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Experimental Geometry, Pythagorean Theorem and Montessori

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ABSTRACT

Background: Some 19th and 20th-century geometry textbooks show methodological innovations in relation to the traditional books that followed Euclid's logical-deductive proposal. **Objective:** To analyse in the book *Psico Geometria: el estúdio de la geometria basado en la psicologia infantil* by Maria Montessori (1934) as the didactic approach of the Pythagorean theorem, proposed by her, fits into an experimental geometry perspective. **Design**: Using documentary analysis, we analysed the books of the following authors: Hoüel (1867), Méray (1874), Laisant (1898, 1906), Calkins (1861), Bert (1886), Prestes (1895), Lyra da Silva (1923), Wentworth and Hill (1901) and Montessori (1916, 1934) to understand the approach to teaching geometry. **Environment and participants**:-The research sources collected in the National Library of France (BNF) and in the Universidade Federal de Santa Catarina (UFSC) Digital Repository refer to the geographic environment of authors from four countries **Data collection and analysis:** The selected and analysed documents can be found at BNF – Gallica in a digitized version, as well as at the UFSC Digital Repository. Based on the documentary analysis of these books, we found that the methodological proposals seek to escape from a deductive presentation of elementary geometry and include in their texts experiences for the introduction of concepts and visual demonstrations of elementary geometry. **Results:** The authors of the analysed works, from different countries, criticized the elementary teaching of geometry, especially the authors of textbooks, for the fact that they approach deductive geometry in the initial classes, favouring a deductive presentation of mathematics. Montessori's book shows an experimental geometry proposal for the Pythagorean theorem. **Conclusion**: We found that Montessori's proposal, besides presenting characteristics of an experimental geometry, also inserts some deductive reasoning.

Keywords: Geometry Teaching, Montessori, Pythagorean Theorem

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Geometria Experimental e o teorema de Pitágoras por Montessori

RESUMO

Contexto: Alguns livros didáticos de geometria dos séculos XIX e XX mostram inovações metodológicas em relação aos tradicionais livros seguidores da proposta lógico-dedutiva de Euclides. **Objetivo**: Identificar em livros didáticos de autores franceses, italianos, americanos e brasileiros a presença de uma geometria experimental e verificar no livro *Psico Geometria: el estúdio de la geometria basado en la psicologia infantil* de Maria Montessori (1934) como a abordagem didática do teorema de Pitágoras, por ela proposto, enquadra-se numa perspectiva de geometria experimental. **Design**: Usando a análise documental, analisamos os livros dos seguintes autores: Hoüel (1867), Méray (1874), Laisant (1898, 1906), Calkins (1861), Bert (1886), Prestes (1895), Lyra da Silva (1923), Wentworth e Hill (1901) e Montessori (1916, 1934) para compreender a abordagem do ensino da geometria. **Ambiente e participantes**: As fontes de pesquisa coletadas na Biblioteca Nacional da França (BNF) e no Repositório Digital da Universidade Federal de Santa Catarina (UFSC) referem-se ao ambiente geográfico de quatro países e os participantes são os autores dos livros de geometria. **Coleta e análise de dados**: Os documentos selecionados e analisados encontram-se na BNF – Gallica em versão digitalizada, assim como no Repositório Digital da UFSC. Constatamos a partir da análise documental de tais livros que as propostas metodológicas procuram fugir de uma apresentação dedutiva da geometria elementar e incluíram em seus textos experiências para a introdução de conceitos e demonstrações visuais da geometria elementar. **Resultados:** Os autores analisados, oriundos de diferentes países, manifestaram em suas críticas um certo descontentamento com o ensino elementar da geometria, principalmente com os autores de livros didáticos pelo fato de eles abordarem a geometria dedutiva nas classes iniciais, privilegiando uma apresentação dedutiva da matemática. **Conclusão:** Identificamos em todos os autores analisados uma proposta de geometria experimental e constatamos que a proposta de Montessori também insere alguns raciocínios dedutivos.

Palavras-Chave: Livros de Geometria; Montessori; Teorema de Pitágoras.

LAUNCHING ANCHORS FOR THE FUTURE

The term experimental geometry is not new. It was used in a 19th century book by Paul Bert[1] (1886). In his view, children in primary school had little taste for geometry and the reason for this lay in the way in which students received the first notions of geometry – an “endless parade of definitions that do not serve at all to interest a child” (Bert, 1886, p. v). It doesn't make much

[1] *Premiers Éléments de Géometrie Expérimentale appliquée* à la mesure des longueurs, des surfaces et des volumes. Paris: Librairie CH. Delagrave, 1886.

sense to tell a ten-year-old child, according to him, that a surface has two dimensions, that a point has no dimension, and then immediately mark with white chalk a large point on the blackboard, which is showing a surface. The criticism was entirely justified because, until then, the definition of dimension was still involved in nebulous discussions. Although Bert does not define what he understands by experimental geometry, it is possible to infer that his presentation does not contemplate theoretical aspects of geometry, but limits itself to appealing to common sense and the evidence of observation.

Currently, an “experimental geometry” can be understood as a didactic approach that introduces geometric concepts and properties with support or articulated with experiences of a varied nature (such as manipulable objects, drawings, dynamic software for teaching, etc.). It contemplates, in particular, the introduction of more abstract concepts such as those that are at the base of geometry, such as point, line and plane and elementary properties such as the Pythagorean theorem with the help of experiments, which make use of manipulable materials such as cutouts of figures in paper or cardboard.

The objective of the present work is to analyze in the book *Psico Geometria: el studio de la geometry based on child psychology* by Maria Montessori (1934) how the didactic approach of the Pythagorean theorem, proposed by her, fits into an experimental geometry perspective.

According to Viñao (2008), textbooks have a direct relationship with the history of school subjects. For Choppin (2000, p.108), school textbooks are, above all, pedagogical tools that aim to assist in learning. They are, moreover, “depositories of knowledge and techniques that at a given moment a society believes that youth must acquire in order to perpetuate its values”. We will use textbooks and school manuals as synonyms.

The books we will deal with were selected from among those whose work we had already visited in previous research (Silva & Silva, 2019, 2020; Silva, 2021), who propose in their books changes in the teaching of geometry (either for pedagogical, epistemological or of mathematics itself), who were preferably mathematicians or mathematics teachers, as well as those identified by Bardin (2020) written mainly by mathematicians.

Condorcet (1743-1794), French mathematician and philosopher; Pestalozzi (1746-1827), Swiss pedagogue; Fröbel (1782-1852), German pedagogue, were three authors who strongly influenced education for decades. In Gasca's view (2015), the common aspect between them was the proposal of the presence of geometry side by side with arithmetic in the teaching of children from the first school stage.

The inclusion of geometric concepts was recommended by Fröbel:

> A truly human education is not possible without mathematics or, at least, without delving into the science of numbers, which must encompass, even if only in a small complement, some notions of shapes and volumes (Fröbel, 2010, p. 106).

In the 19th and early 20th centuries, some authors of mathematics books began to defend the idea of a more experimental teaching of mathematics, especially for elementary geometry, launching fierce criticisms of books intended for initial education. French authors can be named among them: Jules Hoüel, in 1867, in his *work Essai critique sur les princes fondamentaux de la Géométrie élémentaire*; Charles Méray, with the work Nouveaux éléments de géométrie, in 1874; Charles-Ange Laisant, in 1898, in his book *La Mathématique Philosophie Enseignement*; Jules Bert, with the book *Premiers Éléments de géométrie Expérimentale*, in 1886; Norman Calkins, with the book Primary Object Lessons[2], in 1861, Wentworth & Hill, in 1901, in the United States of America and, in addition, the Brazilians Prestes, in 1895, and Lyra da Silva, in 1923, and the Italian Montessori, in 1916 and 1934.

Calkins's book distinguishes itself from the group of French authors, as it is not a mathematics book, but a pedagogy or methodology book or manual for teachers and parents. Calkins seeks to create lessons for the intuitive method as proposed by Pestalozzi, of whom he is a follower.

Hoüel declares, in the preface, that his book is a protest against the texts of elementary geometry, which bring nothing new to improve the exposition of the first principles of this science. He says: "They are content to repeat the traditional phrases and to establish with a false apparatus of rigor the first theorems [...]." (1867, p. iii). According to this author, "geometry is based on the indefinable and experimental notion of the solidity or invariability of figures" (p. 37). He goes on to say that a confusion is made between axioms and abstract truths. Experience teaches properties that enjoy immediate certainty, but there are other, more hidden properties that need reasoning.

Hoüel, when reflecting on the teaching of elementary geometry, takes a clear position: if the objective of teaching mathematics is to offer a model of

[2] Translated into Portuguese by Rui Barbosa in 1886 with the title: First lessons on things: elementary teaching manual for use by parents and teachers.

inflexible logic, applied to certain principles, in order to achieve such a goal, its teaching must never depart from the rigor that distinguishes mathematics from other sciences, as this is an essential condition for this study to be fruitful.

In the study of geometry, for beginners, he suggests, instead of demonstrations, experimental verifications, analogy and induction, as a provisional exposition of the matter:

> [...] the student will be trained in graphic drawings, handling instruments, solving various problems of building plants and carpentry, building three-dimensional figures using wires or plastic clay, representing these figures using their projections, etc. (Hoüel,1886, p. 82).

He also states that geometry can be mixed with the teaching of whole numbers, which are represented by points distributed on a straight line or a plane.

Méray, when dealing with basic notions of geometry, such as the straight line, abandons the Euclidean definition and says that the straight line rests on "the idea of a line that comes from very elongated objects, but extremely thin in all other directions, as a very fine thread, the apparent light trace of a bright dot animated with a high speed [...]" (Méray, 1874, p. 2). In the preface to the first edition he says that: "The first origin of geometric truths is unquestionably experimental" (Méray, 1874, p. xiii). He also proposes that to construct a line passing through two given points, it is enough to use a ruler (p. 7). Méray strongly supports his proposal on the idea of movement. Thus, he presents: "The term displacement of a figure is often understood as a limited movement, by virtue of which it passes from a first called initial position to a second called final position" (p. 9). From the idea of movement, of displacement, he shows a geometry that is not static. Not all authors analyzed highlight this idea of movement, as in Calkins. However, Wentworth and Hill's book brings the idea of motion more clearly – a moving line, in general, generates a surface (Wentworth and Hill, 1901, p. 30).

Charles-Ange Laisant, in 1898, addressed the experimental character of mathematics in the book *La Mathématique Philosophie Enseignement* in which he proposes that the teaching of geometry to beginners should be supported by visualizations, observations and the performance of experiments, avoiding a deductive exposition, that which students are still unable to follow.

Calkins states that "the existence of the (mathematical) notion in the spirit arises from the perception of the similarities and differences between objects” (Calkins, 1886, p. 2). It brings fictional dialogues between teacher and students always based on observations and experience (Silva; Silva, 2020). When dealing with lines, he suggests to the students an experiment of connecting two different points, with a straight line, with a curve and a broken line, as shown in figure 1. With a very thin string, he proposes the measurement of the length of the string for each case. The children themselves must figure out which is the shortest way. And he asks: “Can you draw a line shorter than the straight line between two points?” (Calkins, 1886, p. 71).

Figure 1

Line. (Calkins, 1873, p. 70)

In Calkins' lessons on shapes, the proposal is a simultaneous presentation with flat and solid shapes (Frizzarini & Leme da Silva, 2016). In France, also in 1874, Charles Méray's book *Nouveaux éléments de géométrie,* brings innovations, he says that it is a didactic device to separate geometry into plane and spatial. To enable students to “read in space” (spatial geometry), he brings in a footnote: that the use of three-dimensional figures is essential to guide beginners (p. xii). The author distinguishes in geometry two types of propositions: 1) those that come from experience combined with abstraction[3] and 2) those more numerous that are obtained by reasoning, and which are almost the totality of rational geometry.

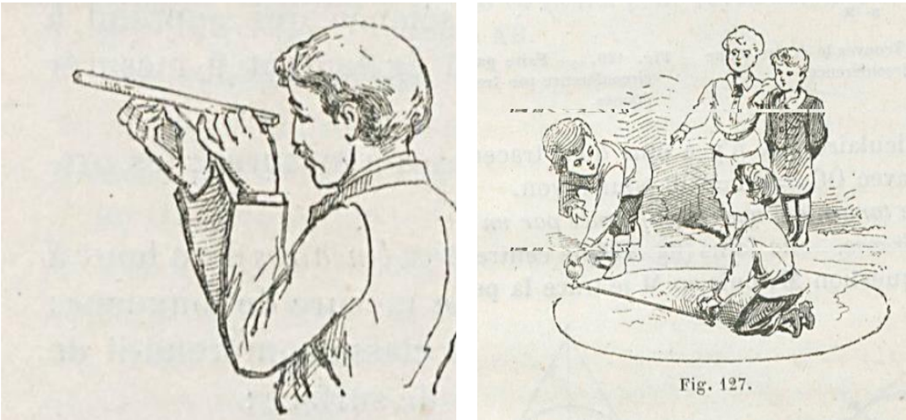
Bert, in 1886, proposes an approach to the first elements of geometry in which definitions appear as they become necessary and the same goes for demonstrations of elementary properties. His goal is to teach useful and practical things, such as measurements of figures, terrain, height of objects, etc. On the left of figure 2, taken from his book, he shows the notion of a straight

[3] In the third edition, he altered this wording a little: In the first type of proposition he placed the axioms, the certainty of which is given by simple observation or abstraction; the second type: theorems, which are unlimited in number end obtained by reasoning (Méray, 1907)

line that he defines as one that can be fixed as with a ruler; to the right of figure 2, children in the yard learning to draw a circle with a stake and rope.

Figure 2

The straight line (L). (Bert, 1886, p. 2). *Circumference drawing (R).* (Bert, 1886, p. 77)



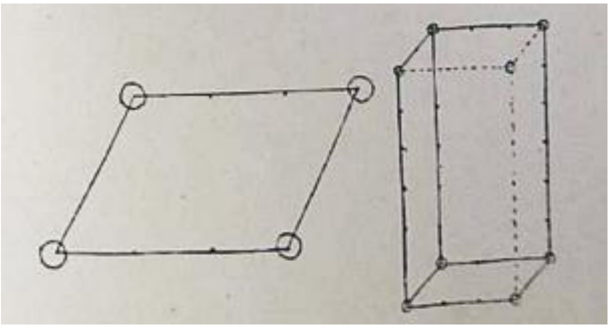
Charles-Ange Laisant, in the book *Initiation mathématique* (1906) returned to the theme of experimental geometry, already defended in 1898. In this book, Laisant makes a deep break with the traditional training of mathematics for the following reasons: different organization of contents, use of accessible language, use of many illustrations and informal representations and confidence in the mathematical mind of very young children (Gasca, 2015). In 1912, when responding to a survey on the role of intuition and experience in secondary education, he was against the approach, because according to him, it is incomplete and insoluble if we restrict the discussion to secondary education only and if we ignore the psychological and physiological aspects of brain development. According to Laisant: "the role of intuition and experience in education in general and in mathematics in particular is a major problem of pedagogy" (Laisant, 1912, p. 528).

At the turn of the 20th century, according to Bardin (2020), John Perry in England, in 1901, defended the teaching of geometry preceded by measurements of figures, where rules were experimentally tested, for example, the calculation of the length of the circumference of the circle. He presented two principles: 1) experimental geometry must precede demonstrative geometry; 2) some deductive reasoning that must accompany experimental geometry.

In Brazil, the author Gabriel Prestes (1895), director of the Escola Normal de São Paulo, in the book *Noções intuitivas de Geometria Elementar*, whose target audience were primary teachers, proposed the work for teaching in the second year, because, in the first, he suggested following intuitive teaching based on the Calkins or Prang system. Appropriating Bert's work, he wrote a book that is similar to the French proposal, as Silva (2019, p. 302) found: "[...] reversal in the march of content traditionally presented by proposing a direct study of line measurements, areas and volumes, use of materials such as threads, string, cardboard, paper cutouts, construction of solids for experimental work". The author did not use the expression experimental geometry in his book, he preferred intuitive geometry, perhaps focusing on the intuitive method, widely disseminated at the end of the 19th century in Brazil. In the preface, he quotes Clairaut and states that the teacher, when starting the teaching of geometry through a series of definitions, principles and postulates, which the child does not understand, contributes to fatigue the student´s spirit.

Figure 3

Construction of the parallelepiped. (Prestes, 1895, p. 77)



In the introduction, he postulates the need to distinguish, in the teaching of geometry, two parts: one achievable, which would be adequate to the level of children's intelligence, and another abstract, based on observations and capable of being developed by deduction. In his book, he shows this “achievable” part, which we interpret as an “experimental geometry”. For example, instead of drawing, in perspective, a parallelepiped on the board, which would confuse the students, he suggests using eight corks and twelve pieces of wire and carrying out the following experiment: with 4 corks and 4 pieces of wire (of equal length), build a square, as shown in figure 3, on the left; on the vertices of this square (in the corks, place 4 wires of equal length and greater than those of the square and secure vertically; at the ends, place another square like the previous one, as shown in figure 3 on the right).

To understand what a cylinder is and that it has 2 flat faces, he takes a log of wood and shows, approaching a piece of wooden board, how they adjust on the faces of the log, as shown in Figure 4.

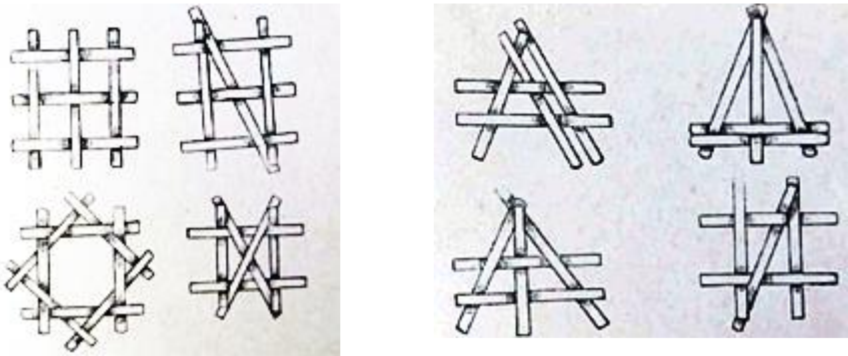
Figure 4

Flat cylinder faces. (Prestes, 1895, p. 69)

Prestes borrows ideas from Fröbel when he proposed the use of wooden stick intertwining. He states that to recap the contents of the relative positions of the lines, wooden stick combinations can be used (Figure 5). This way, they will better understand when two lines are parallel, perpendicular or oblique.

Figure 5

Wooden sticks. (Prestes, 1895, p. 55)



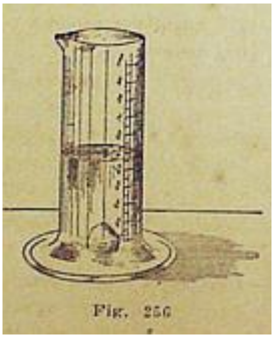
In 1923, Heitor Lyra da Silva, an engineer and professor of mathematics, published the book *Geometria (Observação e Experiência).* The word experience, which he uses in the book's title, appears in proposed activities, which can be exemplified as follows: to draw a right angle, he proposes using a carpenter's instrument or making folds on a sheet of paper; to build a pyramid, he suggests drawing a square on the ground, tracing the diagonals and placing a pole at the central point, driving four stakes into the vertices and covering everything with a canvas cloth – this is how the camping tent is formed; draw a circle, as gardeners do when building circular beds, using a stake and a string, calculate the area of any figure by means of a sensitive scale and cardboard; use graph paper to find approximate areas of figures; calculate the volume of any body, using a graduated cylinder and water; calculate the height of a tree, driving a stake into the ground and comparing the shadow of the tree with that of the stake, by the similarity of triangles (Silva; Silva, 2018).

In order to exemplify Lyra da Silva's proposal, figure 6 shows how he explains the calculation of the volume of any object through the experiment with a beaker or a glass of water:

> [...] immerse the body, mark the water level, remove the body and then pour water until the level returns to the height it was. As each gram of water corresponds to one cm^3, it is enough to check how many grams were poured until the level returned to its original height, in order to know the volume of the body.

Figure 6

Volume calculation. (Lyra da Silva, 1923, p. 138)

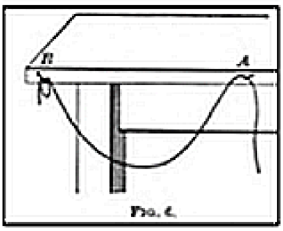


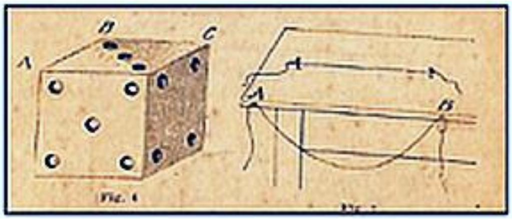
Lyra da Silva has sometimes appropriated formulations and examples that appear in the work of authors Wentworth & Hill, as stated by Silva (2021, p. 7), “To present the concept of a straight line, the figure used is very close to that of the American work”. Figure 7 shows two figures from both books: on the left, the straight line according to Wentworth&Hill and on the right to Lyra da Silva (Silva, 2021)

Figure 7

Straight line. (Silva, 2021, p. 7) (L) and (Silva, 2021, p. 8) (R)

Figura 1 – Linha reta





The aforementioned authors presented different approaches to an experimental geometry and were severe critics of deductive proofs of theorems, such as the Pythagorean theorem. Some of them exemplified how to deal with the Pythagorean theorem within an experimental geometry perspective. Some examples follow.

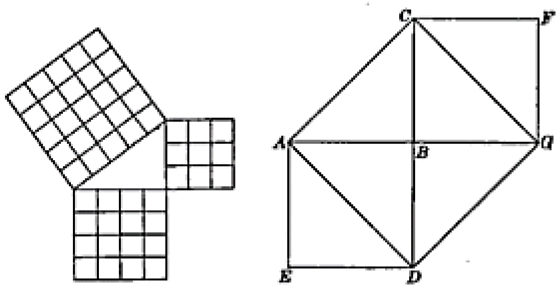
THE PYTHAGOREAN THEOREM

Criticisms of the demonstration of the Pythagorean theorem presented in textbooks at the turn of the 20th century were frequent and mobilized textbook authors to propose more visual and experimental demonstrations, escaping the deductive rigidity that discouraged students from studying elementary geometry. The proposals were varied, however, by way of example, we present only a few authors.

The American authors Georg Albert Wentworth and G. A. Hill published, in 1901, the work *First Steps in Geometry*. According to Silva (2020), there is an explicit concern in this work to present the first steps of geometry, prioritizing practical and experimental teaching. In the case of the Pythagorean theorem, the authors comment that it is a result discovered more than 2000 years ago by the Greeks and present two visual demonstrations that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the legs (Fig. 8).

Figure 8

Pythagorean theorem. (Wentworth & Hills, 1901, p. 104)

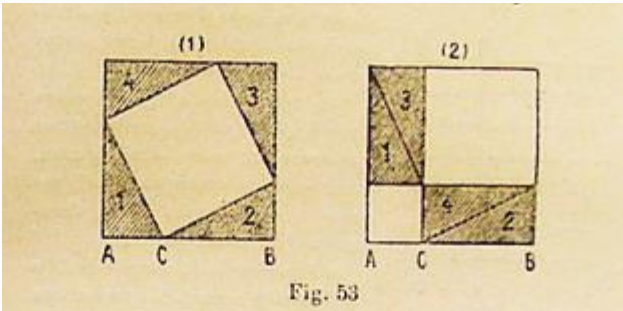


In figure 8 on the left, he says that, if the squares constructed under sides 3, 4 and 5 have respectively 9, 16 and 25 units, and 25 = 9 + 16, the theorem is naturally true. Figure 8, on the right, illustrates the theorem in the case of a right-angled triangle ABC having sides AB and BC equal, in this case, angles C and A each measure 45º. If we construct on these sides the squares ABDE and BCFG, and draw DG and AD and CG, we obtain a figure ACFGDE divided into six equal right-angled triangles, each of which is equal to the triangle ABC. The square built on AC contains 4 such triangles and each of the squares built on CB and AB contains 2 such triangles, so the theorem holds in this case.

Laisant comments, in the final exposition of his book *Mathematical initiation* (Laisant, 1906, p. 167-168), that since the Greeks, therefore for several centuries, a "[...] tiring, anti-rational method that discourages and disgusts students, especially beginners" is used. Supported by Méray's ideas, his book introduces a proposal in which the student no longer passively receives the teacher's knowledge, but is also an active protagonist in this process. He proposes in one of the subtitles of his book the suggestive expression – "we become geometers". In that chapter, after presenting the areas of various polygons, he introduces the Pythagorean theorem stating that this theorem has been "the torment of many generations of students" and the reason for this is that the classical proof is unnatural and difficult to retain. The experience he proposes is a demonstration, like a puzzle, in which the student builds pieces of wood or cardboard in the shape of triangles and squares, as shown in figure 9.

Figure 9

Pythagorean theorem. (Laisant, 1919, p. 77)



Laisant suggests building the puzzle in wood or cardboard, considering the square whose side is AB, where a point C is marked between A and B. Having AC and CB as the lengths of the sides of the right angle, building 4 triangles and designating them as 1, 2, 3 and 4 (as shown in figure 9 on the left). We can see that it forms a drawing that shows a square inside, which has the hypotenuse on its side. Now arranging these so that they occupy the position indicated in figure 9 on the right, we have two squares – the squares built on the sides of the right angle. Hence the two together have the same area as the square of the hypotenuse of the figure on the right.

Méray's book had three editions[4]. In 1903, the second appeared; the third, in 1907. In the first two editions, the Pythagorean theorem is obtained as a particular case extracted from any triangle, using the concept of projection. In the third, in addition to the deductive proof, he praises and cites the visual proof that Laisant proposed, in 1906, of the Pythagorean theorem.

In the 1930s, Maria Montessori released two specific books for teaching mathematics, one of them entitled *Psycho-geometry*: the study of geometry based on child psychology, in which she addressed the Pythagorean theorem (Montessori, 1934). In it, the author presents proofs of this result.

MONTESSORI E O EMPIRISMO

> [...] the mind must first cling, in first place, to some reality and then proceed afterwards in a purely logical field (Montessori,1934, p.8).

Montessori's proposal for the introduction of geometry starts from the following premise – the teacher must start from "things", that is, from concrete representations of geometric objects. Montessori is an empiricist, who values experience as a generator of human knowledge, built a teaching proposal in which student learning takes place from the periphery – with sensory activities and the use of manipulative materials – to the center, the child's mind (Silva, 2021). She fits into the pedagogical current initiated in the 19th century of valuing teaching based on experiences. Montessori as a psychiatrist and pedagogue was concerned with the child's learning, and that is why she emphasizes so much the method that the teacher must follow to succeed in the process of teaching and learning.

[4] We located 3 editions in the digital library BNF: Gallica.

Like the authors analyzed earlier, Montessori was critical of the traditional teaching of geometry. She stated that the teachers' concern was to be able to transmit abstraction to the child's mind as quickly as possible, because, without this, the essence of teaching would be lost, whose purpose is to “raise the mind to the fields of abstraction” (Montessori, 1934, p. 8). For her, it is only exceptionally possible to penetrate the child's mind.

Montessori suggests the use of a material for a study of geometry in schools - a manageable material, which, according to her, makes it possible to induce the mind to reason.

> It is the offer of the periphery and not the direct action on the center, which characterizes our method and differentiates it from the others. Instead of resorting to the power of reasoning comprehension and mental mechanisms to transmit a ready-made thing to the disciple's intelligence, we expose ourselves to its periphery, which is in contact with the environment, the means that lend themselves to a spontaneous exercise of the mind (Montessori, 1934, p. 65).

For Montessori, it is evident that the higher work of the mind has its origin in the material periphery (Silva, 2020).

She asks:

> Was it not from things, from which the first geometers got their knowledge? Was it not the correspondences and relation-ships between things which stimulated some active and interested mind to formulate axioms and hence theorems? How did Pythagoras obtain his famous theorem that infinite generations were content to make use of it to apply it as one who makes use of a received inheritance? It is difficult to understand the proof of that theorem for most schoolchildren, because their mind is passive, closed (Montessori, 1934, p. 64-65).

MONTESSORI MANIPULABLE GEOMETRIC FIGURES

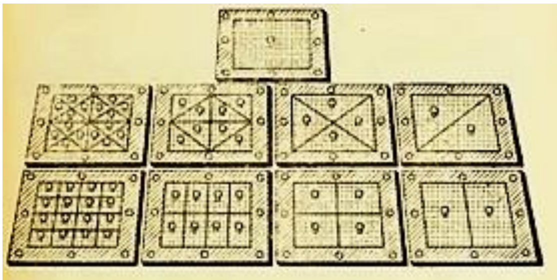
Maria Montessori became very popular with the teaching materials she produced. In 1916, she published *L'Autoeducazione nelle scuole elementari*. The following year, this work was translated into English and published in two volumes. The second volume - *The Montessori Elementary Material* - had a chapter on geometry and teaching this subject with manipulative materials. The book *Psico Geometria* of Montessori is the result of teaching experiences that the author had already carried out at *Casa dei Bambini* with the use of specific material for teaching geometry, built in iron molds and with colorful and interlocking pieces. In the 1916 book, the author explains that the main purpose of this material would be to facilitate children's self-education through geometry exercises, as well as to help solve real problems. The fact that the child can manipulate geometric figures, organize and evaluate relationships holds her attention. According to her,

> The child who exercises long and spontaneously in such means of development, not only continues to fortify his reasoning activities and strength of character, but he acquires superior and clear knowledge which enlarges his mind; in spontaneous and subsequent abstractions, it will have possibility of surprising progress (Montessori,1916, p. 432).

Entre esses materiais, encontra-se aquele que permite compreender a equivalência, semelhança e similaridade de figuras. Assim formado: um quadrado dividido em 2 retângulos, 4 quadrados iguais, 8 retângulos iguais, 16 quadrados iguais, 2 triângulos iguais, 4 triângulos iguais, 8 triângulos iguais e 16 triângulos iguais (Figura 10).

Figure 10

Subdivided squares. (Montessori, 1916, p. 435)

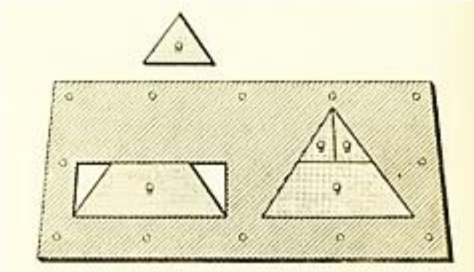


Among these materials, there is one that allows understanding the equivalence, similarity and likeness of figures. Thus formed: a square divided into 2 rectangles, 4 equal squares, 8 equal rectangles, 16 equal squares, 2 equal triangles, 4 equal triangles, 8 equal triangles and 16 equal triangles (Figure 10).

To demonstrate some results, for example the equality between the area of a rectangle and a triangle, with certain conditions, she created specific materials, like the one in Figure 11.

Figure 11

Relationship between the area of triangle and rectangle. (Montessori, 1916, p. 444)



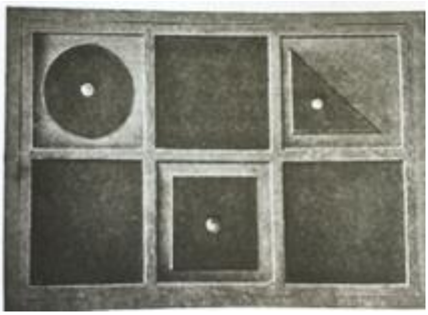
There is a variety of materials built on pieces of metal inserted into a plate, which serve to "demonstrate" geometric results. In this work, she included a section on spatial geometry in which she uses solid geometric figures such as: a quadrangular parallelepiped, quadrangular pyramid, triangular prism, cylinder, cone, ovoid, ellipsoid, cube, octahedron, etc. She concludes by saying that the stimulation given to the child with this material has multiple consequences for the methodical preparation of the child's intellect.

Montessori, unlike the aforementioned authors, prepares specific materials for the proposed activities that, in turn, aim at pre-established objectives. It states that every discipline, including mathematics, must be presented using external objects in a well-defined systematic construction. In this way, each achievement that the child makes with the "method of freedom" - that is, letting the child perform actions with objects at the appropriate time and stay in action until maturity - results in spontaneous abstraction (Montessori, 1916).

In the 1934 book *Psycho Geometry*, Montessori included this material with extensive explanations of usage. According to her, at the beginning of children's learning, we should not worry about definitions, but allow the child to exercise with the objects that surround them, concretizing the outside world through sensations and an intense motor activity. The material is called by her geometric molds. It consists of a larger square piece with a hollow space, which serves as a support to fit the smaller pieces (Figure 12).

Figure 12

Geometric templates. (Montessori, 1934, p. 16)



Como as peças são móveis, se pode retirar e recolocar no lugar oco. Com esse material concreto, a autora chama a atenção que o próprio material apresenta um *controle de erro*, que permite que a própria criança atue de maneira crítica, sabendo quando encaixou certo ou errado. O aparato se presta à comparação das figuras, e portanto "[...] é um material indutivo destas experiências de buscar, tatear e acoplar" (Montessori, 1934, p. 18).

As the parts are movable, they can be removed and replaced in the hollow place. With this concrete material, the author draws attention to the fact that the material itself has an error control, which allows the child to act in a critical way, knowing when it fits right or wrong. The apparatus lends itself to the comparison of figures, and therefore "[...] is an inductive material for these experiences of searching, groping and coupling" (Montessori, 1934, p. 18).

MONTESSORI AND THE PYTHAGOREAN THEOREM

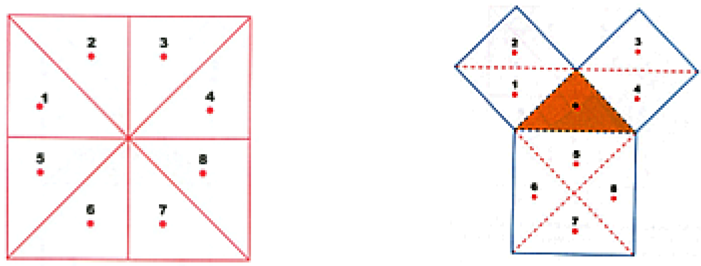
Montessori, when approaching the reasoning about right-angled triangles, in which she will deal with the Pythagorean theorem, she uses the word "demonstration" to show this relationship with her manipulative material. It is not a mathematical demonstration as we currently understand, but it can be read as showing, illustrating.

Although she widely uses the word demonstration, on some occasions she makes the reservation that it is a “material demonstration” (Montessori, 1934, p. 249).

So, start with the first demonstration for a triangle that has two equal legs. The material used by Montessori consists of fittings in an iron frame, with a hollow place where the wooden pieces fit. She uses a square whose side measure is 10 cm, which has been divided into 8 parts (8 triangles), as on the left of figure 13.

Figure 13

Pythagorean theorem. (Montessori, 2021, p. 221-222)



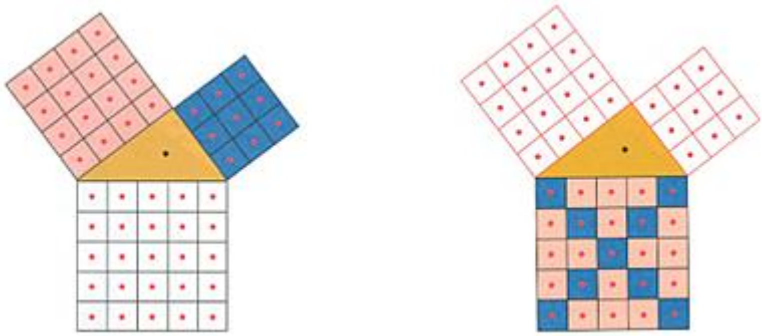
It can be seen that triangles 1 and 2; and 3 and 4 form squares on the sides of the triangle painted in yellow (figure 13 on the right). While triangles 5, 6, 7 and 8 were arranged on the side of the hypotenuse and form a square. The sum of squares 1 and 2; 3 and 4 form a rectangle which is half of the square of figure 13 and also the square formed by triangles 5, 6, 7 and 8 are the other half of the original square. This proves the Pythagorean theorem for the case of the isosceles triangle. The proof in figure 13 is similar to that of Wentworth and Hill.

For Montessori, material is not provided to demonstrate in concrete what is taught in an abstract way, manipulative material leads students to find evident relationships, to demonstrate or reveal, with their approach, other not so evident relationships.

The second demonstration is that of the right-angled triangle which is in a special condition, with measures of sides 3, 4 and 5 (this is also a proof similar to that of Wentworth and Hill, but the colors and arrangement of the colored squares in figure 14 appeal more to visuals). The idea remains to use the already known relations of areas of plane figures and establish for such a triangle the relation of the Pythagorean triple, as 3x3=9 squares, 4x4=16 squares and 5x5= 25 squares, fulfill the relation 25= 9 +16. Figure 14 illustrates the visual demonstration.

Figure 14

Pythagorean theorem. Montessori (2019, p. 223-224)



The material must be colored and made in the quantity that appears in the figures, that is, 16 squares of one color and 9 of another that will cover the square built on the hypotenuse.

The third demonstration: general case is also a visual demonstration. The fitting material needs a space for the right-angled triangle on which the squares will be built, so this space must be the same as in figure 15, and the removable and colored pieces.

Currently, the variability of this example can be visualized very easily with the help of dynamic geometry software.

It is necessary to show that the sum of the two red rectangles, or the red square, is equal to the sum of the square of the legs (in yellow).

The strategy is to remove the smaller square and move the right-angled triangle there so that the vertex of the triangle touches the end of the socket as in figure 16 on the left. The explanation is that the parallelogram (blue color) has the same area as the extracted square (one side of the parallelogram is the hypotenuse and the other the legs) to the edge of the frame (figure 16 on the right). The extracted square has the same area as the blue parallelogram in figure 15, what can be seen in blue is the space that was formed when the right triangle was moved.

Figure 15

Fitting Parts. (Montessori, 2019, p. 225-226)

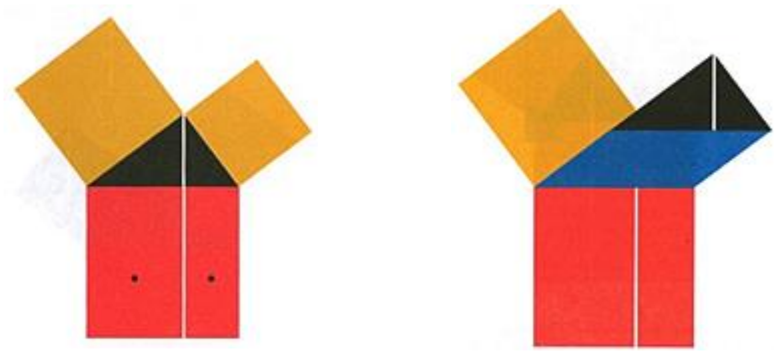
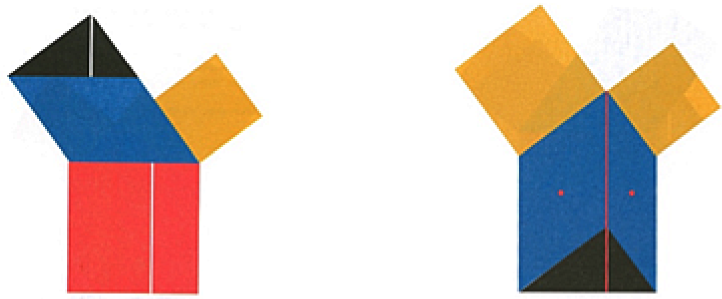


Figure 16

Displacement of triangles. (Montessori, 2019, p. 227-228)

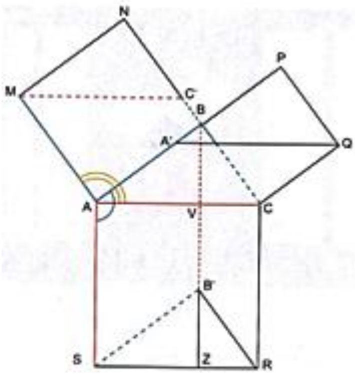


The third movement is to extract the larger square (hypotenuse side) and move the triangle until it coincides with the frame (figure 16). The extracted square has the same area as the blue parallelogram (figure 16). Moving the figures and filling in the spaces, she states that the Pythagorean theorem is "materially demonstrated" (Montessori, 1934, p. 252). It appears that the material demonstration uses the movement of the figures (pieces made of coloured wood) in the background frame. For more details of the demonstration, it is suggested to consult Montessori's book.

However, she does not just make this visual demonstration, she introduces a deductive reasoning, in which she will show that the square and the rectangle are equivalent to the constructed parallelograms. Figure 17 is used to carry out a fully deductive proof (Montessori, 1934, p. 254-255).

Figure 17

Deductive Pythagorean Theorem. (Montessori, 2019, p. 259)



In the introduction to the Russian translation of Montessori's book (2019, p. 8), Serguei Kurganov states that in the pedagogical approach of Montessori's Psychogeometry:

> [...] the teacher alone does not transmit mathematical ideas to the child. The mind of the child that is awakening, opens them each time in its own way, clarifying and deepening the understanding of mathematics.

Departing a little from what is usual in textbooks, she proceeds to generalize the result to other geometric figures such as the equilateral triangle (figure 18) and the isosceles triangle (figure 19) saying: "An equilateral triangle constructed on the hypotenuse is equivalent to the sum of equilateral triangles constructed on the legs" (Montessori, 1934, p. 258). Of course, she is referring to the areas of triangles, however it does not explain why the relation holds in the case of the equilateral triangle. Knowing that the area of a triangle is $\frac{b \times h}{2}$ and that the height h in the equilateral triangle is h= $\frac{l\sqrt{3}}{2}$, we will have for each side a, b and c (being the hypotenuse a and the legs b and c) the following areas: $a^2\frac{\sqrt{3}}{2}$, $b^2\frac{\sqrt{3}}{2}$, $c^2\frac{\sqrt{3}}{2}$. So : $a^2\frac{\sqrt{3}}{2}$=$b^2\frac{\sqrt{3}}{2}$+$c^2\frac{\sqrt{3}}{2}$, o que recai no teorema anterior. which falls under the previous theorem.

Figure 18

Pythagorean theorem. (Montessori, 2019, p. 235-236)

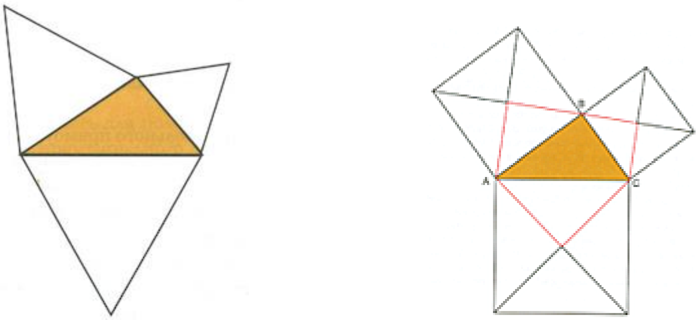
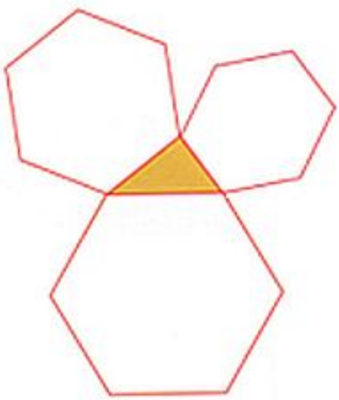


Figure 19

Pythagorean theorem – hexagon case. (Montessori, 2019, p. 23)

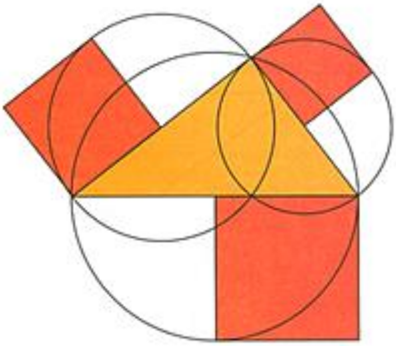


The explanation for the Pythagorean theorem, in the case of taking isosceles triangles built on the sides of the right-angled triangle, is justified from the particular case for the case of squares, since each isosceles triangle is the fourth part of the square built on the sides of the right-angled triangle (figure 18).

The author suggests that the proof for the case of the regular hexagon is made considering that the hexagons can be divided into 6 equilateral triangles and the proof for these is already known.

Figure 20

Pythagorean theorem – semi-circle case. (Montessori, 2019, p. 239)



For the semi-circle it justifies that the area of the circle is in a constant relationship with the square of the radius and if the circles are constructed so that each side of the right-angled triangle is its respective diameter, the Pythagorean relationship appears (Montessori, 1934).

Montessori's objective, when working with these manipulative activities, is to allow children to perceive existing relationships between the figures and, also, that this experimental study can prepare them for a systematic and deductive study of the discipline of geometry (Silva, 2021). The teaching of experimental geometry is propaedeutic for Euclidean geometry.

CONCLUSIONS

The revisited authors: Hoüel, Méray, Calkins, Bert, Laisant, Wentworth &Hill, Prestes, Lyra da Silva and Montessori bring elements of a pedagogical proposal with characteristics of an experimental geometry.

Among them, some present this approach more clearly, with many didactic examples, while others manifest themselves as defenders of such a proposal, but from the didactic point of view they contribute little. The authors analyzed, from different countries, expressed in their criticism a certain dissatisfaction with the elementary teaching of geometry, especially with the authors of textbooks because they approach deductive geometry in the initial classes, privileging a deductive presentation of mathematics. The concern of these authors with the writing of a textbook that effectively aims at learning is in line with the ideas of Choppin (2000).

In the 20th century, Montessori's books (1916, 1934) emphasize experimental geometry, justified by its empiricist theoretical basis and by the development of a systematic method supported by manipulable materials specially constructed for the child to handle and, through activities, reach their own conclusions.

Montessori was sure that she carried out two types of "demonstration", one that he called material ("it is materially demonstrated", she said), when she used the fittings of figures molded in colored wood, in which evidence prevailed over deduction, and another totally deductive, in which she said ("let's repeat the proof with reasoning"). In the case of the Pythagorean theorem, it is noteworthy that Pythagoras paid special attention to the right angle and distinguished it from the others. As an example of applying the concept of equivalence, she established a material and a deductive proof of the result between the sides of a right-angled triangle. In this sense, his methodological approach is similar to that of Pery, in England, starting with an experimental geometry and inserting some deductive reasoning. Montessori, when using geometric molds, figures to fit, manipulable material so that children reach the geometric results, performs what we call an experimental geometry, abundantly illustrated in his book *Psico Geometria*, in which he introduces movement in his boards made of iron and with the possibility of making displacements. Gradually she introduces deductions, so it is not possible to characterize her entire proposal as essentially experimental. Her defense of presenting an experimental geometry is based on child psychology and her understanding of the functioning of the child's brain - activities developed with manipulative material activate the mind and stimulate the strengthening of brain functions.

The hand that touches and handles objects was considered by Montessori to be essential in learning – hence a proposal for experimental geometry (Silva, 2021). In this sense, Montessori remains current because, according to research in neuroscience in recent years, Hughes, (2009), (Almeida and Justino (2020), Fabri and Fortuna (2020), the role of manipulating objects in neuro-psychological development is fundamental in learning. Its active pedagogical proposal is similar to those recent ones that use dynamic software, such as Cabri, Geogebra and others to perform visual geometric demonstrations with characteristics of an experimental geometry. The crucial difference is that the movement in the dynamic software is performed on a flat screen of a monitor, while in Montessori's proposal the hand manipulates objects and performs movements. A combination of both proposals can be an interesting pedagogical path for strengthening brain functions and consequent acquisition of new scientific knowledge.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study are openly available at the National Library of France – Gallica via the link https://gallica.bnf.fr/accueil/fr/content/accueil-fr?mode=desktop and the GHEMAT repository and via the link https://repositorio.ufsc.br/xmlui/handle/123456789/1769 in a format that allows them to be read and processed automatically by a computer.

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