some theorists and their ideas as a basis and, in some cases, articulated them with other studies related to the context of their objectives, sometimes creating their own interpretation of that framework. Other studies were based on a theoretical framework different from the AMT to study advanced thinking in mathematics, such as the Articulation Principle, Advanced Mathematical Knowledge, Mathematical Creativity, and Advancing Mathematical Activity.

The different theoretical references adopted by the researchers led us to question the reasons for those choices, whether they could be the same and in which situations one would be more appropriate than the other. The similarities between theories may indicate the same perspective and the same underlying logic for working with each one. On the other hand, their differences need to be clarified so that researchers in Mathematics Education can choose which one is best suited to their research.

To answer those questions, we established the objective of synthesising three research works involving Mathematical Creativity, Advancing Mathematical Activity, and Advanced Mathematical Knowledge, respectively, and to illustrate possible contributions of those theoretical references in the analysis of a written production.

Using the AMT as a theoretical basis (Tall, 2002; Dreyfus, 2002), we discuss the other three theoretical references linked to the corpus of articles that we defined as the scope of the research: Advanced Mathematical Knowledge (Zazkis & Leikin, 2010), Mathematical Creativity (Nadjafikhah, Yaftian, & Bakhshalizadeh, 2012), and Advancing Mathematical Activity (Rasmussen, Zandieh, King, & Teppo, 2005).

In summary, in the following section, we discuss the methodological issues. After that, we present the synthesis of the articles in the corpus, their theoretical references and relations and interrelations with the AMT. Then, we analyse a written production in the light of those references and, finally, present our conclusions.

METHODOLOGICAL PROCEDURES

In this qualitative (Bogdan & Biklen, 1994), theoretical, and speculative research (Martineau, Simard, & Gauthier, 2001), we used three theoretical references to analyse the procedures of a mathematics teacher to solve questions based on the Brazilian Public School Mathematics Olympiad [OBMEP] (2007). For this, those three theoretical references were discussed from three studies that we synthesised and interpreted in search of relationships based on the AMT. Based on those discussions and the analysis example, we established possible contributions of each reference in the analysis of a written production.

To support the method referring to the relationships that will be established between the theoretical references amid the synthesis we need to clarify it before starting to shorten the articles. For this reason, before describing the theoretical framework, we describe the methodological procedures.

To compare the theoretical references and observe some similarities before and after illustrating them in the analysis of a written production, we carried out a theoretical and speculative study, i.e., we promoted theoretical declarations from other theoretical statements (Martineau, Simard, & Gauthier, 2001). Theoretical and speculative research involves the axes of interpreting, arguing, and telling. “The interpreting axis involves hermeneutics and conceptual analysis; the arguing axis takes us back to rhetoric and, finally, the telling axis involves literary practice” (Martineau, Simard, & Gauthier, 2001, p. 9)[1].

Hermeneutics is the art of interpreting, necessary to avoid misunderstandings as geographic, temporal, or cultural distance separates a text from its reader (Martineau, Simard, & Gauthier, 2001).

> Researchers who carry out theoretical and speculative research are confronted with the texts (books, articles, communications) of other researchers who have addressed the same subject. Thus, even before producing their own text, researchers must interpret those previous texts to overview the investigated field, to specify their research question, and to formulate an original problem. This permanence in the specialised literature is an exercise in interpretation, a work of hermeneutics and conceptual analysis. (Martineau, Simard, & Gauthier, 2001, p. 12)[2].

According to Martineau, Simard, and Gauthier (2001, p. 16), “from the conceptual analysis, he or she [the researcher] will keep in mind the need to

[1] L’axe de l'interpréter implique l'herméneutique et l'analyse conceptuelle; l’axe de l'argumenter nous renvoie à la rhétorique et, finalement, l’axe du raconter englobe la pratique littéraire.

[2] Les chercheurs qui conduisent des recherches théoriques et spéculatives sont confrontés aux textes (livres, articles, communications) d'autres chercheurs qui se sont penchés sur le même sujet. Ainsi, avant même de produire leur propre texte, les chercheurs doivent donc interpréter ces textes antérieurs afin d'avoir une vue d'ensemble du champ investigué, de préciser leur question de recherche et de formuler une problématique originale. Ce séjour dans la littérature spécialisée est un exercice d'interprétation, un travail d'herméneutique et d'analyse conceptuelle.

correctly define the concepts"[3]. The precise definition of the concepts present in the theoretical references of mathematical thinking is important to establish coherent relationships between concepts from different references. Argumentation uses presentations or statements that aim to show the validity of a position. Thus, the theoretical analysis of the corpus and an example of analysis of a written production contribute to the plurality of arguments recommended by Martineau, Simard, and Gauthier (2001). Finally, in the telling axis, we highlight the production of an "unprecedented issue, to propose a new analysis based on the interpretation of previous texts and rigorous argumentation" (Martineau, Simard, & Gauthier, 2001, p. 20)[4].

The analysis of written production allows us to illustrate the differences between analyses carried out with each of the theoretical references, and exemplify the use of those references, aiming to contribute to future research.

In an analysis of written production and educational research in general, it is not possible to isolate one cause and keep the others constant to verify the effects of its variation. Not only the theoretical framework adopted, but the researched subjects, the resolutions, and even the researchers influence the research. Hence, the analysis of written production is subjective, which is characteristic of qualitative research, although the methods adopted help reduce biases (Bogdan & Biklen, 1994, p. 67-68).

As for the procedures of analysis of written production, the choice of a participant with a degree in mathematics is justified by the characteristics of the theoretical references adopted: to analyse the Advanced Mathematical Knowledge, the research participant must have mathematical knowledge obtained at undergraduate and postgraduate levels (Zazkis & Leikin, 2010); training as a mathematics teacher can favour the articulation between advanced and school mathematics so that the presented resolutions allow us to analyse the progression of mathematical thinking, an important aspect in the discussions regarding the Advancing Mathematical Activity (Rasmussen et al., 2005).

Thus, we chose questions that allow multiple solutions to analyse Mathematical Creativity (Nadjafikhah, Yaftian, & Bakhshalizadeh, 2012), and that can be solved by mobilising both school mathematical knowledge and Advanced Mathematical Knowledge. We found questions with those

[3] de l'analyse conceptuelle, il [le chercheur] gardera en mémoire la nécessité de définir correctement les concept.

[4] une problématique inédite, à proposer une nouvelle analyse sur la base de l'interprétation des textes antérieurs et de l'argumentation rigoureuse.

characteristics in the OBMEP (2007) question bank. There, we selected three and modified them according to our objectives.

The modifications were carried out after individual and collective resolutions tested by all authors and discussed, in a process repeated a few times to create favourable conditions for multiple solutions, to encourage analyses in the light of each of the three theoretical references, according to the solutions foreseen, and eliminate possibilities of multiple interpretations of the utterance.

Besides the resolutions, we asked the participant to report in writing the procedures adopted to resolve the issues, from hypotheses to conclusions, including any frustrated solutions and consultations with materials such as books, websites, and videos. So that thinking would not be hindered by the rush, we did not impose a time limit for the resolutions, which were delivered on the participant's initiative after about two weeks the questions had been handed.

The research was approved by the Ethics Committee in Research Involving Human Beings of the State University of Londrina through opinion number 5.144.763.

In this article, we analysed two of the three questions resolved by the participant. Considering that our objective was not to examine the participant's performance but to discuss possible contributions of three theoretical references in the analysis of a written production, we selected the two questions whose resolutions allow a more fruitful discussion with regard to Mathematical Creativity, Advancing Mathematical Activity, and Advanced Mathematical Knowledge.

THEORETICAL REFERENCES

As mentioned in the introduction, some studies that use the AMT theoretical framework make their connection with other studies related to the context of their objectives, sometimes creating their own interpretation of this framework.

For example, Sousa and Almeida (2017) relate cognitive processes focused on the development of modelling activities to AMT processes. This relationship justifies the coherence of using the AMT framework to analyse the development of students' thinking processes in modelling activities.

Likewise, the justifications of other studies for using the AMT are linked to the authors' interpretation of the theoretical framework and relationships in the context of their research. Mateus-Nieves and Jimenez (2020) created a 'holistic scheme' to analyse AMT skills integrated into the Knot Theory, aiming to strengthen the generalisation process. This articulation was used as justification:

> Linking some concepts of knot theory with the development of advanced mathematical thinking skills allows us to expand the process of mathematical generalisation aiming to strengthen the range of didactic strategies that guide the mathematics teaching and learning that enables, reflectively, and innovatively, the interaction with various backgrounds and training levels. (Mateus-Nieves & Jimenes, 2020, p. 66).

Similarly, Bianchini and Machado (2015) justified the importance of mathematics teachers knowing the AMT processes:

> In this way, knowledge about the AMT processes allows the mathematics teacher to evaluate both the difficulties inherent to the concepts and ideas that they want to develop with their students and those presented by the students' lack of habit with the use of the required AMT processes in the construction of such knowledge. [...] the explicit knowledge of the AMT processes can help the teacher to develop activities that aim at the students' appropriation of those processes. (Bianchini & Machado, 2015, p. 29)[5].

Vidotti and Kato (2018) justified the importance of analysis of concept image and concept definition in the diagnosis of difficulties:

> The need to diagnose problems in the teaching and learning process, which go beyond variables such as the teacher, the curriculum, the environment, study habits, and purely mathematical problems, has led researchers to seek clarification on the cognitive processes involved in

[5] Dessa forma, o conhecimento sobre os processos do PMA possibilita ao professor de matemática avaliar, tanto as dificuldades inerentes aos conceitos e ideias que deseja desenvolver com seus alunos, como também aquelas apresentadas pela falta de hábito dos alunos com a utilização dos processos do PMA requeridos na construção de tais conhecimentos. [...] o conhecimento explícito dos processos do PMA pode auxiliar o professor a elaborar atividades que visem à apropriação desses processos por seus alunos.

> mathematical reasoning, whose theoretical bases are supported by studies aimed at understanding how the human brain works.
>
> In this sense, Tall and Vinner (1981) developed the notions of concept image and concept definition [...]. (Vidotti & Kato, 2018, p. 931)[6].

Those are examples of research in which the AMT has adequately contributed to the achievement of objectives. Inspired by those references, in this section we synthetise and relate the corpus researches, linked, respectively, to Mathematical Creativity, Advancing Mathematical Activity and Advanced Mathematical Knowledge. These theoretical references are linked to the AMT, as we establish below, and they can support some research in which the AMT does not fit as in the ones mentioned above. Nadjafikhah, Yaftian, and Bakhshalizadeh (2012) carried out a theoretical work that defines Mathematical Creativity based on several works that addressed it. Rasmussen et al. (2005) elaborated the theoretical framework of the Advancing Mathematical Activity, aiming to offer an alternative characterisation of the AMT. Zazkis and Leikin (2010) defined the concept of Advanced Mathematical Knowledge to investigate how teachers use their mathematical knowledge in teaching.

We started our discussion with Mathematical Creativity. Creativity plays a key role in the AMT cycle, as it contributes to the early stages of theory development, assists in the formulation of mathematics as a system of axioms and proofs, and allows new ideas to be reformulated in previously unknown ways (Ervynck, 2002).

Nadjafikhah, Yaftian, and Bakhshalizadeh (2012) present some of the definitions found in the literature for Mathematical Creativity. The ability to analyse a problem from different perspectives, identify patterns, similarities and differences, and generate multiple ideas and choose a method that is suitable for dealing with an unfamiliar mathematical situation is one way of describing Mathematical Creativity.

Nadjafikhah, Yaftian, and Bakhshalizadeh (2012) support, based on other studies on the subject, the division of Mathematical Creativity into two

[6] A necessidade de diagnosticar problemas no processo de ensino e aprendizagem, que vão além de variáveis como o professor, o currículo, o ambiente, hábitos de estudos, bem como problemas de ordem puramente matemática, levou pesquisadores a buscarem esclarecimentos sobre os processos cognitivos envolvidos no raciocínio matemático, cujas bases teóricas apoiam-se em estudos destinados a compreender como o cérebro humano funciona.
Nesse sentido, Tall e Vinner (1981) desenvolveram as noções de conceito imagem e conceito definição [...].

levels: professional and school. At the professional level, creativity is defined as the ability to produce original work that expands the body of mathematical knowledge and the potential to pose new questions to other mathematicians. At the school level, creativity is associated with problem-solving and can be identified in the process that results in unusual or insightful solutions to a problem and in formulating questions or possibilities that allow a problem to be considered from a new perspective.

For Ervynck (2002), problem-solving makes students deal with failures and get used to the idea that there is no algorithm capable of providing all the answers. In a constantly changing society, being able to apply algorithms is not enough. It is necessary for thinking to be flexible, and tasks that demand creativity contribute to the development of this flexibility.

However, mathematics is generally presented to students as a finished product rather than a process, which is criticised by Tall (2002) and Dreyfus (2002). For the authors, presenting Mathematics as a sequence of definitions, theorems, and proofs shows the logical chain of science but omits that knowledge often results from sequences of trial and error, intuitive formulations, and inaccuracies.

To stimulate students' creativity, Bezerra, Gontijo, and Fonseca (2021) propose that teachers give creative feedback:

> The different instruments used for students to express their thinking constitute rich analytical material. Through them, teachers and students can establish a communicative process that favours the development of creativity and learning in mathematics. We call this communicative process, which is part of the formative assessment, *feedback*. (Bezerra, Gontijo, & Fonseca, 2021, p. 93).
>
> We define feedback intended to develop creative potential as creative feedback. (Bezerra, Gontijo, & Fonseca, 2021, p. 94).

Still, Ervynck (2002) considers the importance of intuition to guide imagination and inspiration that formulate the required results. The author agrees with Tall's (2002) considerations, for which intuition is a product of the concept image (Tall & Vinner, 1981) of an individual, so that the more educated in logical thinking, the more rigorous its intuition.

Creativity is at the heart of mathematical thinking, understood as "a creative activity that brings with it the possibility of human error. Indeed the very possibility of error is what makes the major advances such monuments of human success" (Ervynck, 2002, p. 52). Therefore, activities that stimulate creativity have the power to humanise mathematics as they expose its fallibility.

Teachers must identify, encourage, and improve students' creative ability by proposing, for example, multi-solution tasks, which are tasks that allow for different resolutions based on different representations of mathematical concepts. Another way to stimulate students' creativity is to propose experiments with open problems, providing them with the opportunity to reveal their understanding of a concept. Furthermore, the exercise of creativity requires an interactive environment, i.e., an environment in which students feel safe to share their perceptions and ideas.

The second theoretical reference discussed in this research is the one developed by Rasmussen et al. (2005), who characterise AMT as an Advancing Mathematical Activity and is not limited to specific grades or content levels. With this name, the authors aim to highlight the students' total activity process, not focusing on their final stage but on the progression in mathematical thinking, considering the different mathematical activities derived from social practices.

This progression of thought is hardly analysed by teachers, as Bianchini and Machado (2015, p. 29) observe:

> […] when a mathematics teacher is asked to analyse the knowledge mobilised in solving a problem situation, they focus and describe mainly mathematical procedures, often already automated, and sometimes tacitly accepted. This fact makes it difficult for them to perceive the processes they experience, such as the occurrence of trial and error, comings and goings, visualisations, validations, generalisations, etc., which are part of their knowledge of doing mathematics [...][7].

Rasmussen et al. (2005) understand doing and thinking as being reflexive in nature so that students engage in specific activities, can represent their understanding, and expand their thinking and ways of reasoning in the process. For this purpose, horizontal and vertical mathematising constructs are

[7] [...] quando um professor de matemática é instado a analisar os conhecimentos mobilizados em sua resolução de uma situação-problema, ele enfoca e descreve principalmente os procedimentos matemáticos, muitas vezes já automatizados, e algumas vezes tacitamente aceitos. Tal fato dificulta sua percepção sobre os processos vivenciados, como a ocorrência de tentativa e erro, idas e vindas, visualizações, validações, generalizações etc., que fazem parte de seu saber sobre o fazer matemático [...].

used with examples of symbolisation, algorithmisation, and definition activities to characterise the Avancing Mathematical Activity.

The authors consider that for mathematical learning to occur, it is necessary to participate in different types of mathematical practices. In this way, the horizontal and vertical mathematising form a reflexive relationship in carrying out the activities and are intrinsically associated, allowing the elaboration of comparisons regarding the nature of the students' activity and providing a language to approach the process by which they develop new visions and awareness. To clarify the Advancing Mathematical Activity, Rasmussen et al.. (2005) point out that horizontal mathematising is a broader way to include fields of problems or situations that are, from the perspective of those involved, already mathematical in nature. Therefore, those problematic fields or problematic situations depend on the background, experiences, and goals of those involved in the mathematical activity.

Rasmussen et al. (2005) understand horizontal mathematising as a problem field related to the formulation of a problem situation in such a way that it is friendly for further mathematical analysis. Thus, horizontal mathematising can include, but is not limited to, activities such as experimenting, pattern snooping, classifying, conjecturing, and organising. In turn, vertical mathematising relates the activity carried out by students built from horizontal activities involving reasoning of abstract structures, generalisation, and formalisation to the purpose of creating new mathematical activities for students, promoting a sequence of progressive mathematisations with multiple layers of types of horizontal and vertical activities being interrelated.

Symbolisation, algorithmisation, and definition activities are related to some AMT processes. Symbolisation encompasses representation processes that, according to Dreyfus (2002), include relationships between signs and meanings. The activity of developing an algorithm involves generalising a procedure for solving a problem. In the definition activity, students base themselves on their concept image (Tall & Vinner, 1981) to create a definition and check possible conflicts generated to improve it.

In addition, the activities most linked to vertical mathematising are AMT processes. In fact, the reasoning of abstract structures requires the student to perform abstraction processes that, together with generalisation, according to Dreyfus (2002), are processes that allow managing the complexity of a situation. Formalisation, according to Tall (2002), is a distinctive factor of the AMT in relation to the Elementary Mathematical Thinking.

We emphasise that the AMT characterisations proposed by Tall (2002) and Dreyfus (2002) value the complete cycle of mathematical thinking, and not only the product of mathematical thinking, as the term AMT may suggest when compared to the term adopted by Rasmussen et al. (2005). However, 'Advancing Mathematical Activity' is consistent with Rasmussen et al.'s (2005) proposal of offering an alternative characterisation of the AMT that emphasises progression in students' mathematical activity, focusing on mathematical practices and qualitatively different types of activities within those practices.

The third referential addressed in this discussion is developed by Zazkis and Leikin (2010), who adopted Advanced Mathematical Knowledge (AMK) as a theoretical contribution to investigate how teachers use their mathematical knowledge in teaching. The authors themselves defined AMK as the knowledge of the subject matter acquired in undergraduate courses in Mathematics and made an association with the AMT, which does not have a precise definition. According to the authors, the difference in perspectives regarding the AMT changed the description of its research area to 'tertiary mathematics', and the definition of AMK is in line with this change.

This reflects a notion that the AMT can refer to what is taught in Advanced Mathematics, something that we have identified mainly in previous works and lines of research that used the term 'Advanced Mathematical Thinking'. However, we understand AMT as thinking that, according to Dreyfus (2002), occurs through thinking processes complex enough to manage the complexity of a mathematical situation. On the other hand, AMK is content knowledge. As thinking processes and content knowledge are distinct objects, there is no identity between the AMT and the AMK, although we perceive a collaboration between them, as discussed below.

In their work, Zazkis and Leikin (2010) analysed secondary school teachers' perceptions (corresponding to the elementary school - final years and high school in Brazil) of the use of AMK in their teaching practice. For this, they interviewed 52 teachers who teach mathematics in secondary education, asking to what extent they use AMK in teaching and asking for examples of mathematical topics, teaching situations, and problems in which AMK is essential for teachers.

As a result, the study showed a variation in how much teachers claimed to use the AMK in their practice. Furthermore, claims that the AMK is used all the time did not generally correspond to specific examples of such use. Some teachers exemplify the use of the AMK with general topics, such as calculus and statistics, but they were hardly able to specify situations or problems. Moreover, the authors highlighted non-content-specific examples in teachers' answers, such as ways of thinking, 'good insight' for teaching, making connections between content within and beyond the curriculum, and answering students' questions, seeing a 'better picture' or a 'whole picture' of the subject, sense of terrain, confidence, and 'cross-cutting themes' or 'meta-mathematical issues' such as 'proof', 'language', and 'precision and aesthetics', which can appear in any mathematical content. This indicates that many teachers see AMK as an indirect benefit to teaching practice, and not necessarily a specific benefit for each module studied during graduation. Zazkis and Leikin (2010) conclude that many teachers' difficulties in articulating specific examples of the use of AMK highlight a gap between university mathematics and secondary school mathematics.

We noticed a relationship between some indirect benefits of AMK and AMT processes, especially regarding the issues that Zazkis and Leikin (2010) called 'meta-mathematics'. The proof process is seen by Tall (2002) as the final stage in the development of mathematical thinking; it is when ideas gain precision. At this stage, the language needs precision so that ideas are organised in a logical sequence based on definitions, avoiding inconsistencies. The aesthetics observed by Zazkis and Leikin (2010, p. 274) concerns 'beautiful solutions', which is related to Mathematical Creativity, as it involves the search for different solutions and valuing the most creative ones.

The connections with broader contexts are related to the processes of generalisation and synthesis, which are characteristic of the AMT, according to Dreyfus (2002). In fact, generalisation involves expanding a domain of validity, whereas synthesis means combining parts in such a way that they form a whole. When the teacher is able to see the whole in a single image, he/she can create connections between the elements inside and outside the curriculum to which they are related. Thus, the synthesis process relates to the teacher's competence to make connections between content within and beyond the curriculum, to see a 'whole picture' of the issue and to have a sense of terrain, which are benefits of the AMK according to the teachers interviewed by Zazkis and Leikin (2010).

From these and other thinking processes developed during studies of Advanced Mathematics, we can say that the development of the AMK implies the development of the AMT. On the other hand, Tall (2002) and Dreyfus (2002) argue that the study of advanced processes that occur in the minds of mathematicians when developing their research is important to better understand the same processes that occur with students while learning advanced mathematical concepts. Thus, the development of the AMT also favours mathematics learning in higher education so that we can say that the developments of the AMT and the AMK occur simultaneously, one contributing to the other. Based on Rasmussen et al. (2005), we can add that the experiences carried out by the Advancing Mathematical Activity, both in horizontal mathematising and in vertical mathematising, provide forms of thinking that foster the AMT and the AMK.

The choice of theoretical framework made by Zazkis and Leikin (2010) was consistent with their intention to investigate teachers' perceptions regarding the use of their knowledge of Advanced Mathematics. The AMK is the object whose perceptions of use were investigated, while the AMT does not have such a direct relationship with the research objective. Still, as the authors emphasise, the AMT has different conceptions, some not exclusively related to Advanced Mathematics but also to school mathematics. In addition, we point out that the researchers showed the AMK definition to the teachers interviewed, serving as a basis for the formulation of questions and answers. This was only possible because there was a precise definition, simple enough to be understood by teachers after a quick reading.

There are points of convergence between the mentioned references. For example, Mathematics is seen as a human activity from the perspective of the AMT, Mathematical Creativity and Advancing Mathematical Activity. Thus, those references consider that Mathematics must be reconstructed in the classroom and not given as something finished and polished. In this way, those theoretical references converge in supporting the idea of a favourable environment for learning as an environment in which the students are active in the process of building their knowledge and, for that, open questions are recommended, and with a complexity that goes beyond the mere operationalisation of concepts.

The application of the concept of Mathematical Creativity is, in a way, present in all those references. Rasmussen et al. (2005) report an 'epiphany' that one of the participants of their research described when having an idea of a differentiated procedure to solve a question:

> Joaquin's use of the word epiphany to describe his reasoning indicates that his solution was not the result of a memorized procedure. It appears that Joaquin had developed a highly integrated and complex way of reasoning about the space of solution functions to differential equations and had developed

> effective and dynamic symbolizations to foster and further his reasoning. (Rasmussen et al., 2005, p. 62).

Zazkis and Leikin (2010) describe the contribution of AMK to mathematical ‘aesthetics’ from the speech of a participating teacher, in the sense of presenting beautiful solutions: “I also am aware of the aesthetics that exists in mathematics and try to bring to my classroom examples of beautiful solutions and encourage students finding beautiful solutions. (Dina-2)” (Zazkis & Leikin, 2010, p. 274).

Both the study of AMT processes and Mathematical Creativity are inspired by the thinking of mathematicians when developing their research:

> The critical role of these creative mathematicians, who have been able to create new mathematical insights and ideas, is so Much apparent that there is no need to be emphasized. However, study of processes of their creative thinking is valuable. (Nadjafikhah, Yaftian, & Bakhshalizadeh, 2012, p. 285).

Another highlight of this approximation is the AMT processes (Tall, 2002; Dreyfus, 2002), as we commented on the processes of proof, representation, syntheses, and in terms of intuition and formalisation.

We can also highlight differences between the theoretical references analysed. The choice of terms designates something to emphasise. While the Advancing Mathematical Activity underscores the progression of mathematical thinking, the Mathematical Creativity emphasises differentiated thinking, and the AMK emphasises Advanced Mathematics content rather than advanced thinking.

Next, we discuss a written production based on the theoretical references that we present.

ANALYSING A WRITTEN PRODUCTION

We analysed questions solved by a participant with a degree in Mathematics based on the theoretical references adopted, which allow multiple solutions and can be solved by mobilising both school mathematical knowledge and Advanced Mathematical Knowledge. The two questions selected for analysis were resolved without consulting any materials, as reported by the participant.

The first question selected for analysis has a statement as shown in Figure 1.

Figure 1

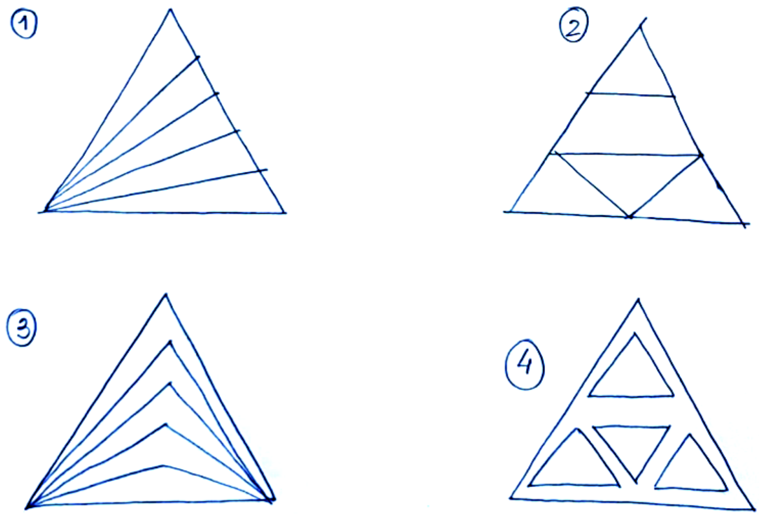
First question. Prepared based on the OBMEP question bank (2007)

1) Consider an area bounded by an equilateral triangle. a) Present drawings, indicating some ways to divide this area into five parts of the same area. b) What knowledge or skills did you mobilise in solving this issue? At what level of your schooling were these topics learned? Or were they learned throughout your professional performance?

First, the participant sought to imagine possible solutions and thus organised four possibilities, which she called configurations, and drew them without strict proportions, as shown in Figure 2.

Figure 2

Configurations Prepared by the Participant



Considering that the original triangle has a height h and a base b, the participant sought to rigorously construct each configuration in a geometric form and prove that the area of each region is $b \cdot h/10$. Starting with configuration 2, she realised that the height of the three lower triangles is the same, so the base of the central triangle would need to be $/2$, which would lead to a division of the original triangle into four equal parts, and not five, as required by the statement.

Configuration 1 was built with ruler and compass, from bottom to top, looking for, in each triangular region, the height that would divide the remaining region into the necessary amount of equal parts.

Configuration 4 was designed with four triangles with base $b/2$ and height $h/2.5$. Thus, the participant used the base of the original triangle to divide it into two bases for two of the triangular regions.

Figure 3

Participant's Argument in Configuration 3[8]

Dessa forma, a região 1 tem área igual a

$$\frac{b.h}{2.5}.$$

A região 2 tem área igual a

$$\frac{b.2h/5}{2} - \frac{b.h/5}{2} = \frac{bh/5}{2} = \frac{bh}{2.5}$$

e assim por diante.

Finally, configuration 3 was classified by the participant as "simplest to execute", as it was sufficient to divide the height of the original triangle into five equal parts. The justification could take into account the area of each pair

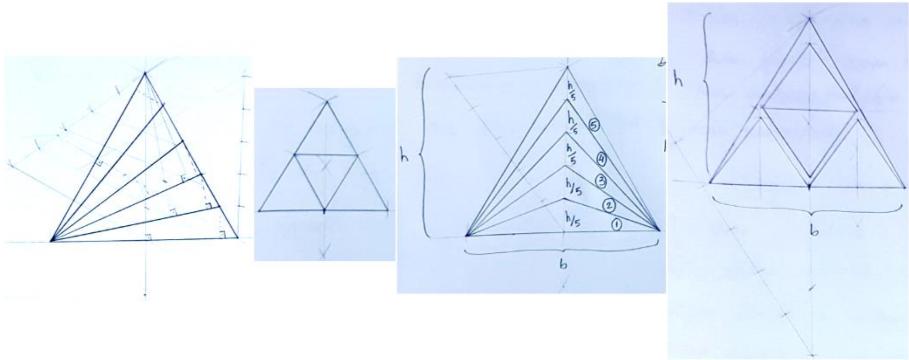
[8] In this way, region 1 has area equal to […]. Region 2 has area equal to […] and so on.

of triangles that form the regions, reducing itself to showing that those triangles have the same area. However, the participant calculated the area of the regions from bottom to top by subtracting the area of two triangles, as shown in Figure 3.

In Figure 4, we gathered the participant's configurations after that rigorous treatment:

Figure 4

Participant-Designed Configurations with Ruler and Compass



From the AMT perspective, we can identify the participant's processes of representation and generalisation. The geometric representation was used throughout the resolutions, while the generalisation of an algebraic pattern was perceived by the participant when expressing "and so on" in the calculations referring to configuration 3, indicating that she realised that the subtraction of the areas of two triangles would have the form $\frac{b\cdot(n+1)\cdot h/5}{2} - \frac{b\cdot n\cdot h/5}{2} = \frac{b\cdot h}{2\cdot 5}$, for $n = 1,2,3,4$.

In addition, the participant mobilised her intuition in formulating the four configurations and used rigour when developing or refuting each possibility. According to Tall (2002), intuition and rigour are not necessarily dichotomous. In fact, we can observe that the logical reasonableness of the result of the participant's intuition shows how refined her intuition is in terms of logic, which means, according to that theoretical reference, that the participant has coherent images of the concepts involved.

Also, the participant placed the arguments in a logical sequence to prove that configuration 2 would not solve the problem, which shows a transition to the AMT, according to Tall (2002).

Considering the theoretical reference of the Mathematical Creativity, we can say that the participant was creative in her resolutions, as she tested several configurations and concluded that one was simpler to execute than the others. This is configuration 3, which is different from all that we had imagined and whose divisions, although they can be obtained by line segments, result in polygons that are not necessarily convex.

Also noteworthy is configuration 4, which provides an infinity of solutions as the smaller triangles move within the larger triangle.

Configuration 1 presented by the participant would be simple to execute if she had considered the segments obtained on the side cut in five as the bases of the smaller triangles instead of finding the triangles by the heights relative to the base of the original triangle. It is possible that the reason the participant did not look at the configuration from this other perspective was that she could prove the configuration possibility on her first attempt.

We can also see predominantly geometric thinking in the way she investigated the issue. She constructed all the solutions geometrically and, just to justify the equality of the areas, she used that the area of each region should be of $b \cdot h/10$. In a given solution, the participant observed that "she could also have used only a millimetre ruler" but preferred to use ruler and compass constructions, showing her ability to use different methods.

With regard to the Advancing Mathematical Activity, we identified horizontal mathematising, especially in the formulation of configurations, and vertical mathematising, mainly in elaborating justifications. We initially perceived a horizontal mathematising in the general exploration of the problem with the formulation of conjectures, represented by four configurations. Then, the participant focused on each of the configurations and sought to geometrically construct the division into five equal parts, in a process in which she had to mathematically think about the geometric implications and the calculation of areas to try to build the solutions, characterising a vertical mathematising. The possible constructions were carried out with a ruler and compass, mobilising the participant's knowledge of Euclidean geometry.

While configuration 2 might seem like a failed attempt at a solution, it did generate a way to divide the area into four equal parts. This way does not answer the problem but is part of mobilising the participant's thinking while investigating solutions to the problem.

As for Advanced Mathematical Knowledge, the reflection the participant shared in answer to question "b" confirms that she mobilised geometric knowledge reviewed in graduation and geometric constructions with ruler and compass, which were used as the main resolution strategy and consist of skills acquired during higher education.

The second question selected for analysis has a statement according to Figure 5.

Figure 5

Second Question. Prepared from the OBMEP question bank (2007)

2) A factory produced an original calculator that performs two operations:
- The usual addition $+$
- The operation $\circledast$

We know that for every natural number a , we have:

i) $a \circledast a = a$

ii) $a \circledast 0 = 2a$

and, for any four naturals $a,\ b,\ c$ and d

iii) $(a \circledast b) + (c \circledast d) = (a + c) \circledast (b + d)$.

a) What are the results of operations $(2 + 3) \circledast (0 + 3)$ and $1024 \circledast 48$?

b) Is it possible to determine the result of $48 \circledast 1024$? Explain your answer.

c) Check for which values of $a,\ b$ and n the following equalities are satisfied:

iv) $b \circledast a = 2b - a$

v) $na \circledast a = (2n - 1)a$.

d) What knowledge or skills did you mobilise in solving this issue? At what level of your schooling were these topics learned? Or were they learned throughout your professional performance?

In this question, the research participant detailed step by step the resolution of the requested items; for the analyses, we will highlight the most relevant excerpts.

In solving the first item, property iii) was used combined with i) and ii). The first operation was easily verified by just applying the properties; the second operation, given by $1024 \circledast 48$, demanded a process of Mathematical

Creativity and Advancing Mathematical Activity to the presented resolution, as illustrated in Figure 6.

Figure 6

Partial Resolution of the Operation **1024⊛48**[9]

De acordo com iii),

$$\overbrace{(a+c)}^{1024} \circledast \overbrace{(b+d)}^{48} = (a \circledast b) + (c \circledast d)$$

Uma das alternativas seria desmembrar 1024 em duas parcelas a e c e desmembrar 48 em duas parcelas b e d tais que $a = b$ e $c = d$. Isso não é possível, pois se $a = b$, então $c \neq d$, já que $a + c = 1024$ e $b + d = 48$. Assim, devemos ter $b = 0$ e/ou $d = 0$ (não adianta ter $a = 0$ ou $c = 0$ porque não sabemos se a operação ⊛ é comutativa). Se $b = 0$ e $d = 0$, não teríamos $b + d = 48$. Então, devemos ter $b = 0$ $\boxed{\text{ou}}$ $d = 0$. Qualquer uma das alternativas resolve a questão.

- Se $b = 0$, então $d = 48$. Assim, devemos ter $a = 976$ e $c = 48$.
  Então $(976 + 48) \circledast (0 + 48) = (976 \circledast 0) + (48 \circledast 48) = 2.976 + 48 = 2000$.
- Se $d = 0$, então $b = 48$. Assim, devemos ter $a = 48$ e $c = 976$.
  Então $(48 + 976) \circledast (48 + 0) = (48 \circledast 48) + (976 \circledast 0) = 48 + 2.976 = 2000$.

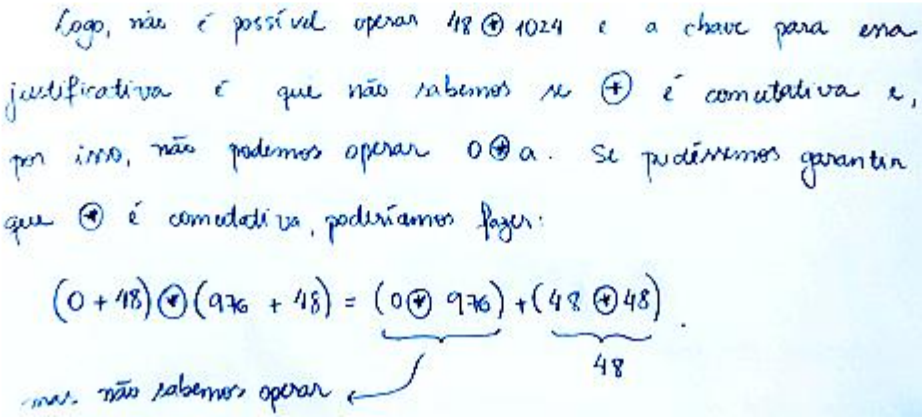
[9] According to iii), [...]. One of the alternatives would be to split 1024 into two parcels a and c and split 48 into two parcels b and d such that $a = b$ and $c = d$. This is not possible, since if $a = b$, then $c \neq d$, since $a + c = 1024$ and $b + d = 48$. Thus, we must have $b = 0$ and/or $d = 0$ (it is not worth having $a = 0$ or $c = 0$ because we don't know if the operation ⊛ is commutative). If $b = 0$ and $d = 0$, we would not have $b + d = 48$. Then, we must have $b = 0$ or $d = 0$. Either alternative resolves the issue. If $b = 0$, then $d = 48$. Thus, we should have $a = 976$ and $c = 48$. So […]. If $d = 0$, then $b = 48$. Thus, we should have $a = 48$ and $c = 976$. So […].

In this part of the resolution, we verify that there is a process of Mathematical Creativity based on Nadjafikhah, Yaftian, and Bakhshalizadeh (2012), as the participant searches for possibilities that can validate the given property. Thus, the answer to be obtained is not evident. Mental processes are required, interacting flexibly and creatively through conjecture formulation and hypothesis testing. The abilities to analyse a problem from different perspectives, identify patterns, produce multiple ideas, and choose a suitable method for dealing with an unfamiliar mathematical situation are indications of Mathematical Creativity.

There is also a formalisation process in the resolution presented. At first, there was a horizontal mathematising and, from interacting mental processes, there was a vertical mathematising and the exposition of the resolution of the question, which represent their understanding of the solution presented through a reflective relationship, in which we verify abstract relationships and formalisation of thought according to the Advancing Mathematical Activity, according to Rasmussen et al. (2005). Next, Figure 7 resumes the resolution of item b.

Figure 7

Justification of the Operation $\mathbf{48 \circledast 1024}$[10]


Logo, não é possível operar 48 ⊛ 1024 e a chave para essa justificativa é que não sabemos se ⊛ é comutativa e, por isso, não podemos operar 0 ⊛ a. Se pudéssemos garantir que ⊛ é comutativa, poderíamos fazer:

$$(0+48)\circledast(976+48) = \underbrace{(0\circledast 976)} + \underbrace{(48\circledast 48)}_{48}.$$

mas não sabemos operar

[10] So, it is not possible to operate 48⊛1024 and the key to this justification is that we don't know if ⊛ is commutative, so we can't operate 0 ⊛ a. If we could guarantee that ⊛ is commutative, we could do: [...] (but we don't know how to operate).

In this way, we verified that the participant exposed her ideas in a logical sequence to prove the arguments and present the solutions and algebraically generalised the values to investigate the possibilities. She showed mathematical knowledge of algebraic structures by inferring that operation $0 \circledast a$ is not defined; therefore, it would not be possible to operate $48 \circledast 1024$. Hereby, we understand that the mental process of synthesis was used to order the conjectures formulated in the resolution process.

By presenting this justification, the participant evidences knowledge of the validity of properties related to Algebraic Structures, that belongs to the AMK, according to Zazkis and Leikin (2010). This fact is reinforced in the justifications given in item d), in which the participant highlights that she resorted to "notions of operations and their properties", stating that perhaps it would not be possible to solve them without this knowledge, especially regarding commutativity. In item c), the participant presents a formal proof for properties iv) and v), in a way that reinforces the use of Mathematical Creativity, Advancing Mathematical Activity, and Advanced Mathematical Knowledge.

We can see that the evidence of the student's mathematical thinking, analysed with each theoretical reference, is different in each question or resolution strategy adopted by her. This result is in line with Sousa and Almeida's (2017) remark when analysing a mathematical modelling activity, that types of thinking appear when required:

> [...] there are interactions between the cognitive processes that seem to reveal nuances of elementary mathematical thinking, on one hand, and on another, nuances of advanced mathematical thinking. Neither one nor another prevails, since these types of thoughts take place when required, or through the cognitive structure of the student, or by the developed activity. (Sousa & Almeida, 2017, p. 721).

Next, we comment on the written production analysed and the theoretical references discussed.

FINAL CONSIDERATIONS

Through this research, we sought to present contributions to researchers in the area, both in the form of dissemination of the theoretical references related to the AMT and more recent ones, and decisions related to the foundation of a research. Our objective was to sum up three studies involving Mathematical Creativity, Advancing Mathematical Activity, and Advanced Mathematical Knowledge, respectively, and to illustrate possible contributions of those theoretical references in the analysis of a written production.

Despite the many tests and resolutions we shared, we were surprised by the resolutions the participant presented as they contained strategies different from those we had predicted. This confirms that we chose well the questions, allowing for multiple solutions and favouring evidence of the participant's thinking.

After analysing the resolutions of the questions from the different theoretical perspectives and the relationships established between them, we realised that these references help us see some of the aspects in the resolutions presented by the participant. None of them sheds light on all the thinking mobilised in the activity, but each one focuses on specific aspects. The AMT highlights the thinking processes developed by the student but does not show the "failed" attempts, the diversity of solutions explored, the evolution of thinking during the activity, or the advanced knowledge mobilised.

Mathematical Creativity, in turn, emphasises the various solution attempts and the mathematical thinking involved in comparing them and looking for a simpler one, in addition to the thinking involved in solutions that stand out for being more different from the usual. The Advancing Mathematical Activity highlights the steps taken and how the resolution was organised and thought. It also highlights the evolution of thinking during the resolution, including the formulation and investigation of conjectures. Advanced Mathematical Knowledge is interesting to discuss elements of the undergraduate curriculum that have been mobilised.

With the synthesis and established relationships, we could verify the researchers' reasons for adopting a particular theoretical reference. The analysis of a written production allowed us to confirm the differences in the perspective highlighted in each referential, leading us to situations in which each referential may be more appropriate.

These conclusions encourage us to be more careful when faced with similar theoretical references. Before inferring that an author is saying the same thing as another with different words, we need to analyse the consequences of adopting one or another theoretical reference in a study, from the epistemological roots to an emphasis or change of perspective caused by the careful choice of terms and definitions.

Beyond the grounds for research in Mathematics Education, we can highlight the implications of using theoretical frameworks for the analysis of students' production in teachers' practice. As illustrated in the analysis presented, the mastery of theoretical frameworks related to learning can help teachers evaluate their students since it allows noticing different thinking processes and knowledge moved by the students. In addition to the evaluation, Bianchini and Machado (2015) remind us that explicit knowledge of the AMT processes, by supporting reflection on knowledge, collaborates with teaching planning.

Furthermore, in teaching practice, teachers do not usually adopt a single theoretical reference to analyse student productions, as we do in research. Instead, teachers' knowledge and knowings interact to support the teacher's perspective of analysis so that different theoretical bases studied in their education can help them. Therefore, we consider that establishing relationships between theoretical references is essential to encourage the use of these references in teacher education.

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AUTHORSHIP CONTRIBUTION STATEMENTS

OI, DCBK, and ACCC organised and analysed the data. AMPDS supervised and led the planning and execution of the research. All authors actively participated in conceptualising the research, funding, investigation, provision of resources, development of methodological procedures, project management, presentation, writing, and review of the work.

DATA AVAILABILITY STATEMENT

The data that support the results of this study will be made available by the corresponding author, ACCC, upon reasonable request.

REFERENCES

Bezerra, W. W. V., Gontijo, C. H., & Fonseca, M. G. (2021). Promovendo a Criatividade em Matemática em Sala de Aula por Meio de *Feedbacks*. *Acta Scientiae*, *23*(1), 1-17.

Bianchini, B. L., & Machado, S. D. A. (2015). Em busca de elementos que propiciem ao professor de Matemática a reflexão sobre seu saber. *Acta Scientiae*, *17*(1), 28-39.

Bogdan, R. C., Biklen, S. K. (1994). *Investigação qualitativa em educação.* Porto

Dreyfus, D. (2002). Advanced Mathematical Thinking Process. In D. Tall. *Advanced Mathematical Thinking*. (pp. 25-41). Kluwer.

Ervynck G. (2002) Mathematical Creativity. In D. Tall. *Advanced Mathematical Thinking*. (pp. 42-53). Kluwer.

Mateus-Nieves, E., & Jimenez, C. A. R. (2020). Mathematical generalization from the articulation of advanced mathematical thinking and knot theory. *Acta Scientiae*, *22*(3), 65-82.

Nadjafikhah, M., Yaftian, N., & Bakhshalizadeh, S. (2012). Mathematical creativity: some definitions and characteristics. *Procedia-Social and Behavioral Sciences, 31*, 285-291.

Martineau, S., Simard, D., & Gauthier, C. (2001) Recherches théoriques et spéculatives: considérations méthodologiques et épistémologiques. *Recherches qualitatives*, *22*(3), 3-32.

Olimpíada Brasileira de Matemática das Escolas Públicas. (2007). *Banco de Questões*. http://www.obmep.org.br/banco.htm

Rasmussen, C., Zandieh, M., King, K. & Teppo, A. (2005). Advancing Mathematical Activity: A Practice-Oriented View of Advanced Mathematical Thinking. *Mathematical Thinking and Learning*, *7*(1), 51-73.

Sousa, B. N. P. A., & Almeida, L. M. W. (2017). Mathematical thinking in mathematical modelling activities. *Acta Scientiae*, *19*(5), 709-724.

Tall, D. (2002). The Psychology of Advanced Mathematical Thinking. In Tall, D. *Advanced Mathematical Thinking*. (pp. 3-21). Kluwer Ac.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with particular reference to Limits and Continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Vidotti, D. B., & Kato, L. A. (2018). Um Estudo sobre Conflitos no Processo de Aprendizagem de Limite de Funções de Várias Variáveis. *Acta Scientiae*, *20*(5), 930-949.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: perceptions of secondary mathematics teachers. *Mathematical thinking and learning*, *12*(4), 263-281.