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Articulating the Blomhøj Modelling Cycle and the Mathematical Working Spaces. Analysis of a Task in Higher Education

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ABSTRACT

Background: The Mathematical Working Spaces (MWS) theoretical framework has developed a growing interest in how mathematical modelling is recognised, analysed, and articulated from its theoretical and empirical scopes. Given the articulations identified in the literature, it is interesting to determine whether new networks with the MWS provide powerful strategies for analysing mathematical modelling task resolution. **Objective:** to characterise the modelling activity from the network composed by the Blomhøj modelling cycle and the MWS in engineering students. **Design:** This work is a case study with a qualitative approach, which analyses the resolution of a modelling task through the proposed network. **Setting and participants:** The experimentation was carried out in an integral calculus course for a computer civil engineering career at a Chilean university. Modelling practices are not usual in this second-year subject, although enhancing their use in professional education is necessary. **Data collection and analysis:** We collected the written records of the students, selecting one to perform the in-depth analysis due to its high representativeness and clarity, as evidenced in the documents. Three steps were followed to characterise the modelling activity in the written record: description, analysis, and interpretation. **Results:** The Blomhøj modelling cycle might present connections with the MWS at the formulation of the problem and not only at the systematisation, which is a novelty in this field of research. **Conclusions:** A novel approach with which to develop the network between the MWS and modelling

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emerges, emphasising the investigation of mathematical problems using the student's reality in higher education.

Keywords: Mathematical Working Spaces (MWS); Blomhøj modelling cycle; MWS-modelling complementarity; modelling tasks.

Articulación entre el ciclo de Modelización de Blomhøj y Espacios de Trabajo Matemático. Análisis de una tarea en Educación Superior

RESUMEN

Antecedentes: El marco teórico de los Espacios de Trabajo Matemático (ETM) ha desarrollado un creciente interés en cómo se reconoce, analiza y articula la modelación matemática desde sus alcances teóricos y empíricos. Dadas las articulaciones identificadas en la literatura, es interesante determinar si nuevas redes con el ETM brindan estrategias poderosas para analizar la resolución de tareas de modelación matemática. **Objetivo:** caracterizar la actividad de modelación a partir de la red formada por el ciclo de modelación de Blomhøj y el ETM en estudiantes de ingeniería. **Diseño:** Se planteó un estudio de caso desde un enfoque cualitativo para analizar la resolución de una tarea de modelación a través de la red propuesta. **Entorno y participantes:** La experimentación se realizó en el curso de cálculo integral de la carrera de Ingeniería Civil Informática de una universidad chilena. En esta asignatura de segundo año no son habituales las prácticas de modelación, aunque es necesario potenciar su uso en la formación profesional. **Recolección y análisis de datos:** Se recolectaron los expedientes escritos de los estudiantes, seleccionando uno de ellos para realizar el análisis en profundidad debido a su alta representatividad y claridad, como se evidencia en los documentos. Además, se siguieron tres etapas para caracterizar la actividad de modelación en el registro escrito: descripción, análisis e interpretación. **Resultados:** El ciclo de modelación de Blomhøj presentaría conexiones con el ETM desde la formulación del problema y no sólo desde la sistematización, lo cual es una novedad en este campo de investigación. **Conclusiones:** Se evidencia un enfoque novedoso para desarrollar la red entre el ETM y la modelación, enfatizando la investigación de problemas matemáticos usando la realidad del estudiante en educación superior.

Palabras clave: Espacios de Trabajo Matemático (ETM); ciclo de modelización de Blomhøj; complementariedad ETM-modelización; tareas de modelización.

INTRODUCTION

The research work on mathematical modelling has more than 40 years of development globally. As a result, multiple communities and scientific investigations have positioned modelling in curricula worldwide (Kaiser, 2020). However, it does not necessarily mean that there is a global consensus on what we mean by modelling. Indeed, there are multiple approaches in Latin America (Arrieta & Díaz, 2015) and Europe (Doerr *et al*., 2017), promoted by a wide variety of educational objectives and a variety of sources both for the construction of mathematical knowledge and for the typology of modelling tasks (Kaiser & Sriraman, 2006). Despite that, there is a general conception of such approaches from the fact that modelling activity crosses the boundaries between reality and mathematics when recognised as knowledge (Blomhøj & Jensen, 2003).

For the model of the Mathematical Working Spaces (MWS) (Kuzniak *et al*., 2016; Kuzniak *et al*., 2022), there is an interest in understanding how reality affects the development of a mathematical task (Lagrange, 2018) or in types of mathematical tasks located in reality to encourage didactic approaches for teaching and learning. This topic has been considered in the last international symposia on mathematical work (Nechache, 2018; Guerrero-Ortiz & Henríquez-Rivas, 2018), generating complementarity between modelling and ETM, promoting the design of modelling tasks based on real contexts. In particular, studies of the complementarity between both frameworks have emphasised the Blum-Borromeo cycle (Borromeo, 2006), obtaining good results when relating the modelling stages with the circulations in the ETM (Cosmes & Montoya-Delgadillo, 2021).

This study proposes to expand the research on the relationship between the MWS and other related models of mathematical modelling, in this case, with the Blomhøj cycle. The goal is to recognise interconnections (Castela, 2021; Bikner-Ahsbahs & Prediger, 2009; Maier & Beck, 2001; Prediger *et al.*, 2008; Radford, 2008) between the stages of modelling with circulations (Montoya-Delgadillo *et al*., 2014), not only in the cognitive dimension of the MWS but also in the epistemological dimension. To this end, we will describe the different scopes of this complementarity from a theoretical and an empirical point of view. Specifically, this study is based on evidence collected from a particular context of interest for research in higher education, namely, mathematical modelling of real research problems in calculus courses for engineering. This aspect has not been addressed enough in the literature because most proposals are aimed at school mathematics (e.g., Borromeo-Ferri, 2007; Blum & Borromeo-Ferri, 2009). The choice for the Blomhøj cycle is because it considers epistemological aspects of the theory underlying modelling, which acquires greater importance when the problems correspond to actual investigations, which require working in task solving with high cognitive demand (e.g. Blomhøj, 2020; 2021).

This article is structured in two parts. The first describes the state of the art of modelling in MWS; to this effect, the MWS framework and the modelling approaches used for the analysis by the MWS are briefly described. After that, we discuss some works of interest on modelling in MWS. Then, we introduce the Blomhøj modelling cycle and a possible theoretical complementarity between the two frameworks, which constitutes the main novelty of this investigation. The second part deals with analysing the data collected in experimentation on the resolution of a task considered as modelling. In particular, we describe the methodological aspects and the experiment implemented in a calculus course in a civil engineering career. Next, we analyse the data to characterise the mathematical work of engineering students when solving an actual research problem modelling task based on the proposal of complementarity between the Blomhøj modelling approach and the MWS. Finally, we present the conclusions that make it possible to further develop the MWS-modelling complementarity, emphasising real investigation problems in higher education.

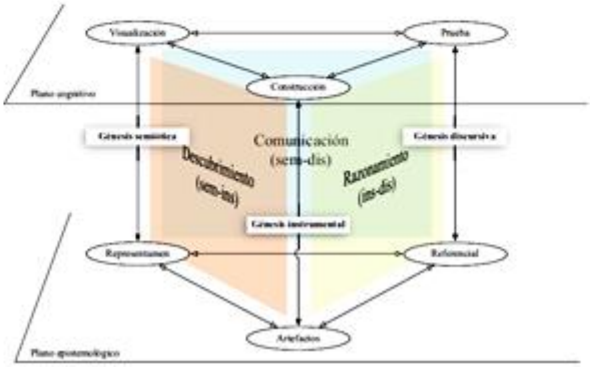
THEORETICAL BACKGROUND

Mathematical Working Spaces (MWS)

The main goal of the MWS analytical approach (Kuzniak *et al*., 2022) is to characterise mathematical work performed in an educational context to facilitate and improve the conditions in which the teaching-learning process of mathematics occurs. For this, the MWS theory considers two dimensions: epistemological and cognitive, represented as horizontal planes in Fig. 1; the first focuses on research oriented from the work paradigms, which depend on the domains of mathematics that structure it and takes into account the diversity of the activity of mathematicians related to the nature of the objects studied, which implies knowing the epistemological foundations of the implied differences. On the other hand, MWS, as a human activity, requires the cognitive dimension, which is associated with the epistemological plane through problem solving. Abstract space, thus conceived, is understood as an organised structure that describe individuals' activities when solving problems.

Figure 1

Plan of the Mathematical Working Space (MWS) (Kuzniak & Richard, 2014)



The epistemological plane consists of three interacting components: the referential (formed by properties, theorems and definitions, among others), the representamen (sign), and artefacts (material or symbolic); in the same way, the cognitive plane is composed of visualisation, construction and evidence. The MWS theory considers among its principles some authors that provide it with a foundation, among which Peirce (1990) stands out due to his definition of the real world through signs. For Peirce, the sign does not necessarily represent an empirical object; it can represent a conventional law, the ownership of a thing, an action or an event. The objects that the signs represent may be perceptible, imaginable and still unimaginable, but they are always known (Peirce, 1990; p.94). The signs allow us to understand and know reality based on reality itself, which would allow us to relate MWS with modelling based on the latter, where the epistemological plane, through semiotic genesis, could be a possible intersection of both theories.

The analysis of mathematical work through the MWS theory allows us to study how it is built progressively, connecting the components of the epistemological and cognitive planes through three geneses: semiotic, instrumental and discursive. According to the scheme depicted in Fig. 1, semiotic genesis, considered a process of decoding and interpreting signs, refers to perceiving a sign (representamen) through cognitive understanding. The inverse relationship, coding or instantiation, occurs in understanding the individual when constructing or specifying a sign. Instrumental genesis allows artefacts to be operational in constructing concepts or objects that contribute to the success of mathematical work. Finally, the discursive genesis of proof is the process by which the properties and results organised in the referential are developed to be available for mathematical reasoning and discursive validations, i.e., those that go beyond graphical, empirical, or instrumented proofs.

Epistemological and cognitive planes structure the MWS theory, providing a model for understanding circulation within the mathematical work. These two levels are articulated by the three geneses and their interactions described by three vertical planes: Sem-Dis, Sem-Ins and Ins-Dis. These emerge naturally in Fig. 1, involving semiotic and discursive, semiotic and instrumental, and instrumental and discursive genesis, respectively.

Finally, we consider the paradigms of real analysis (Montoya-Delgadillo & Vivier, 2016), which guide the mathematical work in this field. In particular, the arithmetic/geometric analysis (GA) paradigm involves work of a perceptual type based on interpretations arising from graphs or numbers, taking into account the role of the figures or different visualisations that activate that work. Thus, it supports early-stage teaching of a given object, such as equations or functions, favouring interpretations with implicit hypotheses based on geometry, arithmetic calculation, or the real world.

The Blomhøj modelling cycle

The modelling activity is interpreted as a cyclical and non-linear process, continuously characterising the relationships between the ideas that live in reality and mathematics and highlighting the sub-processes that relate them both from an epistemological basis defined by the individual with different scopes. It begins with a real-life problem proposing a goal that demands a modelling task based on reality. The problem is located in an investigation domain, in many cases of an interdisciplinary nature, prevailing a dynamic research process that considers as its basis the relationships between mathematical concepts and ideas and the experiences of real life (Artigue & Blomhøj, 2013).

For Blomhøj (2004), the modelling process starts from a real-world situation, often not having an explicit character. For example, for the problem of saving water in the mornings (Blomhøj, 2004), one task would be how to optimise the variables to be considered in the shower or some other relevant, guided by the epistemological characterisation of the individual (Artigue &

Blomhøj, 2013). Starting from an analytical characterisation, as depicted in Fig. 2, the mathematical modelling process has six Blomhøj sub-processes (2004; 2013):

a) Problem formulation: formulation of a more or less straightforward task that guides the identification of the characteristics of the perceived reality that will be modelled.

b) Systematisation: selecting the relevant objects and relationships, among others, within the resulting research domain and their idealisation to make a mathematical representation possible.

c) Translation of these objects and relationships into mathematical language.

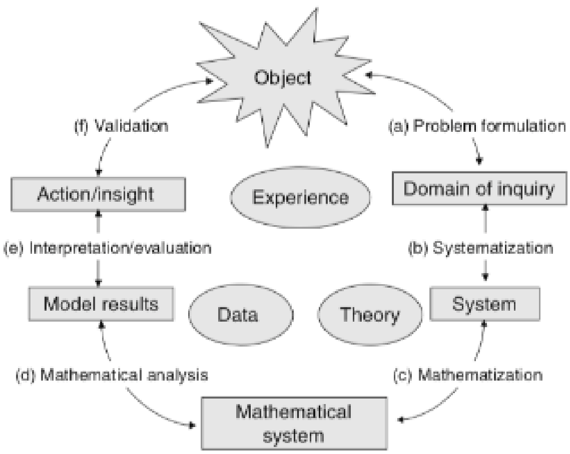
d) Use of mathematical methods to arrive at mathematical results and conclusions.

e) Interpretation of the results and conclusions considering the initial research domain.

f) Evaluation of the model's validity by comparison to data or theoretical knowledge or personal or shared experience.

Figure 2

Modelling Cycle (Blomhøj and Kjeldsen, 2006)



The Blomhøj cycle considers three central islands as the basis for its dynamics (see Fig. 2.): experience, data, and theory. Solving the problem or a given task often leads to data collection, which may be part of a surmised model. Also, the experience can lead to a constructive idea of the model, whose approach is generally inductive. However, the use of mathematical models that exist in theory emphasises their use in solving a modelling task, highlighting a deductive nature. Likewise, the theory island (see Fig. 2) also means the knowledge employed following the research domain to be used in the modelling task, which significantly affects the validation processes of the model and its applications.

The sub-processes implemented to solve a modelling task, characterised in Fig. 2, naturally lead to mathematical ideas drawn according to the choice of research domains. These give to the whole resolution process from an epistemological and cognitive perspective. It proposes a complex system of work, highlighting central aspects of the process (the islands: data, experience, and theory in Fig. 2), which link knowledge of reality and mathematics, making it possible to provide –from the point of view of research– varied analytical forms of idealisation and, therefore, diverse possibilities for building mathematical models. This process, together with the task objective, will make it possible to establish ways of delimiting and characterising mathematical work, particularly circulations through the cycle as the student's personal MWS develops.

Mathematical Working Spaces (MWS) and Modelling

It is currently possible to study multiple works regarding modelling from the point of view of the MWS (Nechache, 2016; Rauscher & Adjiage, 2014, among others). In addition, one may find a diversity of points of interest in the investigations: some consider tasks as a contextualised environment to develop the MWS (Rauscher & Adjiage, 2014; Cosmes, 2018); others analyse the effects of recognising reality as a unique environment for the development of the MWS (Parzycs, 2014); or, also, problematise the theoretical modelling frameworks to confer conceptual results and eventual articulations with the MWS (Nechache, 2016; Derouet, 2016). This last working group is of interest for this study since it aims to facilitate the analysis of modelling tasks, considering a particular approach such as that of Blum-Borromeo (Borromeo-Ferri, 2006). In this sense, this work's contribution is to develop a conceptual articulation between MWS and the Blomhøj modelling cycle, promoting other forms of task analysis and broadening the discussion in the scientific community. It is not a question of recognising some suitable modelling framework for the MWS but of encouraging studies about this topic, stripping us of conceptual uniqueness, and proposing an articulation *vis-à-vis* other perspectives to strengthen the discussion on modelling from the MWS.

Generally, the modelling researches reported from the MWS are qualitative and empirical, focused on the interaction of the real world or other sciences with mathematical models. Also, several investigations are driven by the school context of French secondary education, highlighting the modelling cycle as a common aspect of learning, specifically in geometry and probability (Nechache, 2016, p. 52). In this regard, Nechache addresses this curricular aspect using software for geometric and probabilistic work, characterising the modelling practice for secondary school students. Likewise, Lagrange (2015, p. 317) uses the modelling cycle of the “Casyopée” equipment for dynamic geometry, studying extensions of research towards the modelling of physical problems so that specific types of mathematical functions acquire the sense of solving such problems.

Rauscher and Adjiage (2014) analyse through the MWS the work of solving the problem of the giant –similar to the classic shoe task of Blum and Borromeo-Ferri (2009), through an experiment conducted with students aged 10-11 years. The authors affirm the need to experiment with a modelling process to solve the problem and that this is not an actual research problem for an expert. Despite this, the problem is adequate to develop mathematical thinking from this research for the students who do not have a well-established reference.

Derouet’s doctoral work (2016, p.228-231) proposes a discussion regarding modelling cycles, referring to the cycles proposed by Kaiser (1995), Blum (1996), Blum and Leiß (2007), Coulange (1998) and Henry (2001). The author emphasises that the last two works use the *pseudo-concrete* domain as part of a mathematical domain but situated in another paradigm and the idea of distinguishing between reality and mathematics and considering the choices made regarding the model in reality. Likewise, the author states that, contrary to the Blum cycle (Blum & Leiß, 2007), all the stages proposed by Henry (2001) do not have the same status, for they are characterised as actions or states, which is why the author decides to work with the Blum cycle (Blum & Leiß, 2007).

One work of the MWS, which addresses different modelling approaches, is that of Nechache (2018). The author states that the resolution of probabilistic tasks in the context of French secondary education requires the construction of models involving the implementation of different stages of the process. In addition, it induces uses or changes of probabilistic paradigms (Parzysz, 2011) and domains (Montoya-Delgadillo & Vivier, 2014). In conceptual terms, the author describes a modelling cycle adapted to the probabilistic domain (Nechache, 2016) based on the approaches by Kaiser (1995), Blum and Leiß (2007) and Borromeo-Ferri (2006). Moreover, the author adapts three fundamental stages of the cycle by Henry (1999): description of reality, mathematisation, and external validation. In addition, for the first stage of the cycle, it is observed that it is necessary to make hypotheses based on the probabilistic domain and, therefore, the author makes the MWS of reference of said domain intervene prior to the construction of the mathematical model. In this sense, the MWS may be present in the different modelling stages, not only since the model exists; thus, we believe it is necessary to consider other theoretical modelling approaches.

Complementarity between the MWS and the Blomhøj modelling cycle

In this section, we will try to approach a complementarity of both theoretical constructs, MWS and the Blomhøj cycle, without intending it to be a definitive discussion, but rather with the idea of expanding it, including other scopes of modelling in the MWS. To do this, we draw from the premise that there are similarities associated with the theoretical knowledge, the experience, and the valuation of data (see Fig. 2) since, based on the conceptualisation of signs, it is possible to begin to characterise the relationships with the MWS. Next, we describe a way to relate each sub-process, shown in Fig. 2, with the components of the MWS, shown in Fig. 1.

In sub-process (a), problem formulation, the islands theory, experience, or data would intervene (see Fig. 2), which does not necessarily imply mathematical work. In sub-process (b), systematisation, the theory relating to the problem domain comes directly into play to formulate assumptions and identify variables and their possible relationships. It would explicitly or implicitly produce a circulation or development of reasoning in the vertical plane Sem-Ins (see Fig. 1). Process (c), mathematisation, corresponds to the development of a mathematical model through which the identified variables are constructed. Such construction uses the assumptions and relationships of the previous step, activating various components and the genesis of the MWS. Specifically, the circulation in the vertical plane Sem-Ins (see Fig.1) for interpreting and constructing signs and their relationships, together with the corresponding components of the epistemological plane, explicitly activating a mathematical reference. In process (d), analysis of the mathematical system, the results are determined possibly using the simulation and proof (for example, of existence and uniqueness of analytical and approximate solutions of the model). It activates the circulation in the vertical plane Ins-Dis to construct the results and the possible (instrumented or not) demonstrations. In process (e), interpretation and evaluation, a circulation is positioned in the vertical plane Sem-Ins. For example, visualisation and construction emanating from the representamen, such as a graph or table of results, could lead to questioning the process (b). The results are evaluated by comparing them with data or with experience, which could be related to the discursive genesis or algorithmic processes, activating the instrumental genesis. Finally, the validation process (f), closely related to the previous process, would have the same connection with the MWS.

Given the background and the previous theoretical study on the complementarity of the Blomhøj cycle with the MWS, we think this could expand the analysis of modelling tasks. Therefore, we consider the previous study to investigate the activity of solving a modelling task, which will be presented in the methodology, to try to answer the question: How is modelling activity characterised from the MWS when interpreted through the Blomhøj cycle in engineering students?

METHODOLOGY

This research considered two methodological aspects: first, the design of a task based on a modelling problem with a progressive level of complexity since it comes from the real world, demanding a high cognitive capacity for its mathematical formulation. It is about an actual research problem, an issue mentioned as desirable for modelling tasks at the ETM6 symposium (Montoya-Delgadillo *et al.*, 2018) and highlighted in Blomhøj's theoretical proposal (2004). The second methodological aspect consisted of a qualitative approach to characterise students' mathematical work, delving into ideas and meanings of knowledge (Denzin & Lincoln, 2012). In this sense, the task acquires uniqueness and complexity, given the intention of understanding how circulations occur in the MWS during the modelling process, defining the experimental part of the research as an instrumental case study (Stake, 2007). In addition, *triangulation between researchers* was used to carry out an individualised data analysis to corroborate degrees of similarity, searching for agreement in the dissimilar analyses to increase internal research reliability.

Such a methodological design was feasible given the experience and training of at least two authors who performed such triangulation since they are specialists in mathematical didactics with publications in the area and experience in qualitative analysis.

The proposed task has been catalogued as a modelling task by three scholars (professors and researchers) who have extensive university teaching experience and have taught calculus courses for over three years. These scholars point out that the type of "guided" task is commonly used in university education, given the high cognitive complexity its resolution requires from the students. In this sense, throughout Stewart's textbook (2008), in force in the calculus programs, this modelling task type is encouraged as real problems, requesting a computerised resolution through software, assuming the student has access to it. In addition, this classic calculus textbook attaches great importance to models, seen as applications of functions, in particular, to the dynamics of population growth (sections 1.2, 1.4, 1.5-1.6, 2.8, 3.4, 3.7, 3.8, 4.3, 4.9, 9.1, 9.4, 9.6, 11.1), the subject of the task proposed in this work.

The experimentation consisted in requesting the written and personal task resolution of 31 second-year students of the career of civil computer engineering at a Chilean public university, collected at the end of a 90-minute session. For our study, we have selected an individual for our analysis, which we will call Juan, whose written production is sufficiently complete and of interest for a correct interpretation, both in the understanding of the mathematical work developed and in the deepening of the ideas.

To reduce, organise, and give meaning to qualitative data, we have considered, according to Burns and Grove (2004), three stages of analysis of Juan's written production: description, analysis, and interpretation.

The task proposed to students is as follows:

Table 1.

Modelling task.

Año	Población
1907	3231022
1920	3720235
1930	4287445
1940	5023539
1952	5932995
1960	7374115
1970	8884768
1982	11329736
1992	13348401
2002	15116435
2017	17574003

Table A. Population data

The population has been ageing in the last three decades, and the birth rate has been progressively decreasing. The following table shows the evolution of the population between the years 1907 and 2017, obtained from the information collected in the censuses carried out approximately every ten years (Compendio Estadístico/Statistical Compendium, 2018).

The objective of this problem is to predict population growth from the year 2017 or to estimate it between the values measured in the censuses. For this, we will establish a general formula that models population growth as a function of time. We will start by analysing the simplest model of population growth. We will assume that:

(H1) The relative population growth rate remains constant (independent of population size).

(H2) Individuals reproduce only once over a given period (e.g., every decade to account for data availability).

We will measure the time k in generation units (period between one generation and the next, for example, a decade). Let us denote by x_k the number of individuals in the population in generation k so that x_{k+1} will designate the number of individuals in the population in the next generation, k+1. According to the assumption (H2), there is no population reproduction between the periods k and k + 1. Accordingly, the more general formula to describe population growth between k and k + 1 is of the following type:

$$x_{k+1} = f(x_k), \text{ for } k = 0, 1, 2, \ldots, \quad (1)$$

Where f is a function that determines the number of individuals in the period k + 1 (x_{k+1}) based on the number of individuals in the period k (x_k). The relative growth rate of the population is used to find an appropriate expression for f. The first consists of the difference between the number of individuals in the period k+1 and the amount in period k divided by the last (or the ratio of the number of individuals in period k + 1 and the amount in period k minus 1), i.e.:

$$r_k = \frac{x_{k+1} - x_k}{x_k} = \frac{x_{k+1}}{x_k} - 1 \quad (2)$$

$r_k > 0$ means that the population grows between the period k and $k + 1$; $r_k < 0$ means that the population decreases between k and $k + 1$, while $r_k = 0$ means that the population remained constant between k and $k + 1$.

By finding an appropriate relationship between r_k and x_k, f can be obtained by clearing x_{k+1} as a function of x_k from equation (2). The request:

1 Calculate the relative population growth rate between two consecutive periods, i.e., calculate r_k defined by (2).

2 Calculate the average relative growth rate r from the r_k.

3 According to the assumption (H1), the simplest discrete mathematical model, given by equation (1) is obtained, assuming that the relative growth rate remains constant. Then, assuming this constant equals its mean r, set a formula for the function f by clearing x_{k+1} as a function of x_k from equation (2).

4 Based on the model found, determine the population growth in the periods tabulated in Table 1. To do this, the model begins with the initial population, x_0= 3231022, corresponding to the population in 1907. Then, calculate successively x_1, x_2, ..., starting from x_0, using the formula found in the previous item.

5 To determine the quality of the model found, calculate the relative errors made when approximating the actual population size from the value obtained by the model. That is:

$$\frac{|y_{k+1} - x_{k+1}|}{y_{k+1}} \quad (3)$$

where y_{k+1} corresponds to the actual population size in period k+1 and x_{k+1} is the population size predicted by the model in the period k+1. Also, calculate the average relative error. How good are the model's approximations?

6 Based on the proposed model, predict the population size determined in future censuses, assuming that these will be carried out every ten years. In how many more decades will the size of the population double compared to the size in the year 2017?

DATA ANALYSIS

We divided analyses into two sections: the first consists of the task design, and the second of the student's answers through the MWS in the different phases of the modelling.

Analysis of the task design

The task was designed (see table 1) so that the student could understand the general formulation of the proposed model and answer the questions. In this sense, the student should read the statement several times since calculus courses usually do not pay much attention to modelling tasks, as they focus on mathematical constructs dictated by the plans and programs. So, it would imply that the student would be facing a task with a high level of complexity (Cabassut & Ferrando, 2017).

Likewise, since applying this type of task requires careful teacher's guidance, the design of the statement included the following pieces of the modelling cycle. First, problem formulation follows from the paragraph before the two assumptions, including them (H1 and H2 in table 1). The systematisation is by the variables pre-defined (relative population growth rate) and relationships of interest for the problem's research domain (the relation between the relative growth rate and the number of individuals in the population, table 1). Finally, part of the mathematisation is devoted to deducing the formulas (1) and (2) in table 1. The three previous steps would allow students to approach the proposed model.

Description of the task

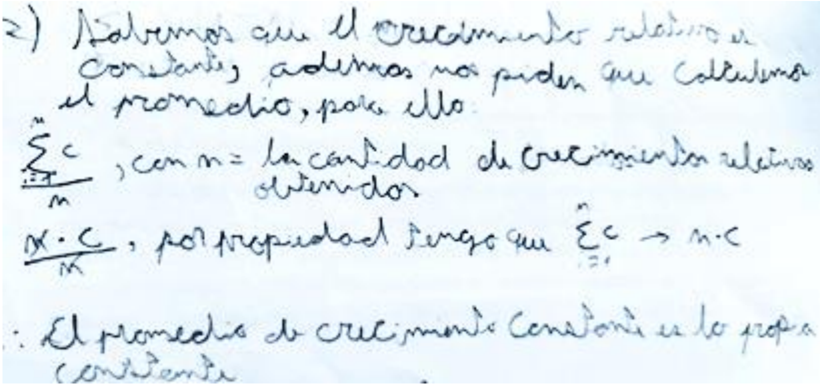
The proposed area of research corresponds to population dynamics, one of the main objects of study of mathematical biology. As the problem is formulated from a discrete model for a single variable (equation (1) in table 1), the mathematical domain is within the real analysis. It is up to the student to establish the system or mathematical model, i.e., the final part of the mathematisation, guided along the statement until reaching question 3. The model equation has to be obtained using both assumptions, mainly guided by H1 (table 1), requesting the student to previously estimate the average relative growth rate (question 2 in table 1). It constitutes a preliminary calculation to establish the model, considering that this variable is constant and equal to its average (question 3 in table 1). Next, the task proposes the calculation of population estimates (question 4 in table 1), a stage of mathematical analysis. It leads to the model results and then to interpret/evaluate the results through the average relative error to advise the model's validity in quantitative form, establishing margins of precision in the predictive calculations required by the task (question 5 in table 1). For the above, the statement provides a table of data on the Chilean population measured in the censuses throughout the period 1907-2017, extracted from the website of the Instituto Nacional de Estadística [National Institute of Statistics]. As a central island of the Blomhøj cycle (see Fig. 2), data may come from the modelling task and be used in the stages of systematisation and mathematisation, both for model construction and validation. In this sense, an eventual task whose goal would be formulating the same problem, with a research domain to discover, should guide the characteristics of perceived reality to model. That could be achieved, for example, through a (semi-logarithmic) graph of the data to visualise population growth, which could guide model construction.

Analysis of the task resolution

Despite students correctly answering questions 1 and 2, we noticed difficulty in specifying a formula for the model in question 3 (equations (1)-(2) combined with H1 and statement in question 3, see table 1). That could be by confusion of assumption H1 with reality, i.e., instead of calculating the relative growth rates in each period (question 1), they only calculated it in the first or subsequent period. Then, they limited themselves to saying that, given that it remained constant, its value was equal to the one they calculated in the period $k = 1$ or $k = k_0$ (for a fixed k_0), from which model results do not fit actual data. Indeed, most students (23 out of 31) mistakenly assumed that $r_k = r_{k_0}$ instead of $r_k = r$, explicitly denoting that $r_k = c$, where c is the value of the relative growth rate in some fixed period, k_0, arbitrarily chosen. A particular case that features this mistake is shown in Figure 3.

Figure 3

Confusion of assumption H1 with actual growth rate values


2) Sabemos que el crecimiento relativo es constante, además nos piden que calculemos el promedio, por ello:

$\frac{\sum_{i=1}^{n} c}{n}$, con n = la cantidad de crecimientos relativos obtenidos

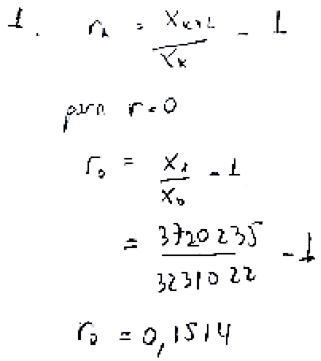
$\frac{n \cdot c}{n}$, por propiedad tengo que $\sum_{i=1}^{n} c \rightarrow n \cdot c$

∴ El promedio de crecimiento constante es la propia constante

This error was foreseeable, given the students' inexperience in this task type and, above all, because of the high cognitive requirement, which constitutes an actual research problem.

Figure 4

Juan's answer to question 1.


1. $r_k = \frac{X_{k+1}}{X_k} - 1$

para r=0

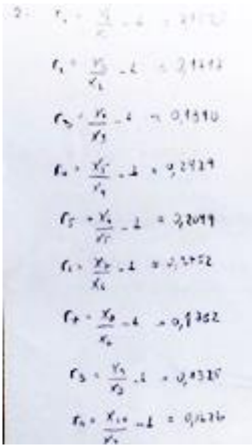
$r_0 = \frac{X_1}{X_0} - 1$

$= \frac{3720235}{3231022} - 1$

$r_0 = 0,1514$

Figure 5

Juan's answer to questions 1 and 2.



As announced in the methodology, we focused the analyses on Juan's answers, representing the portion of students who answered correctly; see Figures 4-5.

Analysis and interpretation

Questions 1 and 2 (Figure 5) require a large margin of manoeuvre, since the relative growth rates r_k and their average r must be calculated. It should be by latent semiotic participation, in the sense of understanding the concepts involved in these notations, which could be considered a connector between the theories of Blomhøj and the MWS since reality is idealised by variables or signs in the systematisation stage. From the answers to questions 1 and 2, we infer an activation of the semiotic genesis since it is necessary to interpret the meaning of the algebraic signs r and r_k. In addition, instrumental genesis is activated to construct and visualise the concepts they represent (relative growth rates and their average). So, there was circulation in the vertical plane Sem-Ins to configure the conceptualisation and understanding of involved notions. The numerical tabulation, without necessarily a validation objective, explains and identifies the components through the semiotic representation mentioned above. The answers are positioned in the GA paradigm, for they involve work of a perceptual type based on numerical tabulation, which encourages the understanding of concepts and allows for interpretations based on the real world, thus connecting the model with the reality of the problem studied. The previous analysis shows a connection between the circulation in the vertical plane Sem-Ins of the MWS with the systematisation and mathematisation phases of the Blomhøj cycle.

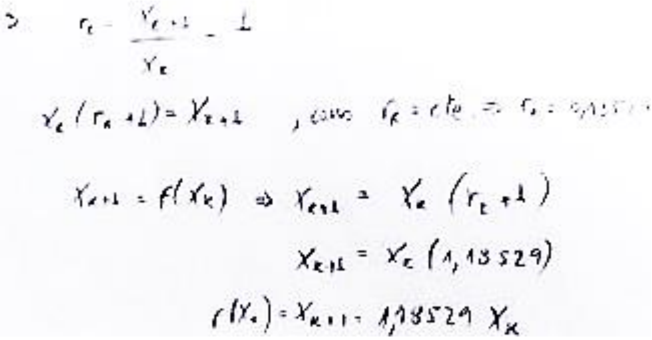
Despite the task difficulty and the fact that most students did not get the correct answer, Juan's group managed to answer what was requested; see Figure 6.

The fact that the task was guided along the statement allowed a better understanding of students who obtained the model. In this regard, question 3 invites assuming, apart from H1 and H2, that the relative growth rate is constant and equal to its average (Fig. 6). We interpret it as a possible connection between the MWS's epistemological plane (Fig. 1) and the Blomhøj modelling cycle centre (Fig. 2). In particular, between theoretical knowledge (referential) with the data and theory of the research domain. In addition, the student would have to decipher a theoretical assumption of the research domain of the real problem. Then, he should translate it by a mathematical work positioned in the

GA paradigm. Also, it requires activating circulations in the vertical plane Sem-Ins to achieve interpretation and construction of the assumption to be able to mathematise it.

Figure 6

Juan's answer in item 3



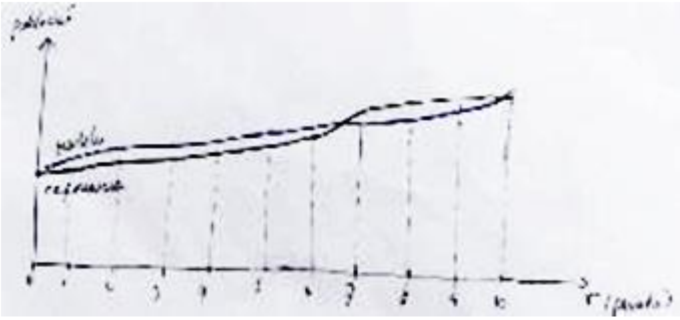
Then, the student calculated the estimates requested in question 4, presenting a tabulation of population sizes calculated from the model. In addition, although it was not asked, he made a freehand drawing of a graph, considered an instrumental artefact. Again, it emphasises reasoning in the vertical plane Sem-Ins and work in the GA paradigm because the construction favours visualisation and interpretation of the results. Although the graph did not provide an ideal visualisation, it is a good approximation since it is possible to see that the actual and simulated growth curves are relatively similar (Fig. 7), giving meaning and usefulness to the model application.

The calculations in question 4 are an essential part of the mathematical analysis of the Blomhøj cycle, an aspect that we consider fundamental to connect the real world with the mathematical work of the student, i.e., with his personal MWS. This connection, in turn, makes it possible to establish that the population will double in approximately four more decades, an inference based on the results, another important aspect of the cycle.

Regarding the epistemological plane, diverse representamens or semiotic representation registers are used (data table, assumptions formulation, variables, and model's general formula). Reference MWS was based on mathematical work in the real analysis field to get results. In particular, it relied on arithmetic calculations and experimental validations by graphical-arithmetic records of the model (Figs. 3-7). On the other hand, the relative errors of approximation of the model to the data were calculated. Also, experimental validation (stage of interpretation/evaluation and validation of the model) and predictions were determined (action-vision). In summary, throughout the student's answers, we identified a work positioned in the GA paradigm, alongside reasoning in the vertical plane Sem-Ins, given that interpretations are favoured through a work instrumented from graphical-numeric records.

Figure 7

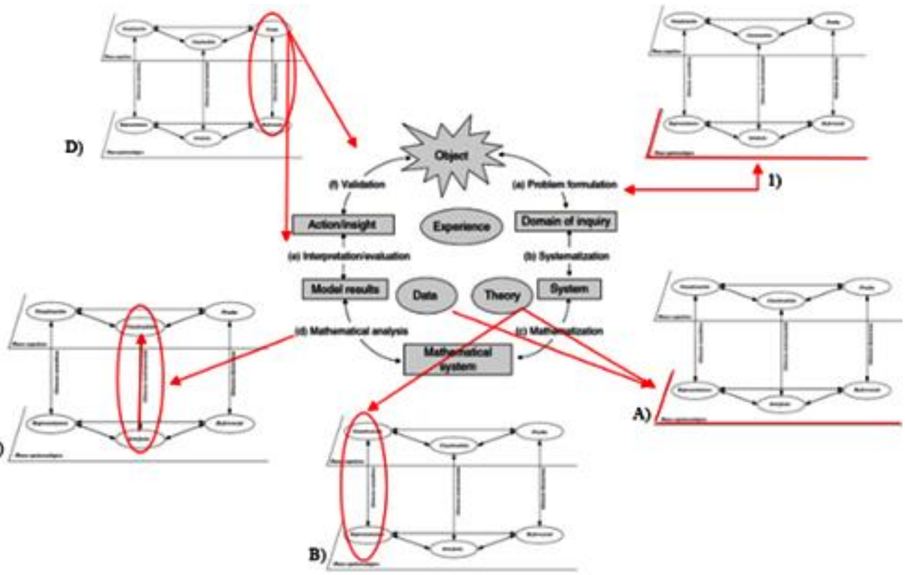
Estimates of the population based on the model



Concerning the cognitive plane, the student had to develop visualisation and construction to interpret and decipher the signs. Thus, he could internally structure the information provided, particularly the object representation (model's recurrence formula, relative growth rate, and population size) and the relationships involved (model's general equation and the relative growth rate). Then, he achieves a formula for the discrete exponential model (mathematisation and obtaining the mathematical system). Additionally, the student had to develop the construction to calculate the estimates/results based on the model (mathematical analysis), i.e., the student activates reasoning in the vertical plane Sem-Ins, as discussed before. Finally, theoretical development is unavoidable, given the existing theory behind the model, which contributes to the understanding of population dynamics. Despite no deductive rationale being applied to obtain either an analytical solution or long-term asymptotic behaviour (convergence/divergence of the model's solution) when computing the prediction of the doubling time for the population, one could consider it as empirical proof of this behaviour. In this sense, the above interaction in the transit per cycle can be observed, mainly through interpretation/evaluation and validation.

Figure 8

The Connection between the Blomhøj cycle and the MWS.



Based on figure 8, in summary:

1 The problem formulation might develop the mathematical work since it considers the data and theory located at the Blomhøj cycle centre.

A The data and the referential connect with the epistemological plane of the MWS. Through mathematisation, it can be connected to the cognitive plane.

B The notions involved are possible connectors with the representamen and visualisation.

C The instrumental genesis activates from the data analysis (centre of Blomhøj's cycle).

D From the (pragmatic) test, according to Balacheff, carried out by the student, interpretation and validation would follow, thus connecting this way to the Blomhøj cycle.

We consider A) and 1) (Fig. 1) two possible connections between the Blomhøj cycle and the MWS, where a student's mathematical work can be developed and then reconnected to the Blomhøj process in D).

We observe that C) has possible similarities with d) of the Blomhøj cycle since mathematical analysis is developed when the instrumental genesis is activated. In particular, one can observe it in Juan's table, used as an instrumental artefact (Fig. 7), to then rely on it in constructing the graph.

CONCLUSIONS

In this work, we have proposed to complement the MWS theory with the Blomhøj modelling cycle, identifying the connections between both theoretical frameworks, in an investigation that intends to expand the discussion of modelling from the MWS. To test this intersection, we designed a modelling task based on an actual problem, a desirable aspect expressed in the sixth symposium on the mathematical work, ETM6. Given the high cognitive requirement to solve it, the task design considered a statement composed of different questions that included problem formulation, systematisation, and part of the mathematisation.

The analysis of the task resolution showed that some students presented difficulties since they were not used to working with models in university and because of the intrinsic difficulty of the problem. In particular, the most critical difficulty found is that they confuse reality with mathematics, suggesting that it is necessary to distinguish data from mathematisation properly. However, the analysis also shows that a portion of the students could develop a mathematical model consistent with the task design, guided by considering the aspects above.

Concerning the intersection of the Blomhøj cycle with the MWS, we observed that some circulations are favoured through the mathematical work between various components of the epistemological and cognitive plane (representamen-visualisation, artefacts-construction), developing, fundamentally, reasoning in the vertical plane Sem-Ins. It was provided by the exploratory approach through the interpretation, construction and validation of the assumptions and model results, favouring a work positioned in the GA paradigm, i.e., of a perceptual type based on the abovementioned approaches.

It could be due to the task design, which did not consider making demonstrations or using analytical calculation rules. It would have led to a work located in the other paradigms of real analysis and to the development of reasoning in the different vertical planes of the MWS.

One of the main contributions of this research is that the results suggest that the Blomhøj cycle may present theoretical connections with the MWS. The first would be from the formulation of the real problem, as this process leads to having to choose the field within the mathematics where the problem is going to be framed visualising the beginnings of an eventual mathematisation that is not only referred to in a purely mathematical environment, but that affects reality. This could be a first entry to the MWS through the referential, which must be chosen through this process. In addition, systematisation would directly connect with the MWS since it leads to the selection of relevant objects, relationships, and idealisation to make the mathematical representation, which would be made explicit at this stage. All other cycle processes are directly related to the MWS (mathematisation, mathematical analysis, interpretation/evaluation and validation). On the other hand, we believe that the central islands in the Blomhøj cycle (data, experience and theory) are closely related to the MWS, as these aspects might contribute to the mathematical work involved in establishing or validating the model.

Finally, in this case, the working paradigms concerning the underlying mathematical field begin to make themselves evident from the mathematisation process since it is where the referential is made more explicit. However, we must recognise that, depending on the problem, these could manifest themselves before, precisely, when the problem is formulated through assumptions and mathematical variables. In this sense, to address the proposed problem, it has been necessary to make some decisions on the variables to be considered. In terms of Blomhøj (2003), this relates to selecting relevant objects and idealising variables to make mathematical representation possible. Precisely, in choosing and idealising variables, we believe that the relationship with the MWS would occur from the systematisation and possibly from the formulation because, as already mentioned, we would be choosing the mathematical field where the model is formulated.

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AUTHORS' CONTRIBUTION STATEMENT.

PVH and JHA have contributed as experts in MWS theoretical frameworks and modelling. PCA has contributed to the data collection and is the creator of the modelling task. PVH conceptualised the original manuscript's analysis, writing, reviewing, and editing and coordinated the research group. All authors have actively participated in discussing the results and conclusions, jointly approving the final version of the work.

DATA AVAILABILITY STATEMENT

The author Paula Verdugo-Hernández will provide data supporting this study's results upon reasonable request.

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