

# Preservice and In-Service Primary Teachers' Knowledge of Mathematical Reasoning Processes in the Context of a Geometry Task

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## ABSTRACT

**Background:** Teachers' knowledge of mathematical reasoning and how to foster it in pupils influence the way they plan and conduct their lessons. In geometry, it implies developing visualisation and spatial structuring. **Objectives:** This article addresses the knowledge of the preservice and in-service primary teachers about reasoning processes, namely the way they relate several reasoning processes when solving a didactical task involving geometry. **Design:** The study reported here followed a qualitative-interpretative approach, adopting a design-based research modality. **Setting and Participants:** The teacher education experiments were developed with 31 preservice primary teachers and 19 in-service teachers of grades 1 to 6. The participants were not selected since they were the unique classes of pre- and in-service primary teachers in the institution. **Data collection and analysis:** Data were collected by audio and video records of lessons, participant observation and the collection of written records of the preservice teachers. We used content analysis of the data using the framework we elaborated on before concerned with knowledge of reasoning processes. **Results:** The preservice teachers identified the process of generalising, relating it with comparing and exemplifying processes. Regarding the process of justifying, participants used the association to understand why a relationship works as a selection criterion for that process. On the contrary, the distinction between justifying and generalising appeared to be more difficult for in-service teachers. **Conclusions:** Collaborative work on didactical tasks that are supported by relevant mathematical tasks and real classroom episodes are promising scenarios to develop teachers' knowledge about mathematical reasoning.

**Keywords:** Mathematical reasoning processes; Preservice primary teachers; In-service primary teachers; Geometry; Spatial structuring.

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## O conhecimento de professores e futuros professores do ensino básico sobre raciocínio matemático no contexto de uma tarefa de geometria

### RESUMO

**Contexto:** O conhecimento dos professores sobre o raciocínio matemático e a forma como promovê-lo influencia a maneira como planificam e conduzem as aulas. Em geometria, implica desenvolver a visualização e a estruturação espacial. **Objetivos:** Este artigo aborda o conhecimento dos professores e futuros professores do ensino básico sobre os processos de raciocínio, nomeadamente a forma como os relacionam, na resolução de uma tarefa didática envolvendo geometria. **Design:** O estudo seguiu uma abordagem qualitativa-interpretativa, adotando uma modalidade de investigação baseada em design. **Ambiente e Participantes:** As experiências de formação foram desenvolvidas com 31 futuros professores e 19 professores em exercício (1º ao 6º ano). Os participantes não foram selecionados pois eram as únicas turmas em formação na instituição. **Coleta e análise de dados:** Os dados foram coletados através de gravações áudio e vídeo das aulas, observação participante e registos escritos dos futuros professores. Utilizamos a análise de conteúdo dos dados recorrendo ao quadro de análise que elaborámos anteriormente sobre o conhecimento dos processos de raciocínio. **Resultados:** Os futuros professores identificaram o processo de generalizar, relacionando-o com processos de comparar e exemplificar. Em relação ao justificar, os participantes associaram-no à compreensão do porquê de uma relação funcionar como critério de seleção daquele processo. Já para os professores, a distinção entre justificar e generalizar pareceu ser mais difícil. **Conclusões:** O trabalho colaborativo em tarefas didáticas suportadas por tarefas matemáticas relevantes e episódios reais de sala de aula constituem cenários promissores para desenvolver o conhecimento de professores e futuros professores sobre o raciocínio matemático.

**Palavras-chave:** Processos de raciocínio matemático; Futuros professores do ensino básico (1º ao 6º ano); Professores do ensino básico (1º ao 6º ano); Geometria; Estruturação espacial.

### INTRODUCTION

Teacher education should give special attention to mathematical reasoning, considering both the ability to reason, and knowledge about the reasoning processes (Stylianides & Stylianides, 2006). Both national and international curriculum guidelines (Ministério da Educação, 2021; MES - Ministry of Education Singapore, 2012; National Council of Teachers of Mathematics, 2000) emphasize the importance of developing mathematical reasoning from the first years of schooling. The teachers' knowledge of mathematical reasoning and how to foster it in their pupils influence the way they plan and conduct their lessons (Lannin et al., 2011; Loong et al., 2017). In

particular, developing mathematical reasoning processes in the domain of geometry, in early years classrooms, implies specifically developing visualisation and spatial reasoning (Moss et al., 2015) since the central processes of generalising and justifying (Rodrigues et al., 2021), in this case, are founded on the geometric properties and on the objects' structure.

This article is part of the *Mathematical Reasoning and Teacher Education* (REASON) project, which aims to study the mathematical and didactical knowledge teachers need to carry out a practice that promotes pupils' mathematical reasoning and to study the ways to foster its development in preservice and in-service teachers of primary, middle and secondary school. In this article we intend to discuss the knowledge of reasoning processes of a group of preservice primary teachers and of a group of in-service primary teachers, when solving a task involving geometry, namely the way they signify and relate several reasoning processes.

## CONCEPTUAL FRAMEWORK

### Reasoning in geometry

Reasoning geometrically about a spatial entity (object, diagram, or concept) implies constituting an adequate mental model that captures its relevant spatial structure and its geometric properties. Battista et al. (2018) state that "spatial and geometric structuring are types of spatial and geometric reasoning, respectively, that play vital roles in the construction of appropriate mental models for geometric reasoning" (p. 202). For spatial reasoning to adequately support geometric reasoning, these mental models must incorporate operational knowledge of relevant geometric properties and concepts, using mental models that integrate geometric properties into their structure and operation (Battista, 2007). Fujita et al. (2020) consider as reasoning skills spatial visualisation and property-based spatial analytic reasoning, being coordinated by domain-specific knowledge. Without this coordination, students' reasoning in problem solving could be influenced by the visual appearance of objects.

Jeannotte and Kieran (2017) consider the procedural aspect of mathematical reasoning to be dynamic and temporal in nature, contemplating different processes. Among these, they establish two categories, one related to the search for similarities and differences and the other to validation. The first category includes the processes of generalising, conjecturing, identifying patterns, comparing and classifying; while the second category includes the

processes of justifying, proving and formal proving, whether or not proceeding to change the epistemic value of a mathematical statement from true to false or more likely.

Lannin et al. (2011) distinguish two aspects in the process of generalising: (i) to identify common elements in different cases; (ii) to extend reasoning beyond the domain for which common elements were initially identified, that is, thinking about a relationship, idea, representation, rule, pattern, or other mathematical property considering it in a broader domain. For example, when a student identifies squares as the figures that have four equal sides, he is making a false generalisation, but it is a generalisation. For these authors, the process of justifying consists of building a logical sequence of statements, each one relying on established knowledge in order to reach a conclusion. Constructing a valid justification for a generalisation is not easy as it has to verify that the generalisation is true for all cases in the domain, resorting to valid implicit relations. A valid justification must explain why by offering a view of the underlying relationships that exist in all cases.

Thus, we consider that the process of generalising is fundamental in mathematics when we intend to "make general statements about properties, concepts or procedures" and that "justification is central to making it possible to mathematically validate" those statements (Mata-Pereira & Ponte, 2018, p. 783). These two processes interact with each other, as in many situations the language used in justification has to be general so that its applicability to the entire domain is clear; on the other hand, as it sometimes happens in geometry, some generalisations are established because, at least implicitly, there is a spatial structuring of the objects that is the fundament for the justifications of the relations to be validated (Brunheira, 2019).

For Jeannotte and Kieran (2017), exemplifying is an auxiliary process of generalising and justifying, which allows inferring data about a problem by generating elements that support those processes. In the process of generalising, it is essential to look for similarities and differences through the production of examples, in which case it is necessary to mobilize the process of comparing. In turn, in the process of justifying, the examples can be critical, for example when we use counterexamples.

Finally, we also look at the process of classifying for its importance in the context of primary school, especially in geometry. For Jeannotte and Kieran (2017), classifying consists of identifying common and distinct points in different objects through the search for similarities and differences, leading to join them or separate them into a class of objects based on mathematical

properties or definitions. This process involves comparing and, by stating that all elements of the class obey certain characteristics, it establishes a generalisation (Brunheira, 2019). For Mason (2001), “classification is not just about making distinctions and describing properties, but about justifying conjectures that all possible objects with those properties have been described or otherwise captured” (p.7). Although the authors cited before (except Brunheira (2019)) are mostly related to algebraic thinking, Mariotti and Fischbein (1997) state in the following way what means to classify in geometry:

A classification task consists of stating an equivalence among similar but figurally different objects, towards a generalisation. That means overcoming the particular case and consider this particular case as an instance of a general class. In other terms, the process of classification consists of identifying pertinent common properties, which determine a category. (p. 244)

Thus, in addition to identifying the different reasoning processes, it is essential to have a deep understanding of the meaning of each one in order to establish relationships between them, thus reaching a high level of knowledge (Rodrigues et al., 2021).

### **Reasoning in teacher education**

Several studies (Lannin et al., 2011; Stylianides & Ball, 2008; Stylianides & Stylianides, 2009) indicate that preservice elementary school teachers must have opportunities to develop their mathematical reasoning if they are to develop it with their students.

In the field of geometry, Battista (2007) states that reasoning is strongly based on the spatial structuring of objects or situations, that is, on mental models that are activated to interpret and reason about these objects or situations. These models form the basis of spatial reasoning, a form of reasoning that is particularly important in geometry but also, as Clements and Sarama (2011) mention, transversal to several areas of mathematics and other sciences. These authors refer to its importance in problem solving and regret its devaluation, which leaves teachers unprepared to teach geometry.

In the context of preservice teacher education in geometry, Brunheira (2019) suggests that processes such as classifying and justifying generalisations about geometric figures are influenced by the quality of spatial reasoning, but they are also promoters of its development. For Lehrer et al. (2013), geometric

concepts such as geometric figures and relationships between them, for example congruence, constitute opportunities to build relationships between their elements.

In addition to developing their own reasoning, Francisco and Maher (2011) refer the need to create opportunities for teachers to learn about how to develop mathematical reasoning in students. In the same sense, Stylianides and Ball (2008) defend the need to develop teachers' ability to plan and implement tasks that promote the development of reasoning in their students. The exploration of tasks focused on (i) classroom practice episodes, and (ii) putting into practice tasks to promote students' mathematical reasoning, both by preservice and in-service teachers may contribute to deepen their knowledge about reasoning processes and about the characteristics of tasks that may promote them (Oliveira & Henriques, 2021, Santos et al., 2022).

Melhuish et al. (2019) state that if we want teachers to promote the ability to generalize in their students, we should provide them with opportunities to analyse evidence of these processes during their education, in order to develop their ability to understand when students are involved in these processes. When teachers select tasks that promote reasoning processes, implement them in the classroom and later reflect on students' productions with other teachers it is a way to broaden their didactic knowledge on how to develop their students' mathematical reasoning (Herbert & Bragg, 2021). In the same direction, Santos et al. (2022) state that if teachers have opportunity to reflect on classroom episodes, analysing them based on theoretical ideas discussed in teacher education context, they deepen the reasoning processes. This deepening consists of identifying and characterizing “the generalising and justifying processes through the analysis, sharing and discussion of ideas and through the analysis of students' solutions of tasks, intentionally designed to promote mathematical reasoning” (p. 13). They further state that, if teachers plan collaboratively with other teachers, put that plan into practice, select episodes from the practice and reflect about students' productions with the other teachers, the professional development may happen.

## **METHODOLOGY**

The study reported here followed a qualitative-interpretative approach (Erickson, 1986) as it aimed to understand the way preservice and in-service teachers signify and relate several reasoning processes. It adopted a design-based research modality (Gravemeijer & Cobb, 2013) aiming at developing a

local instruction theory focused on the mathematical and didactical knowledge teachers need to carry out a practice that promotes pupils' mathematical reasoning. Its context is the teacher education experiments developed in (1) 2019/20 with 31 preservice primary teachers, attending a master's degree certifying for teaching in primary schools (grades 1 to 4) and teaching Mathematics and Natural Sciences in grades 5 and 6; and (2) 2020/21 with 19 in-service teachers of grades 1 to 6. Design research is conducted through three phases: preparation of the teacher education experiment, implementation of the experiment, and retrospective analysis. The experiment, with preservice teachers, implemented in 2019/20 (DBR cycle 1) in a public institution was conducted in 2020/21 (DBR cycle 2) in another public institution, benefitting from the improvements arising from the retrospective analysis. The same happened with the experiment with in-service teachers. So, here we report both the experiments of the DBR cycle 1 (preservice teachers) and of the DBR cycle 2 (in-service teachers) developed in Portugal, in the same institution, where they were the unique classes of pre and in-service primary teachers.

The preservice teacher education experiment, conducted by the last author, took place during six lessons, one per week, each lasting two hours and 30 min. All the tasks were initially explored autonomously by the preservice teachers, organised into eight groups, and were subsequently discussed by the whole class. The in-service teacher education experiment was conducted by all authors of this article and took place online (using Zoom platform) during eight sessions, over four months, each session lasting two hours and 30 min. The tasks were initially explored autonomously by in-service teachers, organised into four groups, and were subsequently discussed by the whole group. They also implemented two tasks in the classroom with their pupils and analysed pupils' mathematical reasoning processes and their own teaching practices. Both teacher education experiments were focused on mathematical reasoning, addressing specialised mathematics knowledge for teaching.

The data were collected through documents gathering (all tasks resolutions of preservice teachers) and participant observation of the lessons or sessions, by the team Project, using audio and video recordings of (i) the autonomous work carried out by two groups of preservice teachers and by all groups of in-service teachers, and (ii) the whole class discussions, both in preservice and in-service education. The data reported here are from one group of preservice teachers and from one group of in-service teachers. These groups were selected because the interaction among preservice and in-service teachers gave us many inputs about the way they understood the reasoning processes. According to the ethical criterion of confidentiality, all the preservice and in-

service teachers signed a free and informed consent form, in relation to the data collection methods, and are given fictitious names<sup>1</sup>.

This article refers to a didactical task about reasoning in geometry (Appendix 1) that was based on the analysis of 3<sup>rd</sup> grade students' exploitation of a geometry task (Appendix 2). It was proposed in the fifth lesson of the preservice teacher education experiment and in the third session of the in-service teacher education experiment. The classifying process was presented by the educators only after the exploitation of the task by preservice and in-service teachers.

We used content analysis (Bardin, 2010) of the data using the framework we elaborated before (Rodrigues et al., 2021) concerned with knowledge of reasoning processes (Table 1).

**Table 1**

*Framework for knowledge of mathematical reasoning processes.* (Rodrigues at al., 2021)

Category	Subcategories
<b>Knowledge of the reasoning process</b>	5. Knowledge of the process fits the definition presented, and includes its relationship with the other reasoning processes
	4. Knowledge of the process fits the definition presented, and is explicitly outlined by enunciating the properties of the process
	3. Knowledge of the process fits the definition presented, and is explicitly outlined through illustrative example(s)
	2. Recognising a reasoning process though considering only 'correct' processes
	1. Knowledge of the process takes on the meaning of the term in everyday language
	0. The process is confused with other processes

<sup>1</sup> The research project does not have an ethics committee. However, this research complies with the principles and guidelines of the Code of Ethical Conduct in Research of CIED (Centro Interdisciplinar de Estudos Educacionais) and the Letter of Ethics for Research in Education and Training of the Instituto de Educação da Universidade de Lisboa. Therefore, this work assumes and explicitly exempts Acta Scientiae from any consequences arising from the absence of ethical evaluation, including full assistance and possible compensation for any damage resulting from any of the research participants, in accordance with Resolution No. 510, of April 7, 2016, of the National Health Council of Brazil.

The categories were identified inductively as they appeared in preservice and in-service teachers discourse and work. They are relative to the knowledge of the reasoning processes worked in the teacher education experiments: generalising, justifying, exemplifying, comparing, and classifying. Each of these categories were divided into subcategories corresponding to six levels of specialised mathematical knowledge of the content, presented in hierarchical form.

## RESULTS

We selected four episodes (two from preservice experiment and two from inservice experiment) which seems to be representative of the work developed.

### Episode 1

In this episode, we analyse the reasoning processes that the group of preservice teachers identified to be used by the 3rd grade students during the solution of the intruder task (Appendix 1). We intended to introduce the classifying process which is the main process in question. In fact, to find the intruder, students must identify pertinent common properties that determines the pyramids as a class (for example, every face is triangular and converge in a vertex) where the prism does not belong and also ignore their particularities (for example, distinct bases) and consider them as representatives of a more general class (Mariotti & Fischbein, 1997).

*Nuno: "Identify the reasoning processes involved". Here, I think that . . . the most obvious common is generalisation. They identify a property that fits all the pyramids.*

*Lara: Yes.*

*Daniela: Yes, Yes.*

*Nuno: So one of them is to generalise. To compare...*

*Daniela: Also to compare. You don't think so?*

*Nuno: Between several...*

*Daniela: Between figures, yes.*

*Lara: To generalise they compare, don't they?*

*Nuno: Yes, yes. They exemplify, here it does not...*

*Lara: No.*

*Daniela: No.*

*Daniela: Compare between what?*

*Lara: Among the different figures so that you can generalise.*

*Helena: For example, here they made a comparison. When they had to select what it was [the intruder].*

*Nuno: Yes, that's a fact. Between the ones that are and the ones that are not [pyramids].*

*Helena: Exactly.*

As expected, the preservice teachers did not report the classifying process, but their analysis clearly identifies the generalising process that is strongly related to that process (Jeannotte & Kieran, 2017; Mariotti & Fischbein, 1997). Furthermore, Nuno explains the substantiation of this generalisation by saying that it corresponds to the property that “fits all pyramids”, an idea that gathers consensus. Associated with the process of generalising, the group also refers to the process of comparing as a support process because, as Lara says, it is necessary to compare the different figures “so that you can generalise”, which is also consistent with the literature.

The group agrees and is sure about the two identified processes, when one of the elements raises the hypothesis that the process of justifying may be also involved:

*Lara: To justify I don't know if it makes sense. Okay, you find properties that you can justify, but properties are generalisations.*

*Nuno: Yes, yes. So, to generalise.*

*Helena: I think it's enough to generalise and to compare.*

*Nuno: Although, in order to generalise, they will also have to justify first, generalising is the most*

*comprehensive of all. They will say first that the pyramid is a pyramid...*

*Helena: Because so, so and so. Exactly.*

*Nuno: It is a quadrangular pyramid, because the base is a square and because the faces are triangles, it has  $x$  vertices...*

*Daniela: I think that, in order to reach the generalisation, they start with justification.*

What started out as a tentative hypothesis from one of the preservice teachers turned out to be a meaningful possibility for everyone. The idea that the process of justifying may be involved derives from the perspective that, in this case, when we generalise, we already have justification in mind or, to put it another way, we generalise because we know why. This idea is consistent with the suggestion by Mason (2001) when he states that the process of classifying also involves justifying conjectures that all possible objects with those properties have been described. Furthermore, in geometry, as Brunheira (2019) states, the justification of generalisations concerning a class of geometric figures is based on a mental model of the class of objects, that is, its spatial structure that often presides over the formulation of generalisations, and so promoting the articulation of spatial visualisation and property-based spatial analytic reasoning (Fujita et al., 2020).

In this way, we consider that the group's dialogues are quite relevant as they identify interactions between the processes of generalising and justifying, also showing understanding about all processes already dealt with, which corresponds to level 5.

## **Episode 2**

In this episode, the same group of preservice teachers analyses the way in which the student's reasoning evolved from her interaction with the teacher (Question 4.1.).

*Lara: She started, she realized first that with 13 [toothpick] it would be left with one, right? That was the first thing she noticed . . . then with 15...*

*Daniela: Yes, but here...*

*Lara: One would be missing.*

*Lara: She only realized the 15... she only gave the answer to 15 so quickly, because she had already done the one for 13.*

*Helena: Because she had already done for 13.*

*Lara: Because she even said there was one missing. Do you understand?*

*Daniela: Yes...*

*Nuno: Then we have to make a comparison with what the teacher was saying. Right here at the beginning, the teacher refers to another example, so she can...*

*Lara: So, she can make a generalisation.*

*Nuno: A generalisation, exactly.*

*Daniela: So, the student began by understanding that with 13 toothpicks, one would be missing to complete the pyramid.*

*Nuno: Then the teacher ... encourages the student to go further...*

*Lara: By giving another example.*

*Nuno: ...and it presents a new example, in this case with 15 toothpicks. Then the teacher encourages the student to go further by presenting a new example. She presents a new example, enabling the student to use the reasoning process, to generalise. She gave this example so that she could later conclude that it couldn't be an odd number.*

In the written record with the answers to the task, the group summarizes the previous ideas and adds:

*Also, the teacher asks why this happens, prompting justification. In a first moment, the student does not justify it, she only describes what happened. After the teacher's insistence, the student points to the material and justifies why her generalisation is valid. (Group's record on question 4)*

In this episode, the preservice teachers elect three processes that are mobilized: to generalise (that there are no pyramids with an odd number of edges), to exemplify (for 13 and 15 edges) and to justify (why an odd number of edges is impossible). Regarding the first process, the group correctly identifies that it is a generalisation when the student extends her conclusion (about 13 and 15) to the domain of odd numbers, corresponding to the definition of the generalisation process that was established. With regard to justifying, it is noteworthy that the group is able to distinguish a simple description of an event (when the student says “Because one is missing or one is left”) from a justification, relating this process with the investigation of the underlying reasons why it is true (Lannin et al., 2011). Finally, about the process of exemplifying, actually the examples used (with 13 and 15 edges) are suggested in the task. However, the group recognizes the support these examples provide for both the processes of generalising and justifying (Jeannotte & Kieran, 2017). In the first case, they consider that it is based on the attempt to build a pyramid with 13 edges that the student quickly concludes that it is impossible to construct a pyramid with 15 edges. In addition, they also realize that these two examples are fundamental to generalise and to justify, as they recognise the understanding of the situation they generate, allowing the student to understand why the number of edges cannot be odd.]

### Episode 3

In the following episode, a group of in-service teachers discusses the possible reasoning processes that students can mobilize throughout the task.

*Antónia: ... But mainly the 4, in which the teacher asks if can be 13 and that one replies no and then “do you think Pedro is right? it’s because?”. Here is clear that they can stop at justifying, but this justifying can also be going further. There may be students who go for generalisation, right?*

*Gina: I think that the difference between justifying and generalising sometimes even depends on a little word or another.*

*Manuela: Exactly.*

*Gina: I sometimes have great difficulty to understand when they are justifying or when ...*

*Antónia: It depends on the level you work at, because they still have some difficulties even communicating what they are thinking.*

*Gina: Well, they can't explain the reasoning and the way they justify it. Now, those who are listening often realize that this is in fact a generalisation, they cannot ...*

*Antónia: They cannot go further to clarify, but the basis is there ...*

The discussion between these teachers highlights several aspects of the way they view the processes of generalizing and justifying. First, the difficulty in distinguishing the two processes is highlighted, evident in Antónia's analysis and consciously expressed by Gina, which may be the result of the interaction between the process of generalising and justifying (Brunheira, 2019). Secondly, it is clear that Antónia conceives a sequential order between the two processes, in which justification comes first and then generalisation. In addition, her expression "Here it is clear that they can stay by justifying, but this justifying can also be going further. There may be students who go for generalisation" seems to indicate that she regards generalisation as a cognitively more complex process than justification. Finally, for Antónia and Gina the identification of the process of generalising or justifying seem to depend, above all, on the difficulties that students may experience in communicating their ideas and not so much on the nature of the processes. Thus, regarding the processes of generalising and justifying, the interventions show that the teachers confuse the processes, placing them at level zero.

The group continues its discussion analysing the possibility of the task mobilizing other reasoning processes:

*Antónia: Apart from justifying and generalising we have more processes?*

*Amélia: Exemplifying.*

*Manuela: These are the situations that justify when it is a generalisation or when it is a justification.*

*Antónia Question 3 is within the exemplifying. He is doing one with nine, does he need more or not? Bring ten. At the outset it's also to justify, isn't it? Or within question 2, for example António's question, who asks*

*to keep eight “what will the base be like?”, he can go and try it, right? And there's exemplifying.*

*Gina: Well, he can exemplify there.*

*Antónia: This process is also possible to exist, isn't it? But there it is, without having what the students said...*

*Amélia: Yes, but as he is with physical material he is trying, he is exemplifying.*

This episode also shows how the teachers face the process of exemplifying. On the one hand, this process is closely associated with the analysis of concrete cases, even when these are part of the task itself. Specifically, in question 3 of the math task, students are asked to answer how many edges a pyramid has that has 9 edges at the base, then repeating the question with 10 edges at the base. It is not the students who generate these examples and, although their analysis can support the production of inferences, this aspect is not mentioned by the teachers. On the other hand, Amélia seems very determined to consider the process of exemplifying associated to the fact that students use material, which favors experimentation. In this way, the teachers' conception of the process seems closer to the current understanding of the term “exemplifying” than to the definition proposed in the literature discussed in the experiment, which suggests level 1.

#### **Episode 4**

The last episode that we present takes place in the context of in-service education and concerns a moment of collective discussion, which takes place after the resolution of the didactical task. In this episode, incident in the analysis of the interaction between the student and teacher (Question 4), another teacher appears (Fernanda) who comes to corroborate the difficulties expressed by the group of episode 3, trying to clarify their doubts:

*Fernanda: After all, where is the generalisation?*

*Educator: So, generalisation is here “with odd numbers it is not possible”.*

*Fernanda: Yes, and the others are justifications...*

*Educator: The rest are justifications, exactly. But do you agree, Fernanda? Does it make sense?*

*Fernanda: I have trouble with this.*

*Fernanda: For me, the last justification is a generalisation.*

...

*Fernanda: I am now being confronted with generalisations and then justifications, because I always justified the whys and then generalised. Today, here I am being confronted with situations that are not what I thought. And that is why it is hard to get in.*

*Antónia: Great, I said this exactly to her when she entered in our room, that we were having an interesting discussion, that for me one of the aspects of this teacher education experiment, for me is the clarification of concepts.*

*Fernanda: It is true, that's right! But and this was something that belonged to my reasoning. First, it is justified, then it is generalised.*

In this episode, Fernanda confirms the difficulties already reported by the group of pre-service teachers referred in episode 3, both about the distinction between the processes of generalizing and justifying, and about their sequencing. However, there is one statement of Fernanda that deserves further analysis: “For me, the last justification is a generalisation”. She refers to “Hmm... Here (points to the base) and here (points to the place where the side edges would be) must have the same number” arising from the teacher's insistence on asking the student why it would not be possible to obtain pyramids with an odd number of edges. In fact, when the student states that the side edges and the base edges are the same number, that statement is a generalization but, in the context in which it is being applied, it constitutes an argument that justifies the inference already made, based on mental models that integrate geometric properties into their structure (Battista, 2007). This construction is consistent with the definition of justification found in the literature, as well as the interaction between this process and the process of generalisation mentioned above. In the case of this task, the use of physical models favors the spatial structuring that the student mobilizes to produce her arguments and develop her reasoning, as suggested by Battista et al. (2018).

## CONCLUSION

The didactical task led preservice teachers and in-service teachers to discuss about the reasoning processes involved in the task focused on the properties of pyramids. Concerning the group of preservice teachers, the discussion shows that the group was able to easily recognize, in context, the characteristics of the process of generalising, as well as its relationship with the process of comparing and exemplifying. However, the richness of the context involved – the establishment of the class of pyramids – enhances the emergence of various reasoning processes that occur in a non-linear way, generating a discussion about the distinction between generalising and justifying. Despite some hesitation, participants used the association between the process of justifying and the understanding of why a relationship works (Lannin et al., 2011) as a selection criterion for that process, which is found to be appropriate. Furthermore, they are also able to understand the supporting role that the process of exemplifying assumes in constructing a justification without confusing the role of empirical examples in establishing a statement, which is very common (Stylianides & Stylianides, 2009). On the contrary, the group seems to recognize that, in geometry, a justification must be associated with the spatial structure of objects (Brunheira, 2019) valuing the articulation between spatial visualisation and property-based spatial analytic reasoning (Fujita et al., 2020) which contributes to a better understanding of geometric reasoning (Battista et al., 2018).

The distinction about justifying and generalising appeared to be much more difficult for the group of in-service teachers who had trouble in choosing which process could be developed by each question of the mathematical task. Also, the teachers were conscious of their difficulties and appreciated the opportunities that the teacher education experiment gave them in order to understand better the reasoning processes.

Although this paper shows that the group of preservice teachers exhibit higher levels of knowledge than in-service teachers, we do not intend to generalize this relationship. However, the fact that the in-service teachers are quite experienced and have had their initial education a long time ago, raises the question of the influence of the mathematical training that they had (possibly following very different practices from those that are currently carried out) may have on how teachers conceptualize reasoning. Also, we may take into account that, by the time that we collected the data, the preservice teachers have had two more lessons on this theme than the in-service teachers. Therefore, preservice teachers had more opportunities to clarify and mature the

concepts involved that possibly had a positive effect in the comprehension of the reasoning processes they showed.

Whether we consider the group of in-service teachers and their difficulties, or we consider the group of preservice teachers who shows a maximum level of knowledge about reasoning processes and its relationships, the episodes highlight the importance and potential of didactical tasks that promote this knowledge, including the idea that different kinds of mathematical tasks can offer different opportunities for reasoning (Stylianides & Stylianides, 2006), and that this didactical knowledge may be promoted using real classroom episodes (Santos et al, 2022).

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## **AUTHORS' CONTRIBUTIONS STATEMENTS**

All the authors are members of the project Reason team and were involved in the design of teacher education experiments since the beginning, so all of them conceived the idea expressed in this paper. LS developed the theory, MR described the methodology, LB conceived the didactical task, did a first selection of data to include and analysed the data. All the authors actively participated in the discussion of results and conclusion, reviewed and approved the final version of the paper.

## **DATA AVAILABILITY DECLARATION**

Authors agree to make their data available upon reasonable request from a reader. It is up to the authors to determine whether or not a request is reasonable.

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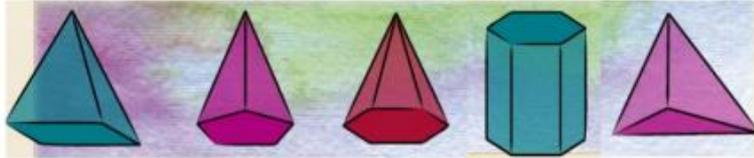
<https://doi.org/10.5951/jresematheduc.40.3.0314>

## APPENDIX 1

### The didactical task

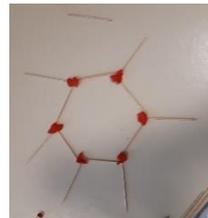
Consider the task *Let's learn about pyramids*, proposed to 3<sup>rd</sup> grade students. In the previous year, the class had already come

into contact with pyramids and prisms, in a first approach to their characteristics. So the teacher introduced the task by projecting the image below and asking *What is the intruder?*



After the initial discussion, the students started solving the task in pairs, using some models of pyramids in cardboard and wood, match sticks, toothpicks and plasticine balls.

1. Analyse the mathematical task and explain which are the properties of the pyramids that may emerge from its resolution.
2. Identify, in the national curriculum, which are the learning objectives that may this task intends to contribute.
3. Identify the reasoning processes involved.
4. Read the following dialog. The students had already analysed the possibility of building a pyramid with 13 edges using the material, as shown in the image. They were currently analysing the same issue for 15 edges.



*Teacher — So, with 15 toothpicks, 15 edges, what happened?*

*Student — It would be missing 1.*

*Teacher — So and how many toothpicks did you put in the base?*

*Student — Eight.*

*Teacher — Eight. And now how many do you have to put on the side edges?*

*Student — Oh my God...*

*Teacher — OK, you can look at what you've done!*

*Student — Seven.*

*Teacher — So, can we build with 15?*

*Student — No... there was a toothpick missing... with odd numbers I couldn't do it.*

*Teacher — Ah! So tell me why it's not possible with odd numbers.*

*Student — Because one is missing or one is left.*

*Teacher — And why does this happen? What happens to the edges in the pyramids?*

*Student — Hmm... Here (points to the base) and here (points to the place where the side edges would be) must have the same number.*

4.1. Discuss how the student's reasoning evolved, relating it to the interaction she established with the teacher.

4.2. Explain the role of the manipulative material in this situation and throughout the task.

5. Analyse the following answer:

5. Record everything you have discovered about the pyramids.

*We found that pyramids always must have 5 or more vertices and they always have a vertex on top.*

Imagine that this answer comes up in your class. What would you say or ask your author? Justify your proposal.

## Appendix 2

*The task Let's learn about pyramids*

1. Start by studying the pyramids your group has and fill in the spaces:

Number of faces _____	Number of faces _____
Number of vertices _____	Number of vertices _____
Number of edges _____	Number of edges _____
Base of the pyramid _____	Base da pyramid _____

2. The group of Marisa, Ana, Pedro and António is building pyramids with chopsticks and plasticine balls, but they have little material.

- a. Ana's pyramid has a figure with 8 chopsticks at the base. On top already placed 5 chopsticks, as shown in the picture.



How many chopsticks are missing? How many chopsticks does she need altogether? And how many plasticine balls does she need altogether?

b. António tells his colleagues: “Keep 8 plasticine balls for me!”. What will the base of the pyramid that António wants to make?

3. Marisa is making a pyramid with 9 toothpicks at the base. How many more toothpicks will she need? What if there are 10 toothpicks in the base? Explain how you thought.

4. At the end of the work, all groups show the pyramids they built. The teacher asks:

— Can someone show me a pyramid with 13 edges?

As no one answers, the teacher asks for another pyramid with 15 edges. Then Peter replies:

— You can't build pyramids with those numbers.

Do you think Pedro is right? Why?

5. Record everything you have discovered about the pyramids.