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A Reference Epistemological Model Regarding the Determination and Construction of Solids for Compulsory Secondary Education

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RESUMEN

Contexto: En el análisis de los saberes geométricos propuestos en el currículo de Educación Secundaria se manifiestan fenómenos como la separación entre las geometrías 2D y 3D y el debilitamiento de la actividad de modelización en geometría. Brousseau considera que la construcción de figuras es un primer ejemplo de modelización geométrica. **Objetivos:** Construir un modelo epistemológico de referencia que explicita las condiciones que permiten determinar la forma y el tamaño de un sólido y buscar qué posibles técnicas permiten construirlo. **Metodología:** Investigación teórica en el marco de la Teoría Antropológica de lo-Didáctico. **Entorno y participantes:** El modelo construido es fruto de varios trabajos realizados en los tres últimos años: análisis de textos escolares y diseño, implementación y análisis de un recorrido de estudio e investigación en torno al diseño de un envase en dos Institutos de Educación Secundaria con alumnos de entre 14 y 17 años. **Recogida y análisis de datos:** El modelo está basado en el análisis de informaciones recogidas de textos científicos de Pólya y otros autores, de textos oficiales y manuales escolares de Educación Secundaria y de las experimentaciones realizadas. **Resultados**: El modelo sustenta el estudio articulado de las geometrías bidimensional y tridimensional y permite guiar procesos de estudio tendentes a abordar de manera coherente el problema de la determinación de un sólido y su construcción. **Conclusiones:** El modelo elaborado contiene cuestiones sobre la problemática de la modelización espacio-geométrica que consideramos como la problemática de iniciación a la geometría en la enseñanza secundaria.

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Palabras clave: Modelo epistemológico de referencia; Determinación y construcción de sólidos; Problema espacial; Modelización espacio-geométrica; Técnicas algebraico-funcionales.

A reference epistemological model concerning the determination and construction of solids for compulsory secondary education

ABSTRACT

Background: The analysis of the geometric knowledge presented in the secondary education curriculum reveals phenomena such as the separation between 2D and 3D geometry and the weakening of the modelling activity in geometry. Brousseau considers that the construction of figures is a first example of geometrical modelling. **Objectives**: To build a reference epistemological model that clearly sets out the conditions that allow determining the shape and size of a solid and looking for possible techniques that enable constructing it. **Design**: theoretical research within the framework of the Anthropological Theory of the Didactic. **Setting and participants**: The model built is the result of several activities carried out in the last three years: an analysis of school texts, and the design, implementation, and analysis of a study and research path regarding the design of a container in two secondary schools with students aged between 14 and 17. **Data collection and analysis**: The model is based on the analysis of information collected from scientific texts by Pólya and other authors, from official texts and secondary education textbooks, and from the experiments carried out. **Results**: The model is based on the structured study of two- and three-dimensional geometry and allows guiding study processes aimed at consistently addressing the problem of determining a solid and its construction. **Conclusions**: The model developed includes questions regarding spatial-geometric modelling considered to be central in the introduction to geometry in secondary education.

Keywords: Reference epistemological model; Determination and construction of solids; Spatial problem; Spatial-geometric modelling; Algebraic-functional techniques

Um modelo epistemológico de referência em torno à determinação e construção de sólidos para o ensino secundário obrigatório

RESUMO

Contexto: Na análise dos conhecimentos geométricos propostos no currículo do Ensino secundário, fenômenos como a separação entre geometria 2D e 3D e o debilitamento da atividade de modelagem em geometria são evidentes. Brousseau considera que a construção de figuras é um primeiro exemplo de modelagem geométrica. **Objetivos**: Construir um modelo de referência epistemológico que explicite as condições que permitem determinar a forma e o tamanho de um sólido e descobrir que técnicas possíveis tornam possível a sua construção. **Metodologia**: Investigação teórica no âmbito da Teoria Antropológica da Didática. **Ambiente e participantes**: O modelo construído é o resultado de vários trabalhos realizados nos últimos três anos: análise de textos escolares e concepção, implementação e análise de um percurso de estudo e investigação em torno do desenho de uma embalagem em duas Escolas Secundárias com alunos entre os 14 e os 17 anos de idade. **Recolha e análise de dados**: O modelo é baseado na análise de informações coletadas de textos científicos de Pólya e outros autores, de textos oficiais e manuais escolares do Ensino Secundário e das experiências realizadas. **Resultados**: O modelo apoia o estudo articulado de geometrias bidimensionais e tridimensionais e permite orientar processos de estudo tendentes a abordar de forma coerente o problema da determinação de um sólido e da sua construção. **Conclusões**: O modelo desenvolvido contém questões sobre o problema da modelagem espacial-geométrica que consideramos ser o problema da iniciação à geometria no ensino secundário.

Palavras-chave: Modelo epistemológico de referência; Determinação e construção de sólidos; Problema espacial; Modelagem espaço-geométrica; Técnicas algébrico-funcionais.

INTRODUCTION AND BACKGROUND

The new official Spanish curriculum, which has just been promulgated, considers that one of the basic knowledge areas to be taught is *spatial sense*. The document published (MEFP, 2022, p. 156, our translation) clearly sets out that:

> Spatial sense addresses the understanding of the geometric aspects of our world. Registering and representing shapes and figures, recognising their properties, identifying relationships between them, locating them, describing their movements, making or discovering images of them, classifying them, and reasoning with them are key elements of teaching and learning geometry.

In the official French curriculum (Eduscol, 2020), in theme D on Space and Geometry, recognising, building, and representing solids is put forward as an objective by the end of cycle 4, which includes secondary education students aged between 12 and 15. The use of dynamic geometry software for said representation is also contemplated.

Guy Brousseau (2000) points out that the construction of figures is a first example of modelling a part of elementary geometry, and Perrin-Glorian and Godin (2014) consider that plane geometry consists in the study of flat shapes and figures. Perrin-Glorian, Mathé & Leclerc (2013) indicate that in order to achieve coherent and functional teaching of geometry in compulsory education, it is necessary to use what Berthelot & Salin (2005) call *space modelling problems or spatial-geometric problems*. Thus, if we consider geometry as a model of physical space, it turns out that the notion of model is inseparable from the study of geometry (Houdement, 2019). Salin (2014) states that geometric knowledge should be introduced as a tool for solving spatial problems, that is, within a spatial-geometric modelling problem.

We postulate that a possible raison d'être of the study of elementary geometry in the case of three-dimensional geometry basically consists in the study of *the determination and construction of solids*.

Furthermore, the search for possible answers to this spatial problem using algebraic-functional modelling techniques will be improved thanks to the use of GeoGebra. It will allow the study of 3D geometry to be connected with that of algebra and functions, as put forward by the Spanish Committee for Mathematics:

> More attention should be paid [in the curriculum] to: using dynamic geometry programmes to work on geometry, relating geometry to algebra and functions, and solving problems (CEMAT, 2021, p. 34).

Rojas and Sierra (2021a; 2021b) enquired about the raisons d’être the study of geometry in secondary education should respond to, particularly two-dimensional (2D) and three-dimensional (3D) geometry and developed and implemented two *study and research paths* (SRPs) regarding the design and construction of a container. These SRPs enabled studying several geometric knowledge areas put forward in the curriculum of Compulsory Secondary Education (ESO in Spanish).

One of the most important tasks of the secondary education mathematics teacher consists in designing and implementing study processes in the classroom related to a certain mathematical organisation (MO) proposed in the curriculum (Chevallard, 2002). The *Anthropological Theory of the Didactic* (TAD) considers it is important for teachers to question themselves about the knowledge to be taught and to enquire about some of its possible raisons d’être. However, finding some of the questions to which geometrical knowledge responds constitutes a complex task that teachers alone can hardly address, since it appears as an open didactic research problem.

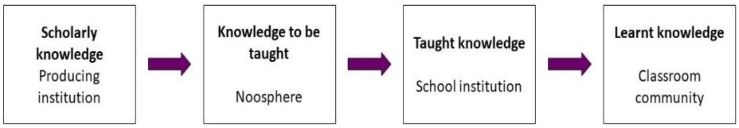
Sierra, Bosch and Gascón (2007) showed that the *didactic tasks* teachers should implement together with their students to reconstruct an MO, the *didactic techniques* used to develop those tasks, and the *technological-theoretical discourse* that allows them to interpret and justify those techniques, mainly depend on the structure of the "mathematical" components and on the raison d'être assigned to said MO.

One of the objectives of this study, apart from providing some of the raisons d'être for school geometry in compulsory secondary education offering a possible connection between 2D and 3D geometry and with other basic knowledge such as algebra and functions, is to set out and develop a proposal that clarifies what is understood by *determining and constructing solids*.

When researchers in didactics of mathematics intend to carry out the praxeological analysis of an MO, they should take into account the empirical data from the different stages of the didactic transposition process. To do this, they should develop their own "reference" epistemological model allowing them to avoid the restrictions that come from the different institutions in which this MO exists (Figure 1).

Figure 1

The different institutions of the didactic transposition process of an MO.



To build this reference epistemological model (REM), the researcher should develop a rational reconstruction of the MO in question. The didactic analysis of the didactic transposition process to which this MO is subjected will enable detecting some of the *didactic phenomena* present in the process (Bosch & Gascón, 2005).

In this study, we present a possible reconstruction of an MO regarding the *determination and construction of solids* for compulsory secondary education based on the search for possible solutions to a spatial problem related to designing and building a container. This reconstruction, which performs the function of an REM, will be elaborated through a spatial-geometric modelling process. An interpretation of the elementary geometry of solids that assigns a new raison d'être to its study will hence be obtained. It is an alternative to the one established by the dominant epistemological model in compulsory secondary education. On the one hand, this reconstruction helps us tackle the didactic phenomenon of the *separation between 2D and 3D geometry*. On the other hand, it guides the design, experiment and analysis of future SRPs to connect specific techniques of spatial-geometric modelling with algebraic-functional models in compulsory secondary education.

In what follows, using the perspective of the ATD: (1) the theoretical framework, in which the general characteristics of REMs are explained, will be described; (2) the general lines of our research problem will be presented; (3) an REM regarding the determination and construction of solids based on the search for possible solutions, through spatial-geometric modelling, to the spatial problem involved in the design and construction of a container will be presented; and (4) some conclusions about the epistemological and didactic functions of the REM built will be formulated.

THEORETICAL FRAMEWORK AND GENERAL CHARACTERISTICS OF REMs

According to the heuristic scheme presented by Gascón (2011), any didactic-mathematical problem defined using the ATD tools usually starts from a teaching problem considered incomplete, to which "it is necessary to at least add the *epistemological* dimension for it to be considered as a problem" (p. 206). Said dimension turns into an REM that constitutes a scientific hypothesis on which to define the MOs involved in the didactic problem being defined. It will thus be possible to establish:

> […] the most appropriate *scope of the mathematical field* to pose the didactic problem in question. […] The *didactic phenomena that will be visible* to the researcher. […] The *types of research problems that may be posed* […] [and] the *tentative explanations that may be proposed*. (Gascón, 2011, p. 209).

This REM may be formulated in terms of questions and answers that lead to the construction of a relatively complete MO (Fonseca, 2004), built from a series of additions and completions derived from a specific MO. In other words, through a process that starts from a specific type of task carried out using a specific technique, and that can give rise to successively broader and more complex (local, regional and global) mathematical praxeologies (Chevallard, 1999). Let us remember that mathematical organisations or praxeologies:

> […] are made up of a practical block or "know-how" composed of the types of tasks and techniques [T/τ], and a theoretical block or "knowing" made up of the technological-theoretical discourse [θ/Θ] that describes, explains, and justifies the practice (Bosch et al., 2004, p. 211).

It should be noted that the components of a mathematical praxeology or MO (*types of tasks, techniques, technologies*, and *theories*) concern the reference institution, in our case the compulsory secondary education institution. Therefore, "what is considered a type of tasks (or a technique, technology, or theory) in one institution is not necessarily the case in another institution" (Bosch et al., 2004, p. 212).

This study is limited to studying and developing the *epistemological dimension* of the didactic problem or of didactic research we explain below. It consists of the construction of an REM considered as a provisional model or hypothesis we rely on to interpret and describe a certain field of mathematics and to "use it as a reference to analyse the didactic-mathematical facts" (Gascón, 2011, p. 208).

To build the REM, inspiration was drawn from: 1) the works on the MO already developed in the texts of scholarly knowledge, related to mathematics and other scientific disciplines; 2) the official curriculum documents related to the MO under study; 3) the proposals of didactic organisations (DOs) that appear in school textbooks with regard to this MO; 4) the possibilities offered by geometric modelling software such as GeoGebra; and 5) the conditions and restrictions that may arise in school institutions in which the MO in question is considered as an MO "to be taught".

It needs to be said that the REM explained here is, like all REMs:

- a *provisional* model, that is, a hypothesis subject to possible permanent changes to be contrasted with experimental data, and
- a *relative* model, developed by the researcher in didactics for specific and limited purposes.

RESEARCH PROBLEM

The first explorations that led us to define this research were based on the analysis of some of the textbooks proposed for teaching mathematics in compulsory secondary education (Rojas & Sierra, 2017). This analysis revealed the general didactic phenomenon of the disappearance of the raisons d'être of the geometric knowledge put forward in the curriculum. The ATD has dealt with facts related to this general phenomenon, such as the rigidity of the MOs studied in secondary school (Fonseca, 2004), or the lack of justifying the appearance of analytical geometry in upper secondary education, and its disconnection with the study of synthetic geometry presented in compulsory secondary education (Gascón, 2003).

The following are some specific facts that underline this general didactic phenomenon in the analysis performed:

- The fragmented view of school textbooks on the mathematical knowledge proposed, since, for example, the study of functions and the calculation of areas and volumes of solids, tend to appear disconnected.
- The almost exclusively numerical treatment of the formulas used to calculate areas and volumes of solids, as, in most cases, it is enough to substitute their elements for specific values provided to find the numerical value of a certain magnitude.
- The types of tasks proposed mainly consist of *direct* tasks (the quantities and unknowns of a problem are never interchanged to formulate *inverse* tasks), and *closed* tasks (there are no *open* tasks that require the student to decide which variables are relevant to solve the problem).
- A repetitive cyclical relationship is observed between types of tasks and associated techniques. For instance, to explain and justify the use of the Pythagorean theorem, problems involving right triangles are presented. Their resolution requires calculating the measure of one of the sides, where the tool, which has previously been explained to solve such situations, is precisely the use of the theorem.

The REM explained in this article allows characterising a certain MO used in compulsory secondary education geometry as well as the existence of didactic phenomena such as *the separation between 2D and 3D geometry*, and the *weakening of the modelling activity in the field of geometry* (Rojas & Sierra, 2021b).

In line with the above, the research problem addressed in this study consists of *clearly setting out the conditions that allow determining the shape and size of a solid and, once this solid is determined, looking for the possible techniques that can be used to build it.* This general problem, which we call *determination and construction of solids,* starts from the search for the solution to a type of *spatial problem* (Salin, 2004), which consists of designing and building a container, addressed within a *spatial-geometric modelling* approach.

The spatial-geometric modelling problem proposed by Berthelot and Salin (1992) starts from a *system* in which a type of spatial problem arises. To solve it, an appropriate *mathematical model* is developed that represents this system using any kind of (geometric, arithmetic, algebraic, functional, etc.) mathematical element. In this model, the answers obtained are validated in the physical space, according to Brousseau's proposal of considering the *study of geometry as a model of space* (Berthelot & Salin, 2001).

Carrying out this process in teaching is highly interesting, since it allows considering the relationship between physical space and geometric space, taking into account the *experimental dimension of geometry*. This approach enables conducting a consistent study of geometry throughout compulsory secondary education, where both types of spaces need to be considered and properly connected. Within the spatial sense in the first three years of compulsory secondary education the study of the following is considered:

> "1. Two- and three-dimensional geometric figures […] Construction of geometric figures using manipulatives and digital tools (dynamic geometry programmes, augmented reality…)" (MEFP, 2022, p. 163).

The REM here described is the result of the contributions developed in several activities carried out in the past three years. First, we created and implemented an SRP for the design of a container on two different occasions in two secondary schools:

- firstly, with 4th year compulsory secondary education students (aged 15 to 16) and 1st year upper secondary education students (aged 16 to 17) during extra-curricular hours (Rojas & Sierra, 2021a), and

- secondly, with 3rd year compulsory secondary education students (aged 14 to 15) during an elective subject called “Mathematics extension”.

As part of the research developed, the conditions required for this type of modelling to exists in compulsory secondary education have been analysed and studied in Rojas and Sierra (2021b).

AN REM REGARDING THE DETERMINATION AND CONSTRUCTION OF SOLIDS

To build the REM, a generating question, deemed sufficiently relevant and fruitful, was considered with regard to the spatial problem of *designing and building a container*. We believe this problem can be modelled geometrically, thus giving rise to a possible connection between 2D and 3D geometry and the use of algebraic and functional models.

The problem was previously considered in the two study processes implemented, and it was endorsed by the scientific community of experts in didactics of mathematics, as may be verified in Rojas and Sierra (2020, 2021a and 2021b). We will show that this spatial problem can lead to both spatial-geometric modelling and to the use of algebraic-functional models.

The generating question of the REM is the following:

Q_G = *How to design and build a suitable container that has a predetermined capacity or volume?*

Answering this question implies asking oneself, amongst other things, what the condition of being *suitable* means. It is related to the function the container has to fulfil and, presumably, to its shape. For instance, if the container is aimed to hold a liquid substance, surely it should respond to certain needs that are different to those of a container designed to contain a solid product. Actually, a broad classification within these two types of containers could be made, since the liquid to be packaged may or may not contain gas, or, if it is a solid product, it could consist of one or of numerous pieces, as is the case of grainy material.

The condition of being suitable is not absolute, and less so in the problem at hand, since the most suitable container could be, for example, the one that is the most visually attractive to the consumer, even if this means an increase in manufacturing costs, or a greater environmental impact. Therefore, designing and building a suitable container implies new questions about (a) its function, (b) the material it will be manufactured with and its optimisation, (c) the environmental impact its use can cause, and (d) the use of space, for instance, during stacking for storage and transport, amongst other aspects. In other words, designing and building a *suitable* container means taking into account several factors that have different levels of importance, which surely involve the use of knowledge from different sectors of mathematics such as geometry, arithmetic, algebra, etc., as well as knowledge from other disciplines such as chemistry, biology, marketing, etc.

With respect to the knowledge that could be useful to respond to Q_G, we decided to start from the review of some of the guides on the design of containers and packaging currently available on the Internet, such as Navarro et al. (2007), Bertomeu-Camós and Fortuny Cuadra (2016), and Ihobe S.A. and Ecoembes (2017), which can help us in the search for an answer to the generating question.

These guides present a wide range of conditions that should be taken into account when designing a container, such as the fact that the design should be doable, desirable, and sustainable. That is, it should be possible to manufacture and should be profitable, it should respond to consumer needs, and the use of resources should be optimised, while reducing its environmental impact. This certainly configures a complex spatial problem including multiple variables, amongst which the shape of the container, its size, and its composition need to be mentioned. However, for now, we will only include the variables that are related to the condition of being suitable, and those that may involve the implementation of several of the geometric knowledge areas proposed to be taught in the new compulsory secondary education curriculum (MEFP, 2022). Thus, when seeking an answer to Q_G, the issues mentioned below will not be addressed:

a. the type of material the container is made of

b. the type of product the container will hold

c. marketing related to the product to be packaged

Eco-design packaging considers a system of three types of packaging: primary or consumer packaging, secondary or grouped packaging, and tertiary or transit packaging. Our focus is on designing and building a primary type container, since the other two types usually have orthohedral shapes, and determining and constructing orthohedral solids is amongst the easiest, and therefore reduces the possibilities of developing the spatial-geometric modelling activity and its relationship with algebraic-functional models.

It is important to note that, although it is clear there is a difference between the container (packaging) and the content (the material the container holds), the problem of designing and building a container will be limited in our study to the case of a container holding liquid. Therefore, and considering the ideal case in which the liquid completely fills the container, we assume that calculating the capacity of the container is equivalent to calculating the volume of the solid associated with the shape its content adopts. On the other hand, the thickness of the *walls* of the container will not be taken into account, given the complexity of calculating the volume of the container (i.e., the material the container is made of) using algebraic techniques, since orthohedral shapes have been avoided. This leaves us with the ideal case of calculating the volume of the container, including its content.

After these initial considerations, we believe the search for a possible response to $\boldsymbol{Q}_G$ implies the approach of other derived questions[1], namely:

$\boldsymbol{Q}_1$ = *What type of solids can be chosen to model this container?*

$\boldsymbol{Q}_{11}$ = *What types of classes of geometric solids are there?*

$\boldsymbol{Q}_{12}$ = *What elements allow us to describe a solid in geometry?*

$\boldsymbol{Q}_{13}$ = *Which are the solids studied in geometry in compulsory secondary education?*

$\boldsymbol{Q}_2$ = *Once the type of solid has been chosen, how to determine the shape and size of the solids that are part of that type?*

$\boldsymbol{Q}_{21}$ = *Which and how many quantities do we need to determine the shape and size of a solid of a certain type?*

$\boldsymbol{Q}_3$ = *How to design and build the container in such a way that it has a capacity of L (in ml), or a volume of V (in cm^3)?*

$\boldsymbol{Q}_{31}$ = *How to design and build a container of a certain shape in such a way that it has a capacity of L (in ml), or a volume of V (in cm^3)?*

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[1] The choice of these questions coincides with some of the questions the students brought up during the implementation of the SRP, the partial results of which were published in Rojas y Sierra (2021a, 2021b).

In what follows, each of the above questions, as well as the ones that inevitably arise throughout the study, are addressed in order to develop a reasoned response to $\boldsymbol{Q}_G$. For example, starting from $\boldsymbol{Q}_{31}$, several specific questions for a particular shape chosen could be derived, namely:

> $\boldsymbol{Q}_{311}$= *How to design and build a container that has the shape of a regular tetrahedron in such a way that it has a capacity of L (in ml) or a volume of V (in cm^3)?*

The search for an answer to $\boldsymbol{Q}_1$ implies dealing with $\boldsymbol{Q}_{11}$ and $\boldsymbol{Q}_{12}$. To do this, the following should be established: a solid figure, unlike a plane figure, is a three-dimensional object that has a certain thickness and occupies a certain place in space (Castelnuovo, 1966). The kind of boundaries that delimits it –its shape and the characteristics of its surface– and their incidence–convexity and concavity–, define the class the solid belongs to (Guillén, 1991). As a result, solid figures may be convex or concave with boundaries consisting of flat surfaces, flat and curved surfaces, or only curved ones. Given the enormous complexity of possible solids, for now, only a reduced class of solids is studied.

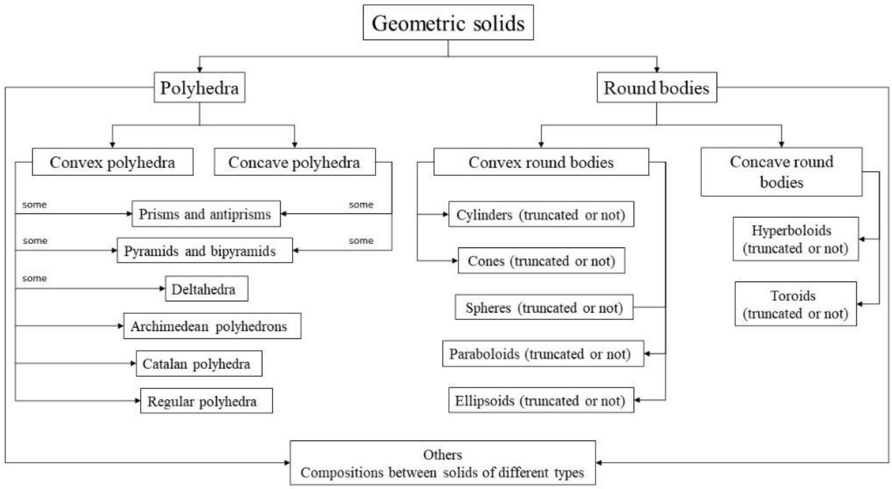
Therefore, a response $\boldsymbol{R}_{11}$ to $\boldsymbol{Q}_{11}$ implies somehow classifying some of the geometric solids. A classification was found (Figure 2), but we noticed that it does not differentiate cases that belong to more than one class of solid figures. For example, the case of the regular octahedron, which is also an antiprism and a deltahedron, or the case of the regular tetrahedron, which belongs to the class of pyramids.

Starting from this classification, in which the shape of the faces of the solids has been taken into account, a response $\boldsymbol{R}_{12}$, to $\boldsymbol{Q}_{12}$ can be elaborated. Other elements that contribute to the description of the solids could be the number of vertices or diagonals. However, this does not define them, unless the relative position and the relationship between the diagonals, for instance, is established. A solid can also be described through its types of symmetries, or, in the case of solids of revolution, the way in which it has been generated from the revolution of a plane figure around a given axis. For example, in the case of the shape of the faces, the regular tetrahedron could be referred to as the solid delimited by four regular triangular faces.

So far, it may be confirmed that a very large number of geometric solids exists, since, only by considering the possible shapes of the faces delimiting them, it is clear that both the number of said shapes and their incidence is overwhelming. Therefore, and in order to elaborate a possible response $\boldsymbol{R}_{13}$ to $\boldsymbol{Q}_{13}$, only some of the solids usually studied in compulsory secondary education will be used. In the school textbooks of 3rd year compulsory secondary mathematics education (Alcaide et al., 2016; Latasa & Ramos, 2022), the chapter on the study of geometric solids concerns the study of some polyhedrons such as regular polyhedrons, prisms, pyramids, and truncated pyramids, as well as some solids of revolution like the cylinder, the cone, the truncated cone, and the sphere. Calculating the surface area and volume of those solids, as well as some solids that can be formed from them, is addressed.

Figure 2

Classification of geometric solids elaborated from the approaches of Guillén (1991).



In unit 3 of the textbook by Bosch et al. (1996), aimed at the study of polyhedrons, the following is proposed: first, the study of the regular polyhedron; second, the transition from cubes to parallelepipeds and prisms; third, the transition from the regular tetrahedron to the pyramid; and, finally, from the regular polyhedron to the dual polyhedron.

Guided by the proposal of Bosch et al. (1996), we propound the study of three classes of solid figures, starting from three regular polyhedrons, taking into account two criteria: a) the number of sides grows indefinitely on one or more of its faces; and b) the regularity of the solid gradually weakens. Each class thus starts with a regular polyhedron, continues with polyhedrons that are less regular until reaching a round solid whose faces are no longer polygons:

- *First class*: regular tetrahedron – right pyramids of any kind of base – cones
- *Second class*: regular octahedron – regular base straight dipyramids – bicones
- *Third class*: cube – right prisms of any kind of base – cylinders

A *fourth class*, starting from the regular dodecahedron and the icosahedron polyhedron, duals of each other, may be considered. They are regarded as more spherical because their volume is similar to their circumscribed sphere, and by truncating their vertices, "soccer balls" (Carena, 2020) come next, to finish with the sphere.

It should be noted that we do not seek to address the entire universe of solids by using these classes, but rather, in a reasoned manner, the majority of the solid figures studied in secondary compulsory education. From these classes of solids, an answer may be elaborated to $\boldsymbol{Q}_2$, which deals with the determination (shape and size) of a solid. To do so, it is necessary to explain what determining a solid means. This involves asking, first of all, about when two solids have the same shape, and how many quantities we need to provide for a person who does not see the solid to be able to build one that has the same shape. Once the shape of the solid has been determined, to finish determining it, we should ask ourselves about determining the size. Another strategy is to jointly determine shape and size.

Let us say that two solids have the same shape if there is a similarity that transforms one into the other. Therefore, if two solids have the same shape, they can only differ in size. What needs to be asked is how many quantities are needed to determine the shape of a particular solid. To determine the shape without considering the size, lengths, areas, or volumes of certain elements of the solid will not be used as measures (because they partly determine the size of the solid). However, relationships between different measures of magnitudes, and the measure of some angles of the solid can be used.

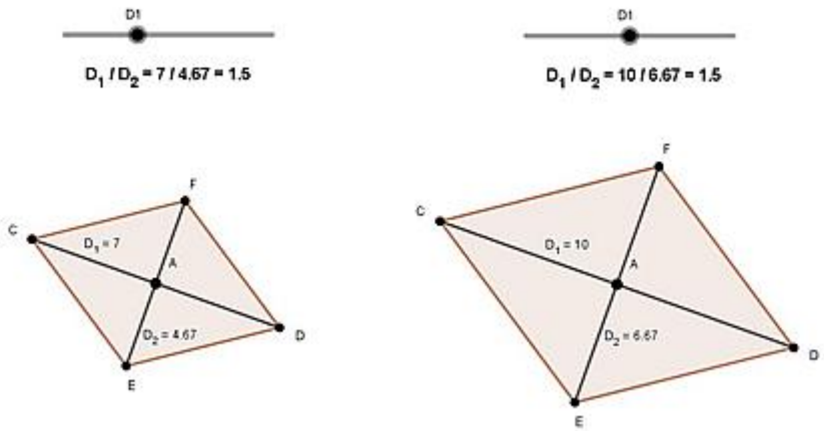
For instance, to determine the shape of a right cylinder without considering its size, it is enough to provide a single measure, the ratio between the diameter of the base, and the height of the cylinder. If we also provide the radius of the base, the cylinder is completely determined, and it will therefore be possible to build it.

Determining and constructing polygons

An analogous example can be found in two-dimensional geometry, in Gascón (2004). In the case of rhombuses, regarded as a type of polygon, to determine the shape of a specific rhombus, without considering its size, it is enough to provide a single parameter (for example, the relationship between the lengths of both diagonals, or the measure of one the angles). Thus, all rhombuses in which the ratio between their diagonals is, for example, $\frac{2}{3}$, have the same shape. That is, if $\boldsymbol{D}_2 = \frac{2}{3}\boldsymbol{D}_1$, where $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are the diagonals of the rhombus, Figure 3 shows that both rhombuses have the same shape regardless of lengths $\boldsymbol{D}_1$ y $\boldsymbol{D}_2$. In order to determine the size, the measure of a length (e.g., the length of the side, or the length of one of its diagonals) also needs to be provided.

Figure 3

Example of rhombuses with diagonals whose ratio is 2/3.



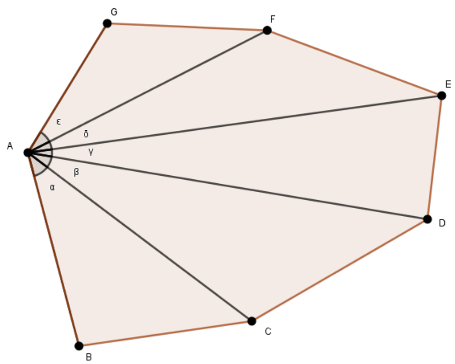
A more general question to ask within plane geometry is: *Which and how many quantities do we need to determine and construct any kind of polygon?*

For any $\boldsymbol{n}$-sided polygon, we will show that a maximum of $2\boldsymbol{n} - 3$ quantities are needed to determine and construct it. Following Pólya (1967), we propose three demonstrations of that fact, each of which suggests a strategy to construct the polygon.

- *First demonstration:* the measures required are the ones of $\boldsymbol{n} - 1$ line segments that start from a vertex where the first and the last lines are sides of the polygon, and the rest are the $\boldsymbol{n} - 3$ diagonals and $\boldsymbol{n} - 2$ angles that determine each pair of previous segments in such a way that the first angle is the one formed by the first side of the polygon and the first diagonal, the second angle is the one formed by the first diagonal and the second diagonal, and so on, up to the angle formed by the last diagonal and the last side. Therefore, $(\boldsymbol{n} - 1) + (\boldsymbol{n} - 2)$, that is, $2\boldsymbol{n} - 3$ quantities will be required. As shown, they allow the polygon to be constructed. (Figure 4).

Figure 4.

*Heptagon **ABCDEF**G determined by line segments **AB**, **AC**, **AD**, **AE**, **AF**, and **AG**, and by the angles between those segments (i.e., **α**, **β**, **γ**, **δ**, **ε**, respectively).*



- *Second demonstration*: The $\boldsymbol{n} - 3$ diagonals starting from a vertex of the polygon, and the $\boldsymbol{n}$ sides of the polygon, which, properly arranged, enable building it, will be required. Hence, in this case, $(\boldsymbol{n} - 3) + \boldsymbol{n} = 2\boldsymbol{n} - 3$ quantities (Figure 5) will be necessary.

Figure 5.

*Hexagon **AEBCDF** determined by diagonals **AB**, **AC**, and **AD**, and by sides **AE**, **EB**, **BC**, **CD**, **DF** and **FA**.*

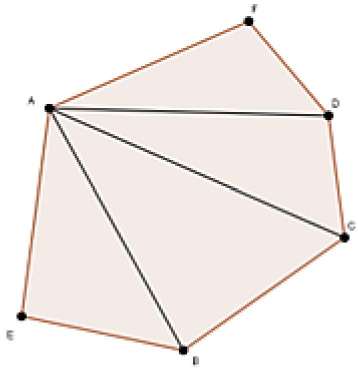
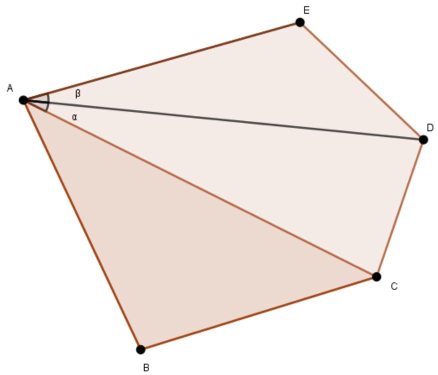


Figure 6.

*Pentagon **ABCDE** determined by triangle **ABC,** which is in turn determined by **AB**, **BC** and **CA**; by triangle **CAD**, determined by sides **CA** and **DA**, and by angle **α** between those sides; and by triangle **DAE**, which is determined by sides **DA** and **EA** and by angle **β** between those sides.*



- *Third demonstration:* any ***n***-sided polygon can be decomposed into ***n*** – 2 triangles. We thus need 3 quantities to construct the first triangle. To build each of the remaining ***n*** – 3 triangles, only 2 quantities are necessary, as they are constructed using one side that was already known to build the previous triangle. Therefore, $3 + 2(\boldsymbol{n} - 3) = 2\boldsymbol{n} - 3$ quantities in total are necessary (Figure 6).

Another way of showing this same result, derived from the third demonstration, without considering the size, is as follows: to determine the shape of an $\boldsymbol{n}$-sided polygon, it suffices to determine the shape of each of the $\boldsymbol{n} - 2$ triangles into which it is decomposed. Since the shape of a triangle is determined by 2 measures (for example, 2 angles), $2(\boldsymbol{n} - 2) = 2\boldsymbol{n} - 4$ quantities will be required to determine the shape of an $\boldsymbol{n}$-sided polygon. Once the shape is determined, it is enough to add one piece of information (e.g., the length of any side of the polygon) to determine the size. In total, $2\boldsymbol{n} - 4 + 1 = 2\boldsymbol{n} - 3$ quantities are required.

Determining and constructing solids

A possible response $\boldsymbol{R}_2$ to $\boldsymbol{Q}_2$ may be elaborated considering, for instance, the first class of solids. Possible responses $\boldsymbol{R}_{21}$ to $\boldsymbol{Q}_{21}$ will also need to be found. The measures necessary to determine the shape and size of these solids will need to be established. The first step is to select some of the kinds of shapes that appear in Figure 7: the regular tetrahedron, right pyramids with a regular hexagonal base, and right cones.

Figure 7.

Some figures of the first class. From the regular tetrahedron to the right cone.

Three new questions whose answers will help elaborate response $\boldsymbol{R}_{21}$ arise here:

> $\boldsymbol{Q}_{211}$ = *What quantities are necessary to determine and construct the regular tetrahedron?*

$\boldsymbol{Q}_{212}$ = What quantities are necessary to determine and construct a right pyramid with a regular hexagonal base?

$\boldsymbol{Q}_{213}$ = What quantities are necessary to determine and construct a right cone?

The search for responses $\boldsymbol{R}_{211}$, $\boldsymbol{R}_{212}$ and $\boldsymbol{R}_{212}$ to the corresponding previous questions leads us to solve the type of tasks $\boldsymbol{T}_{21}$ = *determining and constructing each of those solid figures using GeoGebra tools*[2]. Solving $\boldsymbol{T}_{21}$ will depend on the geometric properties of the solid to be determined and constructed, and on the tools GeoGebra provides, such as tracing 2D and 3D figures, and connecting dynamic objects like sliders (Dos Santos, 2012).

Determining and constructing a regular tetrahedron

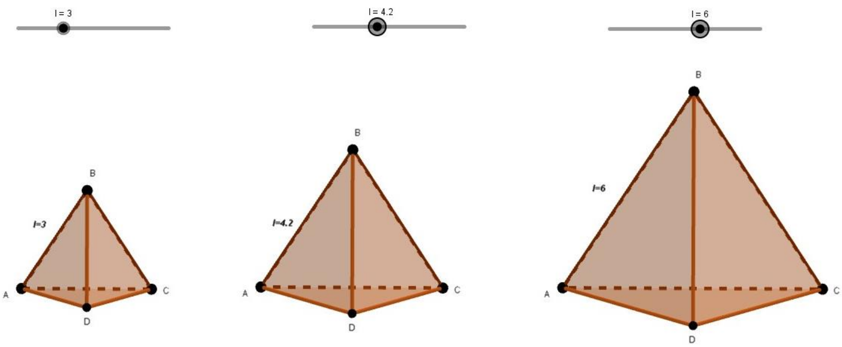
Task $\boldsymbol{t}_{\mathbf{211}} \in \boldsymbol{T}_{21}$, which consists of determining and constructing a regular tetrahedron, is very simple and quite trivial because, to determine a regular tetrahedron, only one measure, which determines its size, is necessary, since all regular tetrahedrons have the same shape. To construct it using GeoGebra, it is enough to provide, for instance, the length of one of its edges. It is constructed in GeoGebra by using this information, and the length of the edge is given by providing the measures of its endpoints.

If we add a slider to our construction, and connect it to the edge of the tetrahedron, when modifying its values, all possible regular tetrahedrons that only differ in size are obtained (Figure 8).

[2] GeoGebra is a free dynamic geometry programme widely disseminated and used in several academic environments for the study, development, and research of synthetic and analytical geometric knowledge, analysis of functions, etc.

Figure 8.

Regular tetrahedron constructed in GeoGebra starting from the endpoints of an edge, connected to a slider that allows varying the distance between said points.

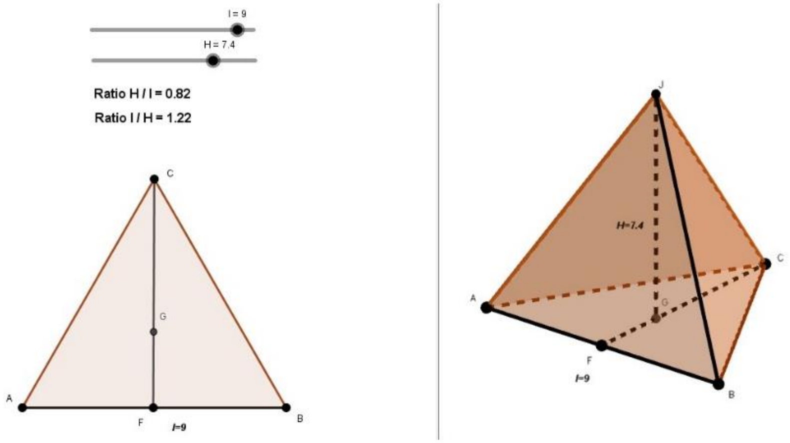


If we consider the regular tetrahedron within the set of right triangular pyramids, and call ***l*** the edge of the base, ***L*** the lateral edge (the side that joins the apex with a vertex of the base), ***H*** the height of the pyramid, and ***a*** the apothem of the pyramid (the slant height of the lateral faces), for a pyramid of this type to have the shape of a regular tetrahedron, one of the possible relations between the measures of ***l, L, H*** and ***a*** should be satisfied. The first relation to be satisfied is that ***l*** = ***L***. In a regular tetrahedron, the relation to be satisfied is that $\boldsymbol{H} = \frac{l\sqrt{6}}{3}$ (to determine this relation, the Pythagorean theorem was applied to two right triangles).

This means that all right triangular pyramids whose ratio between the height and the side of the base is $\frac{\sqrt{6}}{3} \approx 0.816$ are regular tetrahedrons (the converse theorem is also true). This can be verified using GeoGebra. We constructed a right pyramid with an equilateral triangle base, whose side ***l*** = 9 units and ***H*** ≈ 7.4 units. In GeoGebra, we thus obtain the construction of a right pyramid with a regular triangular base connected to two sliders: one that enables modifying the measure of the side of the base, and another that enables modifying the height of the pyramid (Figure 9). It is here possible to verify that all the pyramids in which, when activating the sliders, the relation between the measure of $\boldsymbol{l}$ and $\boldsymbol{H}$ is $\frac{\boldsymbol{H}}{\boldsymbol{l}} = \frac{\sqrt{6}}{3}$, or $\frac{\boldsymbol{l}}{\boldsymbol{H}} = \frac{3}{\sqrt{6}}$ (approximately 1.22), are regular tetrahedrons.

Figure 9.

Right pyramid with a regular triangular base, constructed in GeoGebra, connected to two sliders that allow modifying its height and edge.



Determining and constructing a right pyramid with a regular polygonal base

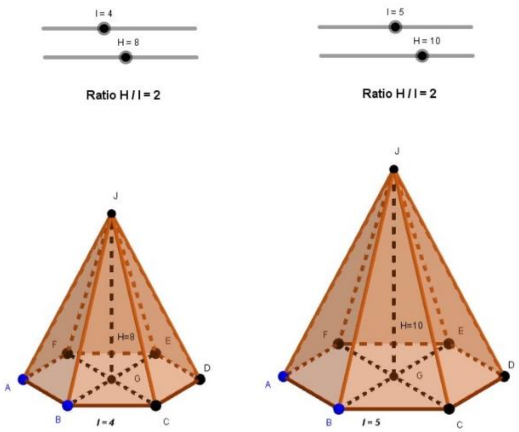
To give a response $\boldsymbol{R}_{212}$ to $\boldsymbol{Q}_{212}$, the type of tasks $\boldsymbol{T}_{212}$ = *Determining and constructing a right pyramid with a regular polygonal base*, need to be solved. It is easy to verify that, to determine the shape of a right pyramid with a regular polygonal base (determining the type of regular polygon that forms the base), one measure is enough. Once the shape is established, other measures are required to determine the size of the pyramid.

We will here stick to the particular case of a right pyramid with a regular hexagonal base. The relation between some of the intra-figural elements such as height $\boldsymbol{H}$ of the pyramid, apothem $\boldsymbol{a}$, edge $\boldsymbol{l}$ of the base, and edge $\boldsymbol{L}$ of the lateral faces will be taken into account.

All the right pyramids with a regular hexagonal base in which the ratio between their height $\boldsymbol{H}$ and the edge of base $\boldsymbol{l}$ is $\frac{\boldsymbol{H}}{\boldsymbol{l}} = 2$ are considered. Those pyramids hence have the same shape. This can be verified using GeoGebra by constructing a right pyramid with a regular hexagonal base connected to two sliders; the first one allows changing the size of side $\boldsymbol{l}$ of the regular hexagon that serves as the base of the pyramid; and the second allows modifying height $\boldsymbol{H}$ of the pyramid. Random use of these sliders enables obtaining an infinite set of regular hexagonal-based pyramid shapes. Connecting a dynamic text that evaluates the ratio between $\boldsymbol{H}$ and $\boldsymbol{l}$ enables verifying for which values of $\boldsymbol{H}$ and $\boldsymbol{l}$ pyramids with the same shape, that is, similar pyramids, are obtained (Figure 10). In this case, it is observed that, to build each of these pyramids, i.e., to determine their size, it is necessary to attribute values to $\boldsymbol{l}$ and $\boldsymbol{H}$.

Figure 10.

Similar right pyramids with a regular hexagonal base whose ratio between $\boldsymbol{H}$ *and* $\boldsymbol{l}$ *is 2.*



Determining and constructing a right cone

As in the case of the right pyramid with a regular hexagonal base, we can proceed in the same manner to elaborate a response $\boldsymbol{R}_{213}$ to $\boldsymbol{Q}_{213}$. The task to solve is $\boldsymbol{t}_{213}$ = *Determining and constructing a right cone.* Relations of the elements of the base of the solid, such as radius $\boldsymbol{r}$ of the base with height $\boldsymbol{H}$ of the cone or generatrix ***g***, are considered, since this allows us to determine the shape of the cone. For instance, all the right cones in which the ratio between their height ***H*** and the radius ***r*** of their base is $\frac{3}{2}$ have the same shape (Figure 11).

Figure 11.

Figure 11.

Right cones of the same shape whose ration between ***H*** *and* ***r*** *is* $\frac{3}{2}$.

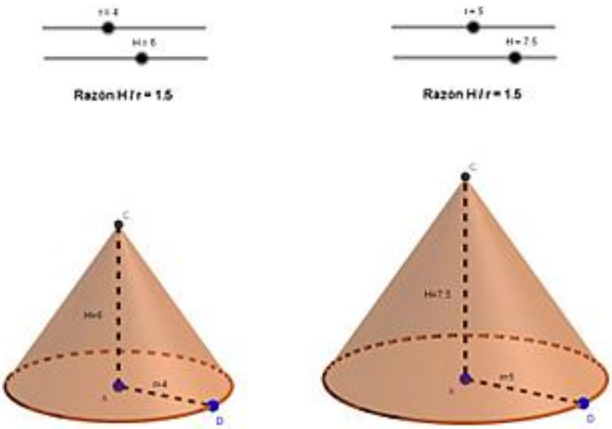
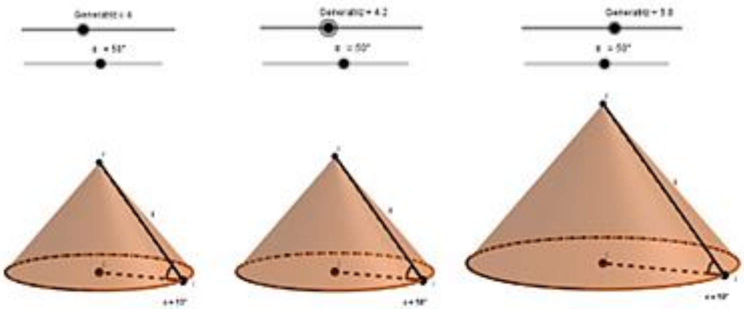


Figure 12.

Right cones of the same shape whose angle between **g** *and radius* **r** *is 50º.*



It could also be considered that all the right cones in which the angle between generatrix ***g*** and radius ***r*** of the base of the cone is, for example, 50º have the same shape (Figure 12).

Determining and constructing a pyramid whose base is an *n*-sided polygon

The following more general question is considered: *How to determine and construct a pyramid whose base is an **n**-sided polygon?* To provide an answer to this question, it is first necessary to determine and build the ***n***-sided polygon of the base as mentioned in the section on determining and constructing polygons. To do this, we need $2n - 3$ measures. In addition, the 3 quantities necessary to determine and construct the vertex of the pyramid, which correspond to the three coordinates of this vertex, are required. To construct it, one vertex of the polygon of the base will be located at the coordinate of the origin (i.e., 0,0,0), and one side of the base at one of the coordinate axes. The rest of the measures[3] will also be put. Once the three coordinates of the apex or vertex of the pyramid are given, the pyramid is already determined and constructed. The required quantities to determine and construct a pyramid whose base is an ***n***-sided polygon are $(2n - 3) + 3 = 2n$ (Figure 13).

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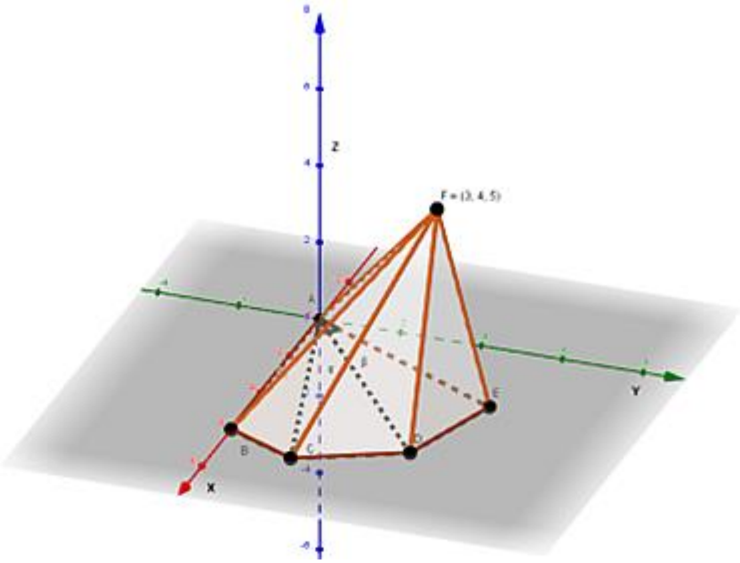
[3] The new Spanish curriculum puts forward the following for first to third year compulsory secondary education (aged 12 to 15): Location and representation systems. – Spatial relations: location and description using geometric coordinates and other representation systems (MEFP, 2022, p. 163).

Figure 13.

*Pyramid with pentagon **ABCDE** base, determined and constructed by triangle **ABC**, with sides **AB, BC** and **CA**, with sides **CA** and **DA** and the angle forming* **α***, and with sides **DA** and **EA** and the angle forming **β**; and point **F**, which is the apex of the pyramid, with coordinates (3,4,5).*

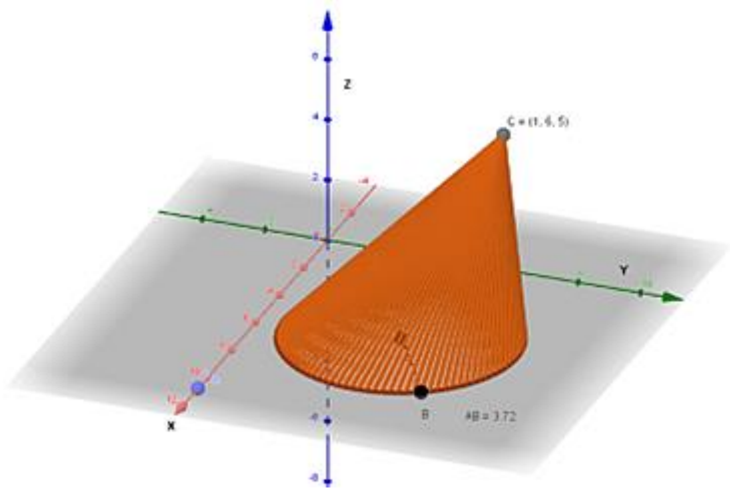


Determining and constructing a cone

The general question is: *How to determine and construct any cone?* The measures to determine the base of the cone, for example, radius ***r***, and the three coordinates of the vertex of the cone that must be located in a plane parallel to that of the base at a distance equal to the height of the cone are necessary. In total, 4 quantities are required (Figura 14).

Figure 14.

*Circular-based cone with radio **AB** = 3,72 units and vertex **C**, with coordinates (1,6,5).*

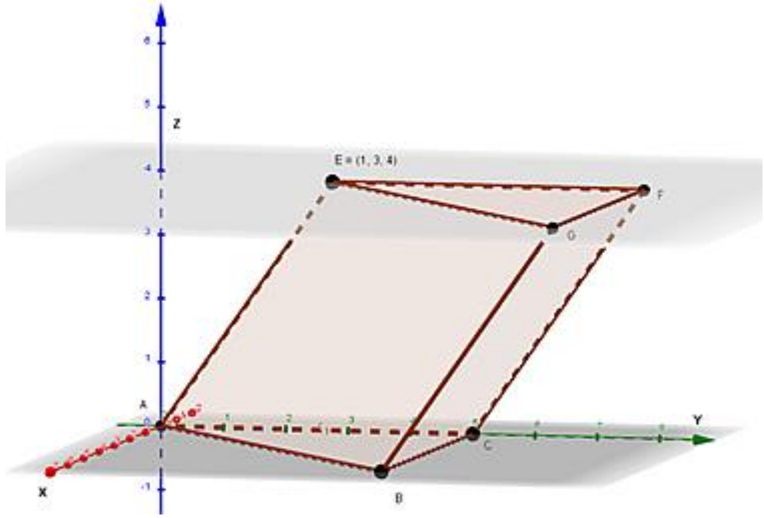


Determining and constructing a prism whose bases are formed by an *n*-sided polygon

The general question: *How to determine and construct a prism whose bases are formed by an **n**-sided polygon?* can also be considered. To provide an answer, we first construct the ***n***-sided polygon of one of the bases, for which 2n – 3 measures are needed. To construct the base, the same procedure as the one in the case of the pyramid is used. Then, the three coordinates of one of the vertices of the other base are required. This vertex will be located in a plane parallel to the base built at a distance equal to the height of the prism. Next, the other base can be built by drawing the different sides in said parallel plane starting from the built vertex, knowing that these sides must have the same length and must be parallel to their corresponding sides in the first base. The measures required to construct a prism whose bases are formed by an ***n***-sided polygon are (2n – 3) + 3 = 2n (Figure 15).

Figure 15.

*Prism whose base is triangle **ABC**, determined by sides **AB**, **BC** and **CA**; and then, by point **E** of the other base whose coordinates are (1,3,4), being* $\boldsymbol{AB \parallel EG}$, $\boldsymbol{BC \parallel GF}$, $\boldsymbol{CA \parallel FE}$, *and* $\boldsymbol{AB \cong EG}$, $\boldsymbol{BC \cong GF}$, $\boldsymbol{CA \cong FE}$.



Designing and building a pyramid-shaped container with a determined volume

In order to elaborate a possible response $\boldsymbol{R}_{311}$ to $\boldsymbol{Q}_{311}$, an answer should be elaborated to the type of tasks $\boldsymbol{T}_{311}$ = *Designing and building a pyramid-shaped container in such a manner that it has a volume of V cm*^{3}.

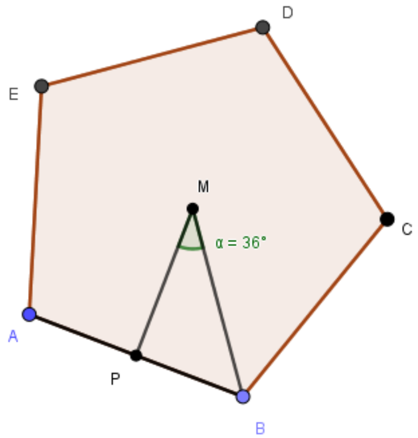
Within the type of tasks $\boldsymbol{T}_{311}$, we chose the following particular task: $\boldsymbol{t}_{311}$ = *Determining and building a container that has the shape of a right pyramid with a regular pentagonal base in such a way that it has a volume of V cm*^{3}. To search for an answer to $\boldsymbol{t}_{311}$, we know that $\boldsymbol{V} = \frac{\boldsymbol{BH}}{3}$, where $\boldsymbol{B}$ is the area of the base of the pyramid and $\boldsymbol{H}$ is its height. The area of the base of a regular pentagon, in relation to edge $\boldsymbol{l}$, is:

$$\boldsymbol{B} = \frac{5\boldsymbol{l}^2}{4\,Tan\ 36º}$$

To obtain this result, the base of the pyramid is regular pentagon $\boldsymbol{ABCDE}$ (Figure 16), whose quantities are: edge $\boldsymbol{l}$, perimeter $\boldsymbol{p} = 6\boldsymbol{l}$, apothem $\boldsymbol{a} = \boldsymbol{MP}$, $measure\ of\ the\ angle < \boldsymbol{PMB} = 36º$, $\boldsymbol{AB} = \boldsymbol{l}$, $\boldsymbol{PB} = \frac{\boldsymbol{l}}{2}$, and with right triangle **MPB**, we know the area of the base[4] is:

Figure 16.

Regular pentagon base of the right pyramid.



$$\boldsymbol{B} = \frac{\boldsymbol{pa}}{2}$$

But,

$$Tan\ 36º = \frac{\boldsymbol{l}}{2\boldsymbol{a}}$$

[4] It is worthy of note that the formula for the area of a regular polygon $\boldsymbol{B}$ is always presented in school textbooks depending on perimeter $\boldsymbol{p}$ and apothem $\boldsymbol{a}$, thus suggesting that both measures are independent. When the polygon is a regular hexagon, the dependency relationship between $\boldsymbol{p}$ and $\boldsymbol{a}$ is obvious, and in other regular polygons it can be shown using trigonometric ratios, as we have just seen when calculating the area of a regular pentagon.

$$\boldsymbol{a} = \frac{\boldsymbol{l}}{2\,Tan\;36º}$$

Substituting the value of ***a***, we obtain:

$$\boldsymbol{B} = \frac{5\boldsymbol{l}^2}{4\,Tan\;36º}$$

By substituting ***B*** in the formula of volume ***V***, we obtain:

$$\boldsymbol{V} = \frac{5\boldsymbol{l}^2\boldsymbol{H}}{12\,Tan\;36º}$$

By separating ***l***, we obtain a first functional relationship between ***l*** and ***H*** for a given volume ***V.***

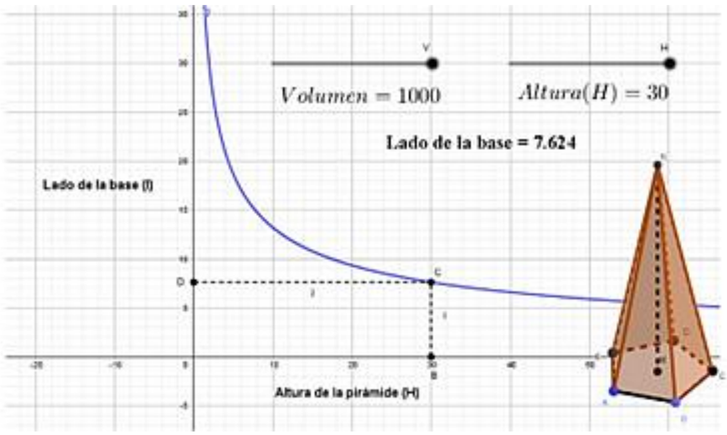
$$\sqrt{\frac{\boldsymbol{V}\;12\,Tan\;36º}{5\boldsymbol{H}}} = \boldsymbol{l}$$

However, if we separate ***H***, a second functional relationship between ***H*** and ***l*** is obtained for a given volume ***V***:

$$\frac{\boldsymbol{V}\;12\,Tan\;36º}{5\boldsymbol{l}^2} = \boldsymbol{H}$$

Figure 17.

Graph for the first case of the right pentagonal pyramid.

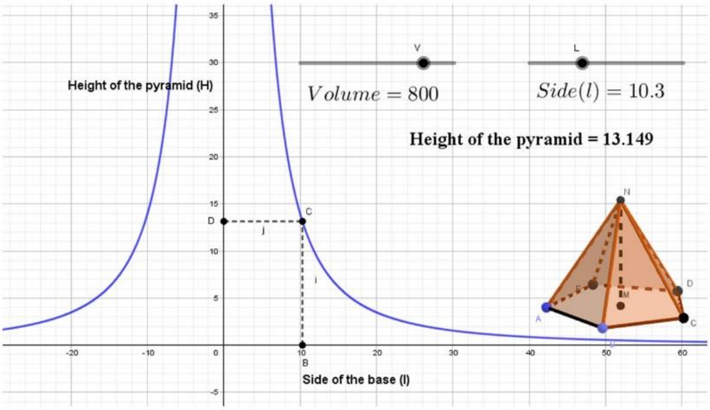


Using GeoGebra, these functional relationships can be represented. To do this, a value is attributed to $\boldsymbol{V}$, using a slider with values between 0 and 1000 cm^3. We will thus see; in the first case, how $\boldsymbol{l}$ varies when we modify $\boldsymbol{H}$ and determine its volume; and in the second case, how $\boldsymbol{H}$ varies when we modify $\boldsymbol{l}$ and determine its volume. Thus, in the first case, for a volume $\boldsymbol{V} = 1000$ cm^3, when $\boldsymbol{H}=30$, $\boldsymbol{l} \approx 7{,}624$ cm (Figure 17).

In the second case, for a volume $\boldsymbol{V} = 800$ cm^3, when $\boldsymbol{l} = 10{,}3$ cm, $\boldsymbol{H} \approx 13{,}149$ cm (Figure 18).

Figure 18.

Graph for the second case of the right pentagonal pyramid.



It needs to be stressed that $\boldsymbol{t}_{311}$ is an inverse and open task, which means the formulas to calculate the volume are dealt with as algebraic or functional models. With the help of GeoGebra, this will facilitate considering the different solutions possible.

So far, an REM has been developed regarding the problems the determination and construction of solids presents. Like all REMs constructed, it is of a provisional nature, awaiting possible permanent modifications, as new questions may appear whose answer will allow extending and completing those problems.

The process followed is based on solving a spatial-geometric problem regarding the determination and construction of solids, starting from a system in which a spatial problem is presented related to designing and building a container with a predetermined volume. To solve this problem, a mathematical model is employed in which geometric, arithmetic, algebraic, and functional elements are used. We have proposed a study of elementary geometry as a model of space where, as Brousseau puts forward (cited in Berthelot and Salin, 2001), *a-didactic situations* regarding the *determination and construction of solids* will allow the development of geometric knowledge.

CONCLUSIONS: SOME EPISTEMOLOGICAL AND DIDACTIC FUNCTIONS OF THE REM CONSTRUCTED

The REM developed includes some of the questions (and associated tasks) related to the *problem of spatial-geometric modelling*, considered to be the problem of introducing geometry in secondary education. Therefore, the REM built stresses *the experimental nature* of geometry, and especially (but not only) of the introduction to the study of geometry.

Open and inverse tasks in the physical space are put forward, and their resolution facilitates the emergence of any kind of mathematical models, which, by using tools such as GeoGebra, help solve them. Therefore, the identification of the mathematical activity with the modelling activity proposed by the ATD, also in the case of geometry, is stressed, advocating that the study activity, in the case of geometry, inevitably involves a modelling activity.

The REM becomes a valuable tool for teachers in charge of study processes to be able to guide the design of their didactic proposals regarding 3D geometry to get students to build new geometric techniques, question their validity, interpret geometric formulas as algebraic-functional models and, consequently, carry out a genuine modelling activity. It will serve as a reference and basis to design, experiment, and analyse different SRPs in compulsory secondary education, and to study the possible conditions and restrictions that may arise during their implementation. In essence, the REM enables supporting study processes for predetermined educational purposes without assuming that the specific path the study community should follow to achieve those purposes is determined in advance.

As shown in the development of the REM, the problem studied with regard to the determination and construction of solids can help connect the study of 2D and 3D geometry, since the determination and subsequent construction of solid bodies is based on the previous determination of plane figures. Once agreed that "doing geometry" at the elementary level consists of determining and constructing figures from some of their elements, it is very difficult to try to determine and construct a solid body without basing it on the determination of the plane figures that determine it.

Furthermore, the development of the REM enables connecting several fields of mathematics, such as the study of geometry with that of algebra and functions. At this point, it is important to note that, from the perspective of the ATD, intra-mathematical modelling constitutes an essential part of mathematical modelling (Bosch et al., 2006; García et al., 2006). This means that, as shown earlier, in a modelling process, the system modelled can be of a mathematical nature, thus obtaining a mathematical model of a mathematical system. This extension of the usual notion of "mathematical modelling" highlights the potential reflective nature of mathematical modelling (a mathematical system can act as a model of its model, as happens, for example, in the case of the relationship between Euclidean and analytical geometry), as well as its *recursive nature* (it is possible to build a mathematical model of the model of a system). In our case, after carrying out the spatial-geometric modelling of a physical object, we built an algebraic model (using an "algebraic formula") and, to solve problems related to the variation of one variable with respect to others, we built and used a functional model.

The REM developed proposes the use of GeoGebra as a suitable tool to efficiently carry out graphic experiences that facilitate the study of the properties of solids from the elaboration of a predetermined container. The use of GeoGebra to respond to the problems raised also allows connecting the synthetic techniques with the analytical techniques of determination and construction of solids.

One of the epistemological-didactic functions of the REM consists of bringing to light characteristics of the dominant epistemological model in school institutions such as the disconnection between 2D and 3D geometry, the isolation of geometry from algebra and functional techniques, and the separation between synthetic and analytical techniques. They constitute indications of didactic phenomena we consider "undesirable" from the perspective provided by the ATD postulates and hence intend to ignore.

Developing this REM also provides the researcher with a tool to question the dominant epistemological model of geometry teaching in compulsory secondary education by means of the analysis of official curriculum documents and school textbooks. The REM thus constitutes a tool for the emancipation of the didactician with respect to the conditioning factors the dominant epistemological model entails (Gascón, 2014).

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AUTHOR CONTRIBUTION

Both authors, CRS and TASD, have made substantial contributions to the conception and design of this manuscript. They have actively been involved in drafting this article.

DATA AVAILABILITY STATEMENT

Some of the data that support this study, such as those derived from the SRPs mentioned, are part of previously published works. However, if necessary, they will be made available by the corresponding author (CRS) upon reasonable request.

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