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Geometry Teaching in the Early Years: A History that Encourages Reflections on the Present

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ABSTRACT

Background: Geometry seems to cause some strangeness and resistance when present in children's education over time and suggests that this characteristic is not exclusive to primary education or the initial years. **Objective**: To historically analyse two discourses that integrate proposals for teaching geometry present in current documents and programs, especially the *National Common Curricular Base* (BNCC) and the *National Pact for Literacy at the Right Age* (PNAIC): the plane-space relationship and the prominence of observation, manipulation, comparison, and visualization in learning geometry. **Design**: Based on themes for teaching geometry, proposals for teaching geometry in the early years and the historical relationships that can be established around them are discussed. **Settings and Participants**: current documents and programs that integrate proposals for teaching geometry in the early years. **Data collection and analysis**: Considering results obtained within the scope of research projects about the teaching of geometry in a historical perspective, located in the field of History of Mathematics Education and also in studies on the BNCC and the PNAIC, we proceeded with a new analytical elaboration. **Results**: Proposals for teaching geometry that last for many generations are highlighted, even with different objectives and purposes in each era. Observation, manipulation and comparison are also verified as strategies for teaching this theme that last over time. **Conclusions**: The dialogue between geometry teaching issues and its history of this teaching makes it possible to build a broader understanding of the difficulties and to elaborate proposals that can take into account the knowledge already produced.

Keywords: geometry; history of mathematics education; BNCC; PNAIC; early years.

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Ensino de Geometria nos Anos Iniciais: História que Incita Reflexões Sobre a Atualidade

RESUMO

Contexto: A geometria parece causar alguma estranheza e resistência quando presente na educação das crianças, ao longo do tempo, e faz pensar que tal característica não seja exclusiva do ensino primário ou dos anos iniciais. **Objetivo**: analisar historicamente dois discursos que integram propostas para o ensino de geometria, presentes em documentos e programas da atualidade, especialmente, a *Base Nacional Comum Curricular* (BNCC) e o *Pacto Nacional pela Alfabetização na Idade Certa* (PNAIC): a relação plano-espaço e a proeminência da observação, da manipulação, da comparação e da visualização para aprender geometria. **Design**: A partir de temas para o ensino de geometria, discutem-se propostas para o ensino de geometria nos anos iniciais e as relações históricas que podem se estabelecer acerca delas. **Cenário e Participantes**: documentos e programas da atualidade que integram propostas para o ensino de geometria nos anos iniciais. **Coleta e análise de dados**: Considerando resultados obtidos no âmbito de projetos de pesquisa acerca do ensino de geometria em perspectiva histórica, situados no campo da História da educação matemática e, também, nos estudos sobre a BNCC e o PNAIC, procedemos a nova elaboração analítica. **Resultados**: São evidenciadas propostas para o ensino de geometria que perduram por muitas gerações, mesmo tendo objetivos e finalidades distintas em cada época. Também se verificam a observação, a manipulação e a comparação como estratégias para o ensino desse tema que perduram ao longo do tempo. **Conclusões**: O diálogo entre problemáticas do ensino de geometria e sua história possibilita construir uma compreensão mais ampla das dificuldades e elaborar propostas que possam levar em conta os conhecimentos já produzidos.

Palavras-chave: geometria; história da educação matemática; BNCC; PNAIC; anos iniciais.

INTRODUCTION

The teaching and learning of geometry in the early years has long been a subject of discussion, and many topics associated with the approach to geometric concepts at this education level have been investigated.

We will analyze historical dimensions of geometry teaching that encourage reflections on the present, based on a number of topical questions: How is geometry currently taught in the early years? What geometric concepts and material are covered? Which methodological strategies are used? Additionally, we will explore other issues related to the way this area of mathematics has already been addressed in schools: Is geometry useful knowledge for students? At what education levels? Which aspects from the history of geometry teaching are still present today and which are not?

The introduction of geometry into Brazilian school culture is associated with its abstract nature and with the relationship between deduction in geometry and the development of logical reasoning, and this seems to have been a determining factor for the inclusion of this subject in the sectional exams[1] required for admission to the first law programs in Brazil, created in 1827 (Valente, 1999).

For elementary school, however, during that same period, practical geometry was part of the curriculum, together with the "reading, writing and counting" triad (Valente, 2012). The practical nature of geometry, demonstrated in this context by the design of geometric shapes, "is connected to the foundations to be acquired in school, for the exercise of professional trades: whether in surveying or in work with drawing, present, as Francoeur mentions, in a wide range of activities in the world of labor" (p. 91).

Based on these findings, our objective is to historically analyze two discourses found in proposals for teaching geometry that are present in current documents and programs, in particular, the National Common Curricular Base (*Base Nacional Comum Curricular - BNCC*) and the National Pact for Literacy at the Right Age (*Pacto Nacional pela Alfabetização na Idade Certa - PNAIC*): the plane-space relationship and the emphasis on observation, manipulation, comparison and visualization in geometry learning.

Considering results obtained in research projects on the teaching of geometry from a historical perspective, located in the field of the history of mathematics education and in studies on the BNCC and the PNAIC, we proceeded to develop a new analytical framework. Our working hypothesis is the enduring nature of certain aspects of the geometry teaching process up to the present day, at least in the propositions of official documents such as the BNCC. As a result, we constructed a new representation about the teaching of geometry in the early years, over time, problematizing the two aspects mentioned above. This representation contributed to the exercise of denaturalizing the current official proposals for the teaching of geometry.

[1] Sectional exams were tests administered by colleges of higher education that covered specific secondary school subjects selected according to the area—law, engineering or medicine—for which passing the specific set of subjects permitted access to higher education without the need for secondary school completion and certification.

DENATURALIZING THE TRAJECTORY FROM SPACE TO PLANE

There is a contemporary trend that proposes that the teaching of geometry in the early years begins with an exploration of the space in which the children are immersed, which will expand as they succeed in performing certain activities.

For Lorenzato (2011), based on the ideas of Piaget, children begin to perceive space based on visual images and then are able to pick up what they see. After that, they are able to move between objects and, finally, perceive themselves within that space. Additionally, they begin this process of exploring spatial relations using notions of vicinity, contour, order, separation and continuity.

Thus, "in the space within the child's reach, it is possible to explore geometric tasks such as putting together, taking apart, composing and decomposing, constructing and deconstructing. In the wider space, reached by the eye, in drawings or photos, we can move, orient and locate ourselves." (Romanatto & Passos, 2011, pp. 25-26). This space cannot be manipulated but rather represented through drawings, maps, sketches, etc.

Romanatto and Passos (2011) also emphasize that it is important to provide situations that allow children to manipulate objects so that they can situate themselves in space in relation to people and objects. To this end, it is possible to design activities that work with the notions of inside, outside, internal and external, which will allow children to learn the concepts of above, below, behind, in front, between, inside, outside, right, left, etc.

A teacher can work with different activities that involve these mathematical concepts, using the classroom itself. For example, the following questions can be asked: What is on the table? Next to the door? Who is sitting behind you? During these activities, it is important for the teacher to expand the student's learning, demonstrating that, for example, right and left depend on which referent is adopted.

It is also possible to work with location and movement in space by asking children to describe the path they must take to move from their desk to the teacher's desk and, expanding on that, how to navigate to the bathroom and to their home from school. They can represent these paths using drawings.

They can also explore spatial geometric figures by recognizing and naming their elements, such as faces and vertices, which are sometimes called points at this level. As Romanatto and Passos (2011) indicate, “by drawing the outline of one face of a cube, it will be possible to recognize that the plane figure that defines it is a square and, furthermore, that the cube has six faces, all equal” (p. 26). It is also possible to paint the faces of the cube and stamp them onto a piece of paper to perform this recognition, i.e., which plane figures compose that spatial figure.

Furthermore, an activity can be carried out with empty containers during which the teacher asks the children not only to separate them according to a certain criterion—such as those that roll and those that do not—but also to open those containers to make them into a plane.

The objective of teaching geometry should be for children to “move from the lived space to the imagined space. In the former, the child observes, manipulates, decomposes, and assembles, while in the latter, they operationalize and construct an interior space based on reasoning” (Lorenzato, 2011, p. 46), i.e., the transition from concrete to abstract. This perspective has been addressed in teacher training programs and in official documents.

In the PNAIC, one of the program’s actions was a continuing education program for early education teachers, which in 2014 focused on mathematics. Booklet 5[2] specifically addressed geometry, and its objectives included:

> To informally represent the position of people and objects and to depict spaces using drawings, sketches, floor plans, maps and models, developing notions of size, laterality, location, direction, meaning and views; to recognize their own body as a reference for locating and moving in space; to observe, manipulate, and establish comparisons between objects in the physical space and geometric objects without the need to use nomenclature, recognizing round and non-round bodies; to turn models of geometric solids into planes; and to construct models of solids from plane surfaces. (Brasil, 2014, pp. 5-6)

It is interesting to note that these booklets included experience reports from teachers who presented activities carried out in the classroom in their

[2] The mathematics material in the PNAIC is composed of 13 Booklets, of which one is an introduction, one is about mathematics education in rural areas, one is about inclusive mathematics education, two are about games and the remaining booklets address the mathematical concepts and material covered in the early years of primary education.

pedagogical practice. The texts that compose Booklet 5 contain proposals to use works of art, manipulable materials, containers, games, maps, etc.

The BNCC (Brasil, 2018) also appears to advocate this proposal, as among the different geometry skills that children need to develop, it is expected that:

> They can identify and establish reference points for the location and movement of objects, construct representations of known spaces and estimate distances, using maps (on paper, tablets or smartphones), sketches and other representations. With regard to shapes, students are expected to indicate characteristics of three-dimensional and two-dimensional geometric shapes, associate spatial figures with their planes and vice versa. They are also expected to name and compare polygons, using properties related to sides, vertices and angles. (p. 272)

In addition, some geometric transformations should also be practiced in the early years. In the PNAIC, one of the objectives of the booklet that specifically addresses geometry is to understand geometric transformations—rotation, reflection and translation—in order to create compositions such as banners, logos and animations (Brasil, 2014). However, no specific grade level is indicated for working with these concepts.

One of the chapters, titled "Dimension, Similarity and Shape," presents what the authors mean by these concepts and how they relate them to the meanings of those words in everyday life and in mathematics. It also discusses the meaning of symmetry, rotation and translation (Brasil, 2014).

In turn, the BNCC presents skills related to geometric transformations only in the 4^{th} and 5^{th} grades. In the 4^{th} grade, it addresses reflection symmetry in figures in general (an image of a butterfly, for example) and in plane figures using a grid and geometry software. In the 5^{th} grade, it introduces a skill related to the enlargement and reduction of polygonal figures on grids, with the use of technology (Brasil, 2018).

This proposal is based on the way that children perceive the world around them and begin to explore the space in which they live and modifies the usual way in which we learn geometry, where we start by focusing on concepts of plane geometry and then work with spatial geometry.

This approach to teaching geometry—to start with the plane—is sometimes associated with the way this area of mathematics is taught in higher education: it begins with Euclid's axioms and then constructs geometry using the primitive concepts of point, line, plane and so on.

Fonseca et al. (2011) argue that addressing these ideas in early education is related to a "study focused on the formal presentation of material to the detriment of the exploration of concepts" (p. 22). For Lorenzato (2011), if children initially perceive space in a topological perspective, to start teaching geometry based on Euclidean axiomatics may lead to difficulties in the learning process.

We thus understand that at this education level, exploring concepts is much more important than working with formality and mathematical rigor, which can be studied in later grades.

This space-plane trajectory was not always seen as the most appropriate for introducing children to the study of shapes and geometry in general. In the late nineteenth century, a period characterized by pedagogical references to the intuitive method[3], there were proposals that advocated both paths: starting with the study of the plane, the point and, from there, lines as a meeting of points; and a different path, which would begin with the study of spatial geometric figures, combining observation and the use of the hands, in an exercise involving manipulation and comparison.

Both approaches were advocated in a publication aimed at teachers, the pedagogical journal *A Eschola Pública* [*The Public School*], in articles published in 1896. The proposal presented by Professor Cardim (1896) was to begin with a point on a blackboard and progress through questions and answers to the study of lines, the formulation of the notion of dimension and so on incrementally. "The observation of the teacher's drawings on the blackboard and the dialogues encouraged by the teacher's questions are the essence of this lesson model for teaching notions of geometry" (Oliveira et al., 2020, p. 143).

In turn, Professor Thompson (1896) recommended that students' first contact be with the sphere, which should be compared to the cube. The comparison was seen as a possibility for understanding through differentiation. The students should experience a "pleasant sensation when holding the 'ball' in their hands, while the same will not occur with the cube. Only then, based

[3] The intuitive method affirms "the primacy of sensory experience as the starting point for knowledge and for the representation of reality" (Trouvé, 2008, p. 239, our translation).

on this feeling and the observation that the ‘ball’ rolls and the cube does not, would the solid be named as a sphere” (Oliveira et al., 2020, p. 143).

In 1926, a translation was published of a manual for elementary school teachers titled *Metodologia de Geometria* [*Geometry Methodology*][4]. Although the author initially presented a discussion of two perspectives for teaching geometry—the first, starting with more abstract entities such as points and lines, moving on to solids and surfaces; and the second, starting with solids and later exploring points and lines—the text chooses the “second approach, based on the idea that starting with solids and surfaces made it possible to ‘concretize’ geometric entities, indicating the path also recommended by the intuitive method: from the concrete to the abstract” (Oliveira et al., 2020, p. 166).

An important figure in Brazilian education, an exponent of the New School[5], Professor Lourenço Filho, in his lectures on the Methodology of Geometry, between 1922 and 1923, situated geometry as the experimental part of mathematics, concluding that it was a science of observation and experience that begins with the study of geometric solids (Bastos & Cavalcante, 2011).

We can formulate a hypothesis of the predominance of the space-plane proposal in the early twentieth century in Brazil, which would be characterized by the introduction to the study of geometry through spatial geometric figures, moving on to plane figures only at a later moment. However, Nascimento Silva’s (2018) analysis of articles addressing geometric knowledge in *Revista do Ensino de Minas Gerais* [*Minas Gerais Teaching Journal*], between 1925 and 1932, indicates two different proposals: one advocating

> the approach that we can call Euclidean, i.e., the path from the two-dimensional to the three-dimensional—from plane figures to spatial ones; as well as the opposite, starting with the study of spatial figures, recognizing and exploring faces, contours as plane figures. (Oliveira, 2018, p. 182)

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[4] The manual was written by the Argentine pedagogue Juan Patrascoiu, a supporter of Herbart’s ideas, and translated into Portuguese by Professor Osvaldo Pilotto. More details can be found in Camara and Pinto (2017).

[5] The New School was an international movement for education that essentially placed the child at the center of the educational process. It was also characterized by the production of educational sciences based on statistical, biological, psychological and medical-pedagogical knowledge.

The Modern Mathematics Movement (MMM), which spread in Brazil in the 1960s and 1970s, introduced proposals into school mathematics that were based, to a certain extent, on references from Bourbakian mathematics and the studies of Jean Piaget. In *La Represéntation de l'Espace chez l'Enfant*, published in 1947, Piaget indicates that children first pass through the topological stage and then through the Euclidean stage.

Adopting the topological perspective as the initial one for teaching geometry, Zoltan Dienes became an important reference in Brazil. However,

> the identification and characterization of plane and solid shapes continued to reflect proposals made since the early twentieth century. The alternation in the order of addressing the shapes, sometimes beginning with the spatial, sometimes with the plane, is explained in each period by pedagogical trends. In the MMM era, plane shapes precede spatial shapes. (Oliveira, 2018, p. 186)

Children's perception of space, conceived based on Piagetian references, still present today, leads us to ask the following: How would that perspective have reverberated in times past? How did Piaget's epistemology transform proposals for teaching notions of geometry in the Modern Mathematics era?

A guide for teachers titled *Ensinando Matemática a Crianças* [*Teaching Mathematics to Children*], published in 1963 by the National Institute for Pedagogical Studies (*Instituto Nacional de Estudos Pedagógicos - Inep*), was aimed at teachers, auxiliary teachers and future elementary school teachers. The introduction to the text emphasizes that the guidelines presented therein took into consideration "the psychological conditions, abilities and interests of the child and the achievements of the Psychology of Learning" (Inep, 1963, p. XIII). Although there are no explicit references to Piaget's psychogenesis or even to other psychology of learning theories, knowledge from this field is frequently mobilized in the text to support and guide teachers in the proposed activities. Two books by Piaget are included in the bibliography: *The Child's Conception of Number* (1950) and *The Initiation to Calculation* (1956). Other national and international books on child psychology are also included.

Figure 1

Excerpt from the guide Ensinando Matemática a Crianças [Teaching Mathematics to Children] (Inep, 1963, p. 52)[6]

Há interêsses básicos, próprios de cada idade, que o professor deverá conhecer para atingir com êxito a seus objetivos.

A criança de 7 anos é ativa, gosta de experiências novas e, principalmente, das que lhe permitem movimentar-se; por outro lado, aprecia trabalhos calmos, por exemplo, escrever, desenhar, recortar, e sente prazer em repeti-los várias vêzes. (Assim, o ouvir uma narrativa, realizar um tipo de jôgo ou exercício).

São motivos de atração, nesse período, os assuntos que se referem a animais, circo, jardim zoológico, histórias (de animais, da vida cotidiana, maravilhosas), brinquedos e jogos, dramatizações, cinema, teatrinhos, canções, desenho, pintura, festas. Demonstra ela também grande empenho por organizar álbuns e livrinhos, ornamentar a sala, construir casas, fazer coleções para utilizá-las nos momentos que desejar.

Partindo dessas preferências, várias atividades poderão ser desenvolvidas.

In this excerpt, possible activities for 7-year-old children are described, considering psychological and physiological aspects and interests for their age.

The Mathematics Program for the 1st grade, from this same book, indicates notions of position, distance, orientation and direction, which involve identifying what is above, below, to the left, to the right, inside, outside, in front,

[6] There are basic interests, specific to each age, that the teacher must understand in order to successfully achieve their objectives.

The 7-year-old child is active, enjoys new experiences and, in particular, those that allow them to move around; they also enjoy quiet tasks such as writing, drawing, and cutting out things, and take pleasure in repeating them several times (for example, listening to a story, engaging in a game or exercise).

Topics that refer to animals, the circus, the zoo, stories (marvelous ones, about animals, daily life), toys and games, dramatizations, movies, plays, songs, drawing, painting and parties are appealing during this period. They also show great interest in organizing albums and little books, decorating the classroom, building houses and collecting objects to use at the times they want.

Based on these preferences, a number of different activities can be carried out.

behind, in the middle, and next to, among other possibilities. Distance, in turn, should only involve comparison—closer or farther away—without concern for the unit of measurement.

This manual presents several suggestions for activities that would make it possible to connect "Children's interests and their use in learning mathematics," the title of Chapter III. Some of the suggested activities involve organizing the classroom, making toys, throwing parties to celebrate special dates, etc. Position, orientation, direction and distance are suggested as learning possibilities when carrying out these activities. For example, students would follow "oral orders" such as "Pick up the object that is on the right side of the table. Put it next to the vase" (Inep, 1963, p. 72).

Almeida (2021), in her analysis, identifies characteristics of the New School era in this manual because it considers observation as an important teaching resource, uses comparison as a teaching strategy, and proposes everyday situations for use in the teaching context as well as the use of everyday objects for teaching and hands-on tasks as part of students' activities. A question then arises: Does learning concepts related to location in space appear in New School proposals for teaching geometry?

Revisiting a few earlier studies that focused on teaching or the professional knowledge to teach geometry to children in the New School era, we found no indications in this regard.

In Nascimento Silva's (2018) dissertation, which analyzed professional knowledge for teaching geometry in *Revista do Ensino de Minas Gerais* in the period from 1925 to 1932, there are no references to location in space or even on the plane, nor to the notion of distance. The study by Cruz (2021), on textbooks published in the 1930s, in which the author sought to identify professional knowledge for teaching geometry in elementary school, also does not mention activities or proposals involving those concepts. Furthermore, Santos's (2022) doctoral thesis, which sought to describe the teaching of geometry in the New School era in the state of São Paulo, notes that "the guidelines for geometry were based on two main elements, observation through comparison and student action, as the guiding principles" (p. 123). These guidelines were focused on teaching plane geometry (lines, points, angles, diagonals, figures and areas) and spatial geometry (the study of geometric solids and their compositions), without any references to aspects related to location on the plane or in space. In Leme da Silva (2021), a study that summarizes the results of different research projects coordinated by the author on the teaching of geometry in elementary school, we also found no references to the study of location in school geometry.

However, we found references to the teaching of location terms such as right, left, below, and above in *Regimento Interno das escolas públicas do estado de São Paulo* [*Internal Rules for public schools in the state of São Paulo*], of 1894, in the subject of geography (São Paulo, 1894). Again for geography, in *Programa de Ensino para os grupos escolares do estado de São Paulo* [*Teaching Program for school groups in the state of São Paulo*] (São Paulo, 1918), there are similar indications but with a more comprehensive scope, involving broader notions of the cardinal points and location. The question is as follows: Would the study of notions related to location in space within geometry have been incorporated from geography?

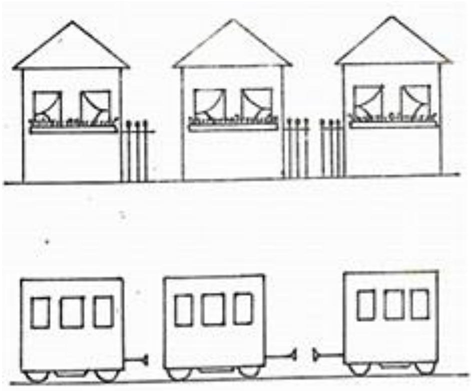
The study of geometry through transformations on the plane—symmetry, rotation, translation, homothety, similarity—was one of the key aspects of the MMM, which at the time encountered a certain level of resistance in the school culture. However, it was treated differently in educational publications and experimental teaching initiatives, particularly for the equivalent of the current final years of elementary school, in the states of Bahia and Paraná (Oliveira et al., 2011).

In elementary school, although the MMM ideology also mentioned the exploration of transformations as a basis for the study of the congruence of figures and the verification of invariants when dealing with movement on the plane and in space, there is little evidence that these proposals were incorporated and appropriated when examining textbooks, pedagogical manuals and magazine articles from the time. Almeida (2021) analyzes the possible use of translation and axial reflection in activities contained in *Introdução da Matemática Moderna na Escola Primária* [*Introduction of Modern Mathematics in Primary School*] due to the absence of statements accompanying the pictures. “The manual includes different drawings of objects, animals, people, and geometric figures, which suggest activities involving sets, increasing order, greater and lesser, addition and subtraction, geometric transformations, comparison and problems” (p. 47).

The two sequences in Figure 2 involve a translation followed by an axial reflection of the initial image.

Figure 2

Activity for observing the movement of the figure (Franchi & Libermanan, 1966, p. 71)



Although we know that geometry based on transformations was advocated in the Modern Mathematics era, there is still little research in Brazil that focuses on Brazilian discussions and appropriations of this topic in elementary school, as Silva et al. (2021) have noted.

GEOMETRY: SCIENCE OF OBSERVATION AND EXPERIENCE

The teaching of geometry in the early years should address aspects such as observation, manipulation, comparison and visualization. According to Romanatto and Passos (2011), activities should emphasize "manipulation, exploration, perception, comparison, connection, classification, construction, transformation and relationship with a wide range of experiences that lead to the discovery of space" (p. 25). Additionally, according to the authors, this work can contribute to the development of creativity, imagination and an aesthetic sense in children.

One activity that can be performed is to take advantage of the objects that are in the classroom or at the school and make comparisons with spatial geometric figures, i.e., "to relate spatial geometric figures to familiar objects in the physical world" (Brasil, 2018, p. 279), which is a skill present in the BNCC for the 1st year of elementary school.

This observation of the world around them and the comparison of objects to spatial figures can help students become familiar with these figures and begin to recognize their properties, for example, which plane figure forms the sides of a spatial figure.

In the early years of education, it is important for children to manipulate objects that help them situate themselves in the space in which they live, in their relationship with people and with different objects.

> This manipulation will allow them to recognize the inside and outside of a domain bounded by a line or a closed surface. Playing inside and outside of boxes (large containers) or standing inside and outside of a hula hoop placed on the floor, for example, are activities that situate them within this movement of recognizing inside and outside. It would be interesting to ask the children to assemble a collection of objects (brought from home or already at school) in order to manipulate them, line them up and step inside them. They need to experiment with cutting out pictures and objects that appear in magazines and recognize where they are located in the real world. (Romanatto & Passos, 2011, p. 26)

Another interesting activity mentioned above is the use of containers that children classify based on a certain criterion of their choosing, which can help in the recognition of properties—such as spatial figures that roll and those that do not.

Observation, manipulation and comparison can be addressed using manipulable materials such as Tangrams, geoboards, cutting out geometric figures from magazines, wooden or acrylic spatial geometric figures, and empty containers, among others.

These materials can also help with visualization, as drawing spatial figures, which are three-dimensional, on the blackboard—which is two-dimensional—can make it difficult to identify the figure or its properties, mainly because it is drawn in perspective and some faces, edges and vertices will be "hidden."

To work with this transition from the plane to the three-dimensional, activities can be proposed in which students are presented with drawings and asked to construct those drawings with sticks, as suggested by Nacarato and Passos (2003). Based on the ideas of Bishop (1979), Nacarato and Passos consider that recognizing the plane representation of three-dimensional objects is not self-evident to everyone and “its reading requires the recognition of some essential, structural and particular elements of the object; in other words, it requires the presence of its mental image” (p. 49).

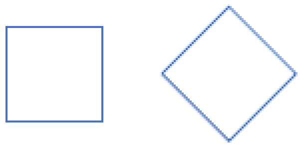
Making three-dimensional geometric figures into planes, in addition to addressing the two-dimensional aspect of the figure and aiding in the identification of the plane figures that form their faces, can help in the recognition of their elements, such as the number of faces and edges. This task is considered important by the BNCC as are the association of three-dimensional figures with their planes and vice versa (Brasil, 2018).

As a rule, the work with plane geometric figures is based on the recognition of the different figures, their elements—side, vertex and angle—and their properties.

An interesting issue to discuss is the position in which these figures are presented in textbooks and by the teacher: they are usually drawn with one side parallel to the line of the notebook or the margin of the page. A classic example is to draw the square as in the first image in Figure 3, which will cause the student not to identify the second as a square but only as a rhombus. The same can occur with a triangle that is always presented with one of the sides parallel to the margin of the page.

Figure 3

Squares in different positions.



Lorenzato (2015) indicates that it is common to begin the study of geometry by highlighting what is constant, permanent and fixed, such as the position of a figure presented frontally, which, for children, reveals acceptable properties. The author argues that “this type of limited teaching will lead to difficulties identifying the height of triangles when the base is not horizontal" (p. 12).

For this very reason, the BNCC indicates that one of its different skills is to "identify and name plane figures (circle, square, rectangle and triangle) in drawings presented in different orientations or in contours of faces of geometric solids" (Brasil, 2018, p. 279).

Furthermore, when plane geometric figures are presented in different positions on the plane, we are working with the figural aspect. Based on the ideas of Fischbein (1993), Nacarato and Passos (2003) explain that this aspect corresponds to a mental image, i.e., "the sensory representation of an object or phenomenon" (p. 61), and is associated with the conceptual aspect, and in the case of figures, they can be manipulated through geometric transformations such as rotation, translation, etc.

With regard to the use of instruments—ruler, compass and squares—for the construction of figures, the PNAIC presents an objective for the early years: "to use the ruler to trace and represent geometric figures and drawings" (Brasil, 2014, p. 6). In contrast, the BNCC (Brasil, 2018) introduces the use of these instruments in the final years of elementary school; in the early years, it only indicates the recognition of right and non-right angles with the use of squares.

However, although both the PNAIC and the BNCC do not focus on the use of these instruments for geometric constructions, it is possible to discuss the measurements of area and perimeter that are part of the BNCC's thematic unit on Magnitudes and Measurements that make it possible to relate geometry and numbers.

Understanding the difference between these two quantities is difficult for students. Different studies, such as those by Facco (2003) and Santos (2011), indicate that students cannot understand that in order to calculate the perimeter of a plane figure, they need to add the measurements of its sides; and in order to calculate the area, it is necessary to find the measure of its surface. One hypothesis for this difficulty is that classroom work only involves the memorization of formulas, without addressing the concepts of area and perimeter.

Based on the ideas of Baltar (1996), Santos (2011) indicates that the differentiation between area and perimeter can be accomplished using different perspectives, and two of them seem particularly interesting to us: the topological and the dimensional. The topological perspective proposes that the "concepts of area and perimeter correspond to distinct geometric objects, the area being associated with the surface and the perimeter with the contour," and the dimensional perspective proposes that "a surface and its contour are mathematical objects of different natures with regard to dimensions, which has immediate consequences on the use of units adapted to express the measurements of area and perimeter," i.e., area is two-dimensional and perimeter is one-dimensional (p. 19).

The calculation of the area of a plane geometric figure requires the use of a different formula for each figure and, if performed in the classroom only through memorization, can also become a challenge for students.

Lorenzato (2015) emphasizes that one way to approach this material is by using manipulable materials, making it possible to arrive at these formulas through decomposition. The author stresses that for the main quadrilaterals—rhombus, trapezoid, parallelogram, rectangle and square—it is possible to perceive that all of them can be divided into two triangles, which suggests that their areas can be calculated using the area of the triangle.

As we have learned from one of the exponents of New School proposals in Brazil, Lourenço Filho (cited by Bastos & Cavalcante, 2011), "geometry is a science of observation and experience" (p. 161). Emphasizing experience can be seen as giving meaning to the manipulation of everyday objects, spatial geometric figures, based on the New School approach. However, observation, manipulation and the mobilization of the sense organs constitute the essence of knowledge according to the intuitive method.

The publication in Brazil, in 1886, of Rui Barbosa's translation of the manual *Primary Object Lessons: A Manual of Elementary Instruction for Parents and Teachers*, written by Norman Allison Calkins, is considered one of the most important initiatives for the dissemination of proposals related to the intuitive method (Saviani, 2007; Valdemarin, 2004).

The text discusses the shape of objects, and the observation of their characteristics and properties would be part of the exercises to educate the eye. Size would be evaluated based on the nature of the shape, i.e., length, width, area and volume, notions that would be mobilized for the comparison of shapes, based on the observation of objects (Calkins, 1886).

Oliveira (2016), analyzing the role of geometry in elementary schoolteacher training in Brazil in the early twentieth century, a period in which elementary school was structured around the intuitive method, concluded that geometry was seen as a knowledge to be taught, regardless of the material related to it—such as shapes, for example—because the intuitive perspective of knowledge was essential. Indeed, observation and manipulation enable the mobilization of sense organs, which is essential for knowledge in the intuitive method. As such, geometry would provide one of the central references for child development following the intuitive method.

According to the analysis by Oliveira et al. (2020) regarding the development of elementary mathematical knowledge, in the late nineteenth century and the early decades of the twentieth century, knowledge for teaching geometry and drawing

> is characterized by the study of the shapes of objects, through observation and drawing; shapes occupied an important role in elementary school. The study of geometric solids was central, and observation, touch and comparison were elements for learning. (p. 197)

According to the New School proposals, observation, which had already been a central element in student learning since the dissemination of the intuitive method in the late nineteenth century, began to be accompanied by student manipulation of objects or action upon them, which would be observed, compared and analyzed.

Santos (2022) also emphasizes observation through comparison as one of the guiding principles of geometry teaching in the state of São Paulo during the New School era—more precisely, the paper covers the period from 1920 to 1960.

> In the guidelines for teaching geometry, it was observed that it follows the approaches that begin with teaching centered on objects and undergoes a transition to consider learning guided by the child themselves, in their actions, but without ever relinquishing the central focus on the materials and the instructions given by the teacher. Observation is very important, and its objective is to compare objects. (pp. 82-83)

Observation and comparison are strategies to teach notions of geometry, to explore geometric figures and to perceive their characteristics and their properties. The study by Nascimento Silva (2018) also demonstrates the importance of observation, manipulation and comparison for the study of geometry in the state of Minas Gerais. A significant number of articles from *Revista do Ensino de Minas Gerais*, published between 1926 and 1932, refer to these strategies for teaching geometry.

One article is devoted to *Os exercícios de observação no ensino modern* [*Observation exercises in modern teaching*], written by Maria Luisa de Almeida Cunha (Minas Gerais, 1926b, pp. 76-78, cited by Nascimento Silva, 2018, p. 47). According to Nascimento Silva's analysis, by observing and comparing geometric solids, it would be possible to establish a connection between matter and thought, which would lead to teaching in a reliable way.

When examining the elementary school programs from Minas Gerais dated 1961, likely developed in the 1950s, we do not notice any deviation from the New School proposals, particularly the emphasis on the importance of responding to the needs of everyday life.

The arithmetic and geometry program begins with considerations about teaching, the first paragraphs of which are intended to demonstrate the importance of arithmetic in everyday activities, such as estimating, measuring, comparing, evaluating, calculating. For the teaching of geometry, it is recommended that it be undertaken based on the observation of shapes present in children's daily life and related to manual labor and agricultural work—in this case, in the form of land measurement for the study of area. Geometry problems should be based on real circumstances; for example, to study angles, the observation of objects such as scissors or the hands of a clock is proposed.

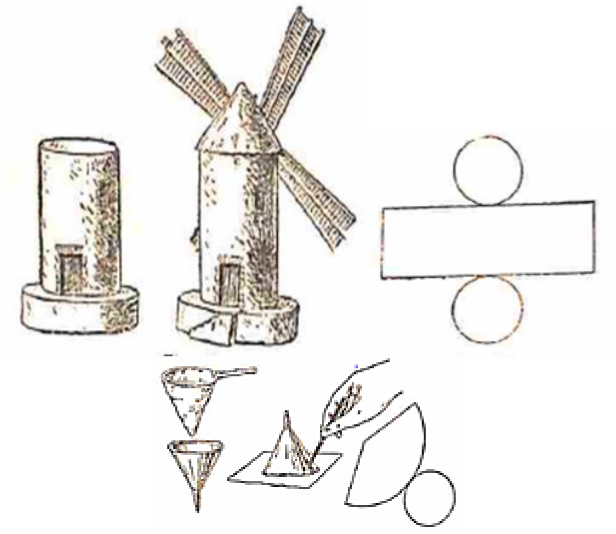
Even in the Modern Mathematics era, observation remains—along with manipulation—an important strategy for learning geometry. In this period, analyzed by Mendonça and Oliveira (2019), an important manual for elementary school teachers proposed that spatial geometric figures should be studied with their planes. The recommendation was to perform observation activities with students in such a way that through the decomposition and composition process - by turning spatial figures into planes and then turning those planes back into spatial figures -, children could observe the elements of figures, such as vertices, faces and edges.

The decomposition and composition of those spatial figures seems to be a permanent tenet of geometry learning. Numerous articles on the teaching of geometry, published in *Revista do Ensino de Minas Gerais* between 1926 and 1932, analyzed by Nascimento Silva (2018), demonstrate that turning the cube, parallelepiped and other solids into planes and observing their elements can be an exercise in understanding the elements that characterize those geometric figures and some of their properties, for example, observing similarities and differences between the cube and the parallelepiped, in the most general case.

During the New School era, a book published in 1934 by a Spanish author, Margarita Comas, titled *Metodología de la aritmética y la geometría* [*Methodology of Arithmetic and Geometry*], circulated in Brazil. In the author's proposal, an entire chapter was devoted to the study of spatial figures, which is carried out through the construction of a village in which the houses, churches and other buildings are formed by solids. To construct the village, the solids are assembled using their planes.

Figure 4

Construction of a mill (Comas, 1934, pp. 50-51)



Rodrigues (2018) analyzes the book by Margarita Comas and notes that "most of the proposals for teaching geometric knowledge present in the text are of an experimental nature, related to everyday situations but without any practical application" (p. 59). Furthermore, according to Rodrigues, the teaching of geometric knowledge in this text is not at the service of arithmetic, as in other works analyzed by the author, which, to a certain extent, is a characteristic of the time.

As mentioned above, in the first legislation for elementary school in Brazil, in 1827 (Valente, 1999), practical geometry is part of reading, writing and counting. This practical geometry is configured, in part, through land measurement problems involving the notions of area and perimeter. Although these notions are not limited to calculation, some studies point to the emphasis on it, which would imply a geometry that is at the service of arithmetic.

Barros and Oliveira (2016) analyze the practices and references of the Minas Gerais teacher Alda Lodi, who worked to train elementary school teachers in her state from the 1920s until at least the 1940s. Based on an analysis of different materials produced and used by the teacher, the authors conclude that

> geometry from the perspective of the teacher's pedagogical practice and her conceptions was a tool to support the teaching of arithmetic and should be at the service of the study of arithmetic material, the decimal system, etc., in a predominantly practical approach, far removed from a deductive focus. (p. 214)

Geometry was seen by the teacher, an expert in the methodology of arithmetic, as an application for the study of arithmetic.

But this conception cannot be understood as general nor as a legacy of earlier times. In one chapter of her work, Leme da Silva (2021) addresses the relationship between geometry and measurement. Referring to the importance of measurement in the nineteenth century, the author analyzes tacheometry—a speed-oriented knowledge for calculating measurements—as a proposal for teaching practical geometry.

According to Leme da Silva (2021), appropriations of tacheometry in Brazil had important French publications as references and exhibit their uniqueness in a text by Gabriel Prestes, *Noções Intuitivas de Geometria Elementar* [*Intuitive Notions of Elementary Geometry*], published in 1895.

> Prestes does not consider speed in the study of geometry but rather organizes a study in which measurements are given a special emphasis, following a pedagogical order: first direct measurement, considered intuitive, concrete, practical, in which students perform them with material they have constructed, i.e., measurements of objects; only afterwards is there a systematization of measurements, corresponding to indirect measurement, as an abstract process of synthesis, in which the formula, in this case, makes the calculation much faster. (Leme da Silva, 2021, p. 159)

In the section on intuitive measurements, Leme da Silva (2021) describes the didactic process for teaching measurements from the perspective of the intuitive method: it starts with estimating the measurement using the naked eye, involving a comparison between sizes, proceeds to the students creating instruments to perform measurements and culminates in the establishment of standardized units using the decimal metric system. Children would experience each of these steps in the process of learning to measure.

Leme da Silva (2021) also demonstrates that this proposal is not universal, as there are contemporary textbook authors who guide teaching towards faster access to formulas for calculating measurements, even if it involves some effort to make them understandable, i.e., to ensure that the formulas would result in explanations accessible to the students.

The use of drawing instruments, such as the ruler and compass, appeared in publications devoted to teaching linear drawing[7] in the early decades of the nineteenth century. In this early period, it was more related to verifying the accuracy of freehand constructions than to their realization. The aim was to improve freehand constructions by checking them against the instruments. This can be seen in the work *Princípios do Desenho Linear compreendendo os de Geometria Prática, pelo método do ensino mútuo* [*Principles of Linear Drawing, including Practical Geometry, using the mutual instruction method*], based on L. B. Francoeur, by Antônio Francisco de Paula and Iollanda Cavalcanti d'Albuquerque, published in 1829.

Other linear drawing manuals appeared throughout the nineteenth century, proposing other ways of mobilizing drawing instruments. On this topic, Oliveira (2019) demonstrates, for example, that in publications by Gama (1880) and Pacheco (1881), constructions and the study of figures appear together in the former and separately in the latter: "the separation is between what is studied without instruments, only by 'looking at it,' and what is constructed with instruments—a ruler, compass, etc." (Oliveira, 2019, p. 9).

[7] In the nineteenth century, linear drawing was a subject in primary education not only in Brazil but also in Europe, particularly in France. Francoeur's manual was translated and adapted into Portuguese by Cavalcanti d'Albuquerque in 1829. It involved the study of lines and plane figures drawn freehand but seeking to satisfy relationships and properties. Linear drawing contributed to the institutionalization of mutual instruction.

The use of drawing instruments strengthened the connection between drawing and geometry. In the late nineteenth and early twentieth centuries, discussions around drawing took a different path.

> Everything indicates that the organization of the artistic class is crucial to the introduction of natural drawing in Brazilian primary school, in line with the international debate. Demands from outside the school, such as professionalization and International Design Conferences, influence and alter a long-established school culture. (Frizzarini et al., 2015, p. 56)

In the New School era, education advocated that the best way to educate was through the senses and, consequently, that drawing should precede any graphic exercise, although not drawing with instruments, which was even critiqued by Lourenço Filho (Oliveira et al., 2020).

Analyzing the teaching of geometry in the Modern Mathematics era in Minas Gerais, Mendonça and Oliveira (2019) indicate that,

> although the initiation to geometry through a topological approach was one of the important proposals of Modern Mathematics for the early school years, this perspective does not seem to have had any impact. Euclidean geometry studied with construction activities involving a ruler, compass and protractor, the use of didactic or manipulable materials—flat patterns, among others—transformed the way geometry was taught/learned. (p. 1078)

We can say that the use of drawing instruments—a ruler and compass, in particular—resurfaces in new proposals for activities, with different objectives and aims for teaching geometry, when compared to the nineteenth century.

PAST AND PRESENT OF TEACHING GEOMETRY TO CHILDREN: A POSSIBLE SUMMARY

Over time, many objectives and aims, often divergent, have been mentioned to justify teaching geometry to children—initially focused on learning practical and useful notions for future professional training, important for development due to the mobilization of the senses it elicits, necessary for the perception and understanding of space in its different dimensions: three- and two-dimensional.

The study of geometric shapes, regardless of dimension, is part of the historical trajectory of teaching geometry to children and is configured in different eras as a journey in teaching from the space to the plane and vice versa or even a movement back and forth between spatial and plane shapes.

The activities proposed for this study, although specific to each era, do not seem to deviate from the observation, manipulation and comparison of objects and figures. Turning solids into planes, which is also a recurrent feature, makes it possible to observe the elements and properties of the figures involved and can occur in the sense of composing a three-dimensional figure or even decomposing it.

It appears that the relationship between geometry and the perception of space has been established over time, strengthened by the observations of Piagetian psychogenesis on the topic, and it is considered central in the present day.

It is possible to argue that while topics such as the study of shapes appear as a constant element in teaching geometry to children, others must be assimilated and appropriated by the school culture in order to be incorporated. This seems to be the case for the geometry of transformations, particularly notions such as rotation, isometry and translation, which, proposed during the Modern Mathematics era, were resisted by the school culture. Today, these topics seem to have been reconfigured and recreated in order to become part of the school culture of the early years and are part of teacher training programs and official documents (PNAIC, BNCC, etc.), proposed activities, textbooks—in short, teacher practices.

Geometry should contribute to understanding space, location, movements, characteristics and properties of the shapes that surround us, as well as to enabling the construction of an aesthetic perception and analyzing natural and man-made shapes, among many other possibilities.

Furthermore, the observation and manipulation that were part of the New School proposals are still present in geometry teaching today, as are aspects related to visualization.

Activities with geometric solids and their planes, a proposal that appears in New School textbooks and official documents as well as those in the Modern Mathematics era, can still be found today. This study is also necessary in order to enable students to observe the elements of solids—faces, vertices and edges.

The relationship between geometry and arithmetic is important and the interplay between them is still being examined in the present day. At certain moments in history, geometry was seen as an application of arithmetic. As such, in those periods, topics such as area and perimeter were placed at the service of arithmetic calculations, working with changes in units, among other aspects.

We can observe that there has been a constant emphasis, over time, on the study of area and perimeter focusing solely on calculations, which may be an indication of the difficulty faced by students in differentiating between these two quantities. Some intuitive proposals to address these concepts through students' own creation of measurement units, experimentation, and comparison, which are then followed by the introduction of the metric decimal system, may not have been incorporated into the school culture.

Finally, the drawing instruments that seem to remain in the background in today's official documents can be found with different perspectives in earlier eras. In the first decades of the nineteenth century, the use of a ruler and compass was more related to verifying the accuracy of freehand constructions, but other linear drawing manuals that appeared during that century proposed other forms of use.

The reflection enabled by this dialogue between contemporary issues related to the teaching of geometry in the early years and the history of geometry teaching in primary schools allows us to construct a broader understanding of the difficulties of teaching geometry in the early years and thus develop proposals that are able to consider the knowledge that has already been produced on the subject.

AUTHORS CONTRIBUTION STATEMENTS

M.C.A.O. conceived the idea of the submitted study. The two authors, M.C.A.O. and R.F.C., actively participated in the development of the theory, organization and analysis of the data, discussion of the results and approval of the final version of the paper.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article, as it is a survey of publicly available literature.

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