




Specialised Knowledge of the Mathematics Teacher to Teach through Modelling using ICTs

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Received for publication 31 Aug. 2022. Accepted after review 9 Nov. 2022

Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Background: The study of the knowledge of the mathematics teacher has focused attention on differentiating mathematics preservice teachers from other types of professionals who know the area to establish specific characteristics that can identify the professional licensed for the teaching of mathematics. **Objectives:** To characterise the specialised knowledge of the teacher who incorporates information and communication technologies (ICT) in the teaching of mathematics using modelling. **Design:** Qualitative research based on an instrumental case study. **Setting and participants:** The study is carried out in a secondary basic education course, with a teacher with training and experience in teaching mathematics with technological resources. **Data collection and analysis:** The analyses focused on three subdomains of the mathematics teacher's specialized knowledge model, namely: knowledge of the topics, knowledge of mathematical practices, and mathematical knowledge for teaching, by adapting indicators to the categories of knowledge of these subdomains, integrated with aspects of modelling. **Results:** The relationships that arise between categories of the subdomains were found and are evidence of the need for the teacher who teaches mathematics to know the subject discipline in depth and to relate it to their didactic-pedagogical knowledge to develop mathematical modelling processes using the ICTs. **Conclusions:** The construction of knowledge indicators of the KoT, KMP, and MKT allowed us to understand and interpret the specialised knowledge of the teacher when teaching conics, particularly, the circumference, through mathematical modelling with the specialised GeoGebra software.

Keywords: Modelling; Specialised knowledge; MTSK; Mathematics knowledge, Mathematics teaching.

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Conocimiento especializado del profesor de matemáticas para enseñar a través de la modelación utilizando las TIC

RESUMEN

Antecedentes: el estudio del conocimiento del profesor de matemáticas ha centrado la atención en diferenciar a los licenciados en matemáticas de otro tipo de profesionales que tienen conocimiento del área, de manera que se puedan establecer características específicas que identifiquen al profesional licenciado para la enseñanza de las matemáticas. **Objetivos:** caracterizar el conocimiento especializado del profesor que incorpora las tecnologías de la información y comunicación (TIC) en la enseñanza de las matemáticas utilizando la modelación. **Diseño:** investigación de carácter cualitativo basada en un estudio de caso de tipo instrumental. **Contexto y participantes:** el estudio es realizado en un curso de Educación básica secundaria, con un profesor con la formación y la experiencia en la enseñanza de las matemáticas con recursos tecnológicos. **Recopilación y análisis de datos:** Los análisis se enfocaron en tres subdominios del modelo del Conocimiento Especializado del Profesor de Matemáticas, a saber: conocimiento de los temas, conocimiento de las prácticas matemáticas y conocimiento de la enseñanza de las matemáticas, mediante la adaptación de indicadores a las categorías de conocimiento de estos subdominios, integrados con aspectos de la modelación. **Resultados:** Se encontraron las relaciones que surgen entre categorías de los subdominios y que son evidencia de la necesidad que el profesor que enseña matemáticas, conozca en profundidad su disciplina y la relacione con su conocimiento didáctico-pedagógico para poder desarrollar procesos de modelación matemática usando las TIC. **Conclusiones:** Se concluyó que la construcción de indicadores de conocimiento del KoT, KPM y KMT permitió comprender e interpretar el conocimiento especializado del profesor al enseñar cónicas, y en particular la circunferencia, mediante modelación matemática con el software especializado GeoGebra.

Palabras claves: modelación; conocimiento especializado; MTSK; conocimiento de las matemáticas, enseñanza de las matemáticas.

Conhecimento especializado do professor de matemática para ensinar por meio de modelagem usando TIC

RESUMO

Antecedentes: O estudo dos conhecimentos dos professores de matemática centrou-se na diferenciação dos licenciados em matemática de outros tipos de profissionais que têm conhecimentos da área, de modo a que possam ser estabelecidas características específicas que identifiquem o licenciado profissional para o ensino da matemática. **Objetivos:** caracterizar o conhecimento especializado do professor que incorpora tecnologias de informação e comunicação (TIC) no ensino de matemática por meio de modelagem. **Desenho:** de natureza qualitativa com desenho de estudo de caso

do tipo instrumental. **Contexto e participantes:** o estudo é realizado numa turma do ensino secundário, com um professor formado e experiente no ensino de matemática com recursos tecnológicos. **Coleta e análise de dados:** As análises focaram em três subdomínios do modelo de Conhecimento Especializado do Professor de Matemática, a saber: conhecimento de tópicos, conhecimento de práticas matemáticas e conhecimento de ensino de matemática, por meio da adaptação de indicadores às categorias de conhecimento desses subdomínios, integrados com Aspectos da modelagem. **Resultados:** Foram encontradas as relações que surgem entre as categorias dos subdomínios e que são evidências de que é necessário que o professor que ensina matemática conheça a fundo sua disciplina e a relacione com seu conhecimento didático-pedagógico para desenvolver processos de modelagem matemática utilizando TIC. **Conclusões:** Concluiu-se que a construção de indicadores de conhecimento do KoT, KPM e KMT permitiu compreender e interpretar o conhecimento especializado do professor no ensino de cónicas, e em particular a circunferência, através da modelação matemática com o software especializado GeoGebra.

Palavras-chave: modelagem; conhecimento especializado; MTSK; conhecimentos de matemática; ensino de matemática

INTRODUCTION

Various researchers have made efforts to demonstrate and define mathematics teachers' profession to reinforce its key differences with any professional with disciplinary knowledge of mathematics, such as mathematicians, physicists, statisticians, engineers, and accountants, among others. Therefore, the works by Shulman (1986), Fennema and Franke (1992), Rowland et al. (2005), Ponte (1992), and Ball et al. (2008) served as the basis to conceptualise the mathematics teacher's specialised knowledge -MTSK- (Carrillo et al., 2018). This model approaches elements of both disciplinary and didactic knowledge linked to specific mathematics content that should be available to any teacher who teaches mathematics, regardless of the level of schooling they teach.

In Colombia, where this research was done, since 2002, mathematics teaching has been developed by professionals who do not necessarily have a degree in mathematics. This situation was validated by the Ministry of National Education in Colombia (Gobierno de Colombia-MEN, 2002) and has been contemporary with the implementation of the programme of the Ministry of Information and Communication Technologies (Min-TIC) 'computers to educate' (Gobierno de Colombia -MinTIC, 2016), whose idea is based on supplying computers to Colombian schools and training teachers in competencies on Information and Communication Technologies (ICT) to improve educational quality. However, the results of standardized tests such as

the programme for the international student assessment (PISA) in the report presented by the Organization for Economic Cooperation and Development (OCDE) in the case of mathematics and language show that the countries that have invested the most money and ICT resources in the educational context have not obtained good long-term results (OCDE, 2019).

Thus, the need for teacher education in the didactic-pedagogical component leads us to think about the need for teachers with disciplinary and pedagogical formation in Colombian schools to be responsible for teaching mathematics (Padilla-Escorcía & Acevedo-Rincón, 2022). This could enhance the importance of the teaching profession by selecting a profile for the teacher who teaches mathematics with training in teaching areas -pedagogy and didactics-, which other professionals related to mathematics do not have. A professional with these characteristics has concrete specific knowledge of mathematics, which allows them to deepen the possible use of ICT for teaching mathematics, with a didactic-pedagogical sense, because they will be able to use technological resources as a means to carry out the teaching of the contents of mathematics in a more interactive, practical way and where more mathematical skills can be evidenced, such as the case of modelling (Padilla-Escorcía & Acevedo-Rincón, 2020; Padilla-Escorcía, & Acevedo Rincon, 2021; Padilla-Escorcía, 2022).

A previous study (Padilla-Escorcía, 2020) questioned whether Colombian mathematics teachers are prepared to teach the mathematical contents that require from the student a high degree of abstraction through ICT, especially in modelling, which represents a relevant place within the research agendas at an international level, due to the interest generated by how the teaching staff implements this competency in teaching (Verschaffel & De Corte, 1997; Villa-Ochoa, 2015; Granados-Ortiz & Padilla-Escorcía, 2021).

Several works propose that, through the insertion of technological tools, it is possible to make modelling competency visible through the use of different representations of mathematics, such as objects, the analysis of mathematical models, and algebraic graphs that are linked to reality through virtual environments (Villa-Ochoa, 2007; Villa-Ochoa & Ruiz, 2009; Villa-Ochoa, González & Carmona, 2018; Molina-Toro, Rendón-Mesa, & Villa-Ochoa, 2019).

In that order of ideas, this research seeks to characterise the specialised knowledge of the teacher who incorporates the ICTs in the teaching of mathematics using modelling. This because most of the studies that have been carried out around the Mathematics Teacher Specialised Knowledge (MTSK)

have been focused on exploring the mathematical and didactic-pedagogical knowledge, which requires a teacher to teach elementary, high school and even higher basic education content. However, no aspects related to the necessary knowledge to teach mathematics content through technology had been studied. Therefore, in this research article, an approximation is made from the MTSK model (Carrillo et al., 2018). Therefore, we formulate the following research question: What are the characteristics of the specialised knowledge of the teacher who incorporates ICTs in the teaching of mathematics using modelling?

In this way, the theoretical foundations of the study are shown, focused on MTSK as a model of analysis of the professional knowledge of the mathematics teacher and modelling in mathematical education. The methodological design of the study developed and the categorisation based on the indicators of Knowledge of Topics (KoT), Knowledge of Mathematics Teaching (KMT) and knowledge of practice of mathematics (KPM), which contribute to the analyses developed in the following section, is described below. Finally, the results and conclusions of this article are shown.

MTSK THEORETICAL FRAMEWORK

Since the 1980s, the field of research has studied, discussed, analysed, and evaluated the knowledge of the mathematics teacher. Thus, works such as Ball, Thames, and Phelps (2008), Rowland et al. (2005), and Godino (2009) have delved into the components of this knowledge from different theoretical models that conceptualise the teacher's knowledge. Carrillo et al. (2018) propose a mathematics teacher specialised knowledge model -MTSK- to constitute an analytical tool to understand the knowledge that a mathematics teacher uses when developing tasks linked to their profession. This model preserves the structure in two domains proposed by Ball et al. (2008), differentiating the mathematical knowledge, linked to the deep knowledge of the discipline, from the didactic knowledge of the content, understood from Shulman's perspective (1986).

The domain of mathematical knowledge is further divided into three subdomains. The first of them, the knowledge of the topics (KoT), reflects a knowledge of the local nature of the mathematics taught and encompasses both the knowledge related to definitions, procedures, and properties, as well as meanings, representation registers and phenomenological aspects, in which the set of situations is included, within which the teacher can locate a topic (extra-

mathematical knowledge) to generate mathematical knowledge through the use and applications of the contents in real life.

The second subdomain is knowledge of the structure of mathematics (KSM), which reflects the knowledge that the teacher has of the different mathematical topics as a set of connected and related elements. Thus, in this subdomain, we find simplification, complexification, transversal, and auxiliary connections (Montes et al., 2016). Finally, the knowledge of the practice of mathematics (KPM) contemplates the mathematics teacher's knowledge from a syntactic perspective (Schwab, 1978).

On the other hand, pedagogical content knowledge (PCK, according to the terminology proposed by Shulman, 1986) is structured into three subdomains. The first is knowledge of mathematics teaching (KMT), which corresponds to the teacher's knowledge of manipulative or digital resources for mathematics teaching, teaching theories that allow them to structure their teaching, and tasks and examples they can use in their classes. The second, knowledge of features of learning mathematics (KFLM), focuses on the teacher's knowledge of different aspects of the student's learning, such as strengths or difficulties they may have, their usual ways of dealing with mathematical content, or emotional aspects in learning mathematics. This subdomain also includes knowledge of learning theories, whether personal or stemming from research in mathematics education. Finally, the knowledge of mathematical learning standards (KMLS) covers the knowledge that the teacher has about both the official curriculum standards and other professional associations. Also included in this subdomain is knowledge of sequencing proposals of different mathematical topics or knowledge of the level of conceptual or procedural development that a student would be expected to have.

Finally, the MTSK model considers a third domain linked to the beliefs of the mathematics teacher as an element that permeates the teacher's knowledge (Carrillo et al., 2018). This domain will not be addressed in this paper.

Modelling in Mathematics Education

To define modelling in the field of mathematical education, it is necessary to address the concepts of mathematical model and mathematical modelling as key elements in the processes of mathematical modelling. Villa-Ochoa (2007) defines mathematical modelling as “a scientific activity in mathematics that is involved in obtaining models of the other sciences” (p.65).

Such models have a mathematical nature and are axiomatic systems made up of indefinite terms that are obtained from the abstraction and qualification of ideas from the real world, characterized as graphs, symbols, simulations, and experimental constructions (Maki & Thompson, 1973; Giordano et al., 1997; Leal, 1999; Villa-Ochoa et al., 2009), which mostly aim to explain, predict, and solve aspects of a contextualised phenomenon or situation.

In this sense, mathematical modelling can be an opportunity to generate the abstraction and application of the facts of everyday life for educational purposes to the contents of the mathematics taught by the mathematics teacher. Thus, different authors (Cetinkaya et al., 2016; Galbraith et al., 2007; Parra-Zapata et al., 2018; Romo et al., 2019; Rosa & Orey, 2019; Villa-Ochoa, 2007; Villa-Ochoa et al., 2009; Villa-Ochoa et al., 2020; Villa-Ochoa & Ruiz, 2009; Villarreal et al., 2018) propose that modelling in mathematical education is conceived as a process in which mathematics and the rest of the disciplines are related, or, from a more informal perspective, the ‘real world’.

In this way, modelling helps students build a mathematical concept by creating their own mental images associated with the contexts of each content, which motivates the interest in learning mathematics, given the relationship of this subject with real life. Mathematics education modelling, in turn, explores the study of situations and solving problems of everyday life that are related to mathematical knowledge and that serve as a resource in the teaching and learning of mathematics by the teacher, since the use of real data allows students to give and reference meaning to this process (mathematise), in other words, to pose and represent relationships between different objects and quantities.

That is why teachers must relate the different problems that can be productive as an object of study, i.e., that they can be modelled in the classroom and not be necessarily specific to mathematics but to other areas of knowledge such as physics, economics, administration, and engineering, among others (Çetinkaya et al., 2016). The teacher’s knowledge of mathematical modelling is not exclusively mathematical, in the sense of the MTSK but also implies knowledge that promotes mathematical modelling in the classroom, referred to as extra-mathematical knowledge by Villa-Ochoa et al. (2020), focused on developing one’s own modelling or its integration with other disciplines (Carmona-Mesa, Cardona Zapata, & Castrillón-Yepes, 2020).

In view of the above, integrating aspects of modelling within the categories of knowledge of the subdomains of MTSK seem not only possible, given that in this model of knowledge of the teacher, disciplinary aspects of mathematics and didactic-pedagogical type are deepened, which are also

present when you want to develop mathematical modelling processes in the classroom, but also necessary to feedback the MTSK model itself. For Villa-Ochoa (2007), the modelling process can be understood from the modelling phases that the teacher must consider in the choice and analysis of the daily contexts and phenomena that will be the objects to be modelled. These phases are:

Observation and experimentation: consist of identifying a phenomenon or problem of daily life that is prone to be modelled so that the teacher considers the previous concepts that the students must have internalised to address the situation to be modelled.

Delimitation of the problem: it consists of reducing all the variables with which a problem or phenomenon of the real world that one intends to model. Thus, not all the data involved in the phenomenon are considered, so one makes simplifications and assumptions that eliminate some data and allow the construction of the desired model representing the study phenomenon.

Selection of strategies: consists of teacher's selection of resources and methodologies to organise didactic sequences that contribute to the representation, manipulation, and treatment of both intermediate models and the desired or intended final model.

Evaluation and validation: consist of assessing the relevance of the model used, considering that the conditions imposed have been complied with, in accordance with the experienced set of data. In this phase, the discussions between peers, and between students and specialists in the subject, as well as the influence of behaviour change of the variables that make up the model are especially relevant.

Connection with other models and situations: consists of identifying other phenomena in which relationships can be established between the same concepts but under another type of interpretation.

Therefore, taking as a reference the knowledge indicators that are proposed from the categories that make up the subdomains of the MTSK (Carrillo et al., 2018) and according to the modelling phases (Villa-Ochoa, 2007), knowledge indicators that the teacher requires for teaching conical

sections using ICTs are proposed from the development of classroom modelling processes. Each of the phases presented, which are necessary to carry out the mathematical modelling process in the classroom, requires the teacher's expertise, both mathematical, to know how to relate the contents of mathematics with problematic situations that can be solved with said mathematical content, through modelling, and of the didactic type, to know how to establish the sequentiality that leads to the representation, manipulation, and treatment of the model that captures the situation that is related to certain mathematics content.

Thus, in the case of the observation and experimentation that mathematics teachers must consider to develop modelling processes, there is the relationship that this has with the KoT in terms of the categories of definitions, phenomenological relationships, and representation records; in KFLM, in relation to the category of motivations of students for mathematics learning; in KMT, in teaching techniques and resources; and in KPM, in the categories of ways of proceeding and modelling.

At the time of delimitation of the problem, it seems plausible the relationship with the development of modelling processes with the KoT in the categories of properties and definitions. Similarly, when selecting strategies, it is related to KMT from the categories of resource potentiality and of activities, tasks, examples, and strategies for teaching mathematics; and in the KoT, to the category of representation registers.

Finally, to develop processes of evaluation, validation, and connection with other models, the teacher will mobilise KoT in the categories of definitions, properties, phenomenological relationships, and procedures; and KPM, with the categories of ways of modelling and proceeding.

In this research, we focus solely on the KoT and KPM subdomains of the mathematical domain, and the KMT subdomain of the PCK domain. This decision was made because they are the subdomains of the model in which their respective categories are more related to the phases of the modelling proposed by Villa-Ochoa (2007). Thus, the versatility that this model has to establish relationships between own categories of the same subdomain, between categories of different subdomains of the same domain of knowledge, and between categories of different subdomains of different domains of knowledge is an example of the projection of the MTSK to the phases of mathematical modelling, a practice not addressed in its depth in the categories of the KPM, but substantial within the specialised knowledge required by a teacher who teaches a mathematics content, as is the case of the conical sections, which can

be applied to various situations of daily life, and which, through the use of tools and/or technological resources, facilitates its teaching through this practice. Therefore, the mathematics teacher is required to have mathematical knowledge and didactic-pedagogical knowledge, for their knowledge to be specialised for the teaching of the modelling of certain content using technology. Although indicators of the KSM and KFML (PCK), and KMLS (MK) subdomains could be identified and analysed in future research, they will not be studied in this paper.

METHODOLOGY

This research work is developed under a qualitative approach (Stake, 2010) with an instrumental case study (Stake, 2005) since, through a particular case study, we can understand, interpret, and analyse what it implies to integrate elements of modelling within the practice of a teacher who *teaches conical sections using ICTs*.

Participants and context

Based on Simons' (2011) criteria, from a set of cases related to research interests, we selected teacher Pablo (fictitious name), who has a teaching degree in mathematics, is a specialist in mathematics didactics and has a master's degree in mathematics didactics. He also has three years of teaching experience, teaching conical sections in the tenth grade of secondary education using specialised mathematics software such as GeoGebra, and Geo Tic, among others, and has published a scientific article on his practice.

Data collection tool

The instrument for collecting information in the research corresponds to 13 non-participant observation units applied to Pablo, 11 under the virtual and 2 under the face-to-face modality. These units of observation were transcribed verbatim from the recorded teacher Pablo's classes, in which he made use of ICTs for the teaching of the following contents that correspond to the conical sections: the circumference and its parts, straight line tangent to a circumference, canonical equation of the circumference, demonstration of the general equation of the circumference, a graph of the circumference, the ellipse, empirical construction of an ellipse and on the flat surface, parts of the ellipse,

demonstration of the general equation of the ellipse, calculation of the parts of the ellipse, the hyperbola, parts of the hyperbola, calculation of the general equation of the hyperbola and graph of the hyperbola. For this research article, we selected an excerpt from an episode of the observed class session #2, from lines 37 to 57 (2. S2-L37-57), related to the content of the circumference. The encoding used for the units of analysis is as follows: session, session number, and session transcript lines. We decided to choose that episode because it shows evidence of the specialised knowledge of the teacher of the subdomains to be studied at present, the relationships between them, and the approaches of the teacher to develop mathematical modelling in the teaching of the content of the conical sections, specifically of the circumference.

Data Analysis

As for the analysis of the data, the indicators to the knowledge categories of the subdomains of the MTSK proposed by Carrillo et al. (2018) are used as an analysis tool, adapted to the moments of the modelling that must be considered by the mathematics teacher as proposed by Villa-Ochoa (2007) to develop mathematical modelling (Tables 1, 2 and 3). These indicators are linked to subdomains, the subject of analysis in this research article, KoT, KPM, and KMT. The analysis is developed in the form of content analysis to an excerpt of an episode related to the content of the circumference. In this analysis, with inductive characteristics, the data are organised into categories and, from their analysis, be constantly reviewed until reaching a saturation point at which no further contributions can be made to those categories since they are completely defined and described (Bardin, 1996; Krippendorff, 2009; Carter, 2020). That is why the analysis of the interactions between Pablo and the students aimed at characterising the teacher's specialised knowledge is done by subdomains of the MTSK. In this way, the following tables show the proposal of indicators to the knowledge categories of the KoT, KPM, and KMT subdomains of the MTSK as an adaptation of those proposed in the study carried out by Carrillo et al., (2018) and their relationship with the teacher's moments to develop mathematical modelling in the classroom as proposed by Villa-Ochoa (2007), in this order: KoT, KMT, and KPM.

Table 1 shows the teacher's moments to develop mathematical modelling that most relate to categories of the KoT subdomain, especially those corresponding to the observation and experimentation phase and the validation and evaluation phase, which is understood as in the latter definitions,

procedures and properties of mathematics are used to model real-life situations and generate the mathematical model (s) from them.

Table 1.

KoT Indicators

<i>Domain</i>	<i>Subdomain</i>	<i>Category</i>	<i>Indicators</i>	<i>Moment of Villa-Ochoa (2007) modelling</i>
MK	KoT	Knowledge of definitions	Know the definitions of conic sections in the construction of mathematical models using ICTs (GeoGebra)	Observation and experimentation Evaluation and Validation
		Knowledge of the type of phenomenology or application in everyday life of mathematical contents	(IC2) Identify daily living situations from mathematics or related areas of knowledge that can be modelled from school mathematics (conical sections).	Observation and experimentation
		Knowledge of content representation records	(IC3) Represent the contents of school	Observation and experimentation

	<p>mathematics (conical sections) in each of its records (graphic, numerical, algebraic, pictorial, and natural language).</p>	<p>Strategies selection</p>
<p>Knowledge of the properties and fundamentals of the contents to be modelled</p>	<p>(IC4) Apply the properties of conic sections in modelling contextual situations using ICT tools.</p>	<p>Problem delimitation Evaluation and Validation</p>
<p>Knowledge of the content procedures of school mathematics in the modelling of mathematical contextual situations.</p>	<p>(IC5) Apply the procedures of the conical sections in the modelling of situations within mathematics or related areas of knowledge (why is this done? How do you do it? When is it done? Using ICTs.</p>	<p>Observation and experimentation Evaluation and Validation</p>

Knowledge of the possible phenomenological relationships between mathematical contents when modelled in contextual situations.	(IC6) Recognise the phenomenological relationships that arise within mathematical contents (conical sections) and contribute to modelling real-world situations.	Connection with other models
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Table 2 shows that the teacher's time to develop mathematical modelling that is most related to categories of the KMT subdomain is the selection of strategies. This is understood since, from the knowledge of the examples, the strategies and the potentiality of the resource as a teaching strategy allow us to carry out processes of modelling daily life situations using mathematics in the construction of models.

Table 2.

KMT Indicators

<i>Domain</i>	<i>Subdomain</i>	<i>Category</i>	<i>Indicators</i>	<i>Moment of Villa-Ochoa (2007) modelling</i>
PCK	KMT	Knowledge of the type of content that students like.	(IE1 ¹) Identify situations that are of interest to the	Observation and experimentation

¹ Note: IE represents knowledge indicators of KMT from 1 to 6.

	student in mathematical modelling through mathematical software	Strategies selection
Knowledge of the potential of the virtual resource and/or material in the modelling of the contents in contextual situations.	(IE2) Know the effectiveness of GeoGebra commands and Microsoft programs as support tools for the mathematical modelling of daily living situations of conical sections.	Strategies selection
Knowledge of the potential of the virtual and/or material resource in the teaching of the contents.	(IE3) Know the effectiveness of ICT tools (GeoGebra and Microsoft programs) to dynamically perform mathematical representations of conic sections.	Strategies selection
Knowledge of activities, tasks,	(IE4) Know the variety of	Problem delimitation

<p>strategies, and examples that contribute to the teaching of mathematical content.</p>	<p>activities, exercises, examples, and tasks to be performed with ICT tools that contribute to the construction of mathematical models related to conic sections</p>	<p>Strategies selection</p>
<p>Knowledge of the type of aid provided to students to cover their needs in the teaching of mathematical contents.</p>	<p>(IE5) Know the type of aids to provide students when they have difficulties in capturing a real situation through mathematical modelling related to conic sections.</p>	<p>Observation and experimentation Strategies selection</p>
<p>Knowledge of the type of institutionalised theories as support to perform mathematical modelling</p>	<p>(IE6) Know how to relate theories and pedagogical processes in the teaching of mathematical</p>	<p>Connection with other models</p>

processes about 1 modelling
 contextual of real
 situations. situations
 and/or
 contexts of
 conical
 sections
 using
 GeoGebra.

Table 3 shows that the teacher's time to develop mathematical modelling that is most related to categories of the KPM subdomain is evaluation and validation and the connection with other models. This is understood since, from the teacher's knowledge about the ways of modelling, demonstrating, and proceeding using mathematical algorithms, he can develop processes of mathematical modelling of real situations and, in turn, relate his mathematical models as a simulation to such situations with already existing mathematical models.

Table 3.

KPM Indicators

<i>Domain</i>	<i>Subdomain</i>	<i>Category</i>	<i>Indicators</i>	<i>Moment of Villa-Ochoa (2007) modelling</i>
MK	KPM	Knowledge of the practices of mathematical work (modelling of real situations of mathematical contents)	(IP1 ²) Interpret real-world situations, whose solution is determined by building mathematical	Observation and experimentation Evaluation and validation

² Note: IP represents knowledge indicators of KPM from 1 to 6.

<p>Knowledge of how to proceed in mathematics using mathematical models in real situations using algorithms.</p>	<p>models using GeoGebra. (IP2) Meaningfully use mathematical algorithms in the modelling of real-world situations through area software (i. e., GeoGebra) of the conical sections.</p>	<p>Evaluation and validation</p>
<p>Knowledge to relate mathematical models to each other.</p>	<p>(IP3) Relate mathematical models built using software (GeoGebra) as a solution to real-world situations related to conic sections and other existing mathematical models.</p>	<p>Connection with other models</p>
<p>Knowledge of validations of mathematical content applications by simulating these</p>	<p>(IP4) Verify the resolution of a contextual situation of the conical sections in a theoretical way, by simulating</p>	<p>Evaluation and validation</p>

<p>Knowledge of the demonstrations of the abstract mathematical contents to be modelled through real situations.</p>	<p>them built from a mathematical model in GeoGebra. (IP5) Demonstrate concepts, formulas, axioms, and/or theorems of conic sections, which are related to everyday living situations and can be modelled by mathematical software</p>	<p>Evaluation and validation</p>
<p>Knowledge of how to solve mathematical problems according to their levels of abstraction.</p>	<p>(IP6) Recognise the different levels of mathematical abstraction to carry out mathematical procedures related to daily living situations and solved by mathematical modelling through GeoGebra.</p>	<p>Evaluation and validation</p>

For this work, we selected an episode in which elements of modelling (Villa-Ochoa, 2007) and elements of the specialised knowledge of the teacher who teaches mathematics intervene (Carrillo et al., 2013). The selected episode focuses on the process that a student must do to, according to the rotational movement of the radii of a circumference with respect to the axis y and through the GeoGebra software, draw a tangent line (2. S2 - L37-57) related to the content of the circumference and shown below:

- 37 **Pablo** *That triangle that is formed in GeoGebra, what function does it fulfil? What does it describe?*
- 38 **Student C** *Does it describe the movement of the triangle?*
- 39 **Pablo** *That indicates that when the triangle measures 90° out there at that point is going to pass the tangent lines that I mentioned, where will the right triangle be located?*

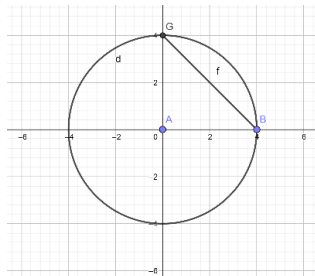


Figure 1. Graph of the triangle that is formed in the continuous movement of the radius of the circumference in GeoGebra.

- 40 **Student B** *So, how will it indicate the tangent line to the point?*
- 41 **Pablo** *That is, if I want to make a line tangent to the circumference, I place in use the GeoGebra commands, do you see where it says tangent line?*
- 42 **Student B** *Yes.*

- 43 **Pablo** *Well, then, with the cursor, I select: point or line in GeoGebra.*
- 44 **Student B** *Well, and then I select where it says the circumference, conic, or function.*
- 45 **Pablo** *Exactly, the reference point will be the G point, I look and trace the tangent line.*

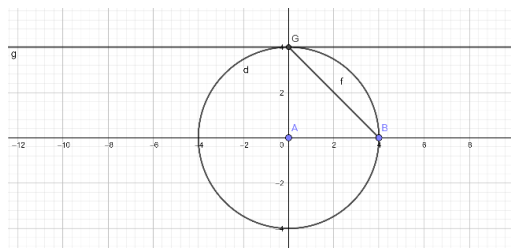


Figure 2. Graph of the tangent line to point G of the circumference that, in turn, forms a right triangle.

- 46 **Student B** *Yes, teacher, I understand.*
- 47 **Pablo** *What triangle was formed?*
- 48 **Group** *A 90° right triangle.*
- 49 **Pablo** *Yes, of course, since a perpendicular is formed, that is a right angle.*
- 50 **Student B** *And if I want the tangent line to pass through the point now, is a right triangle also formed?*
- 51 **Pablo** *Yes, look, I'm going to place it at point D that the tangent line passes through that point on the circumference, so now it starts to move [pause] and every time it passes through the quadrants what is marked? [Pause the GeoGebra software and show the representation], a tangent line. Did you see it?*
- 52 **Group** *Yes, teacher.*
- 53 **Pablo** *At how many points does the tangent line intersect?*
- 54 **Student H** *In one.*

- 55 **Pablo** *OK, what if the line didn't cut the circumference at a single point?*
- 56 **Student C** *So, it wouldn't be a tangent line.*
- 57 **Pablo** *OK.*

This episode begins with a diagnosis that Pablo makes to his students of the knowledge that they have of the meaning of the rotational movement of the radius of a circumference in GeoGebra, as well as the shape that the triangle that is formed in the movement of the radius of a circumference with the axis y takes. After this, Pablo introduces the concept of a line tangent to a circumference and defines the triangle formed in the movement of the radius as a right triangle (a measure of one of its angles equal to 90°). Then, he states that when the motion of the radius forms an angle of this measure, one of the many lines tangents to the circumference passes through the endpoint of the hypotenuse side of the triangle. Finally, Pablo asks questions about the perpendicularity and characteristics that these types of triangles have.

RESULTS Y ANALYSIS

Results

Characteristics of Pablo's specialised knowledge

In this section, the characteristics of Pablo's specialised knowledge found in the target subdomains of analysis of this research article are described, first, the characteristics of the KoT subdomain, in which Pablo shows knowledge of the procedures, representation records, and definitions of the content corresponding to the circumference. Then, there are the characteristics of the KPM subdomain, where the teacher's knowledge of the mathematical practices (modelling) of the line tangent to a circumference is evidenced. After this, we investigate the characteristics of the KMT subdomain, where the teacher's knowledge of the potentiality of the virtual resource and strategies for teaching the conical section regarding the circumference is evidenced. Such features are described below.

KoT Subdomain Features

Pablo presents three characteristics of the KoT subdomain, each identified from the corresponding indicators. This teacher has knowledge of the

procedures of the contents of school mathematics in the modelling of mathematical contextual situations identified from indicator IC5 (Table 1). In this case, it is determined by the patterning of the circumference, the movement of its radius with respect to the axis y , and the construction of the tangent line formed from that movement. So, is the question Pablo asks student B: “*How is it going to indicate the tangent to the point?*” a sample of Pablo’s knowledge of how to trace a line tangent to a point on the circumference? This is not only evidence that he knows how to make the tangent line that should pass through a single point of the circumference (point G), but also knows that the movement of the radius forms a right angle and, therefore, a right triangle, apart from knowing how to capture the situation mentioned through the commands of the GeoGebra software.

Pablo also shows knowledge of different *representation registers* (KoT) (Macías-Sánchez, 2014) to carry out the teaching of the conical sections. This is related to the IC3 of Table 1: “represent the contents of school mathematics (conical sections) in each of its registers (graphic, numerical, algebraic, pictorial, and natural language)”. In the case of representations of the natural language (2. S2- L39), he shows he can introduce the concept of the tangent line to the addressed situation of the circumference in the GeoGebra software without delving into what a tangent line is in mathematics (Definition-KoT), but he does so in the occasional use that this line has in the modelling of the rotation of the radius of the circumference dynamically in GeoGebra.

In the case of pictorial and graphic representation (2. S2- L45), Pablo uses point G as a pictorial representation [see Figure 2] to indicate the point where it passes in the plane, the line tangent to the circumference and that is inferred by Pablo so that the students build knowledge relative to that point as the coordinate of the axis y will form the right triangle that is required to build the line tangent to the point. As for the graphic representation, the graph of the circumference, the radius, the right triangle inscribed on the circumference from the drawn radius, and the tangent line to a point of the circumference are used, so, based on these elements, he analyses with his students that the tangent line only cuts a point of the circumference, or that the position of the radius with the y axis, when rotated by one of the commands of the GeoGebra software, must form a right triangle so that the tangent line can pass through a point of the circumference.

Thus, Pablo’s knowledge of the representation records is significantly evidenced in this episode. This aligns with the IC3 indicator of the KoT, since the teacher uses three types of representation registers, the natural, pictorial,

and graphic, in the teaching of conical sections, particularly of the tangent line to a circumference in GeoGebra, which, in turn, is related to the moment of selection of strategies proposed by Villa-Ochoa (2007), as shown in Table 1, because the use of pictorial, verbal, and graphic language is used for the benefit of the modelling of the triangle that is formed in the continuous movement of the radius of the circumference in GeoGebra, in the case of pictorial representation using GeoGebra points as a visibility strategy that shows the exact moment where the right-angled triangle is formed as a result of the dynamic movement of the tangent line to the circumference at that point in the software, or of the graphic representation, as a visibility strategy of the tangent line that cuts exactly to a point of the circumference.

Furthermore, Pablo evidences knowledge of the definitions (KoT) in the teaching of the conical sections, specifically in this episode of the circumference (2 Cor. S2- L47-47), which is aligned with IC1 of Table 1, which states that the teacher must know the definitions of the conical sections in the construction of mathematical models using ICTs, which is evidenced when Pablo affirms through questions such as: Which triangle was formed? At how many points does the tangent line intersect? What if the line didn't cut the circumference at a single point? In this sense, we believe Pablo performs them so that students interpret the application of the definition of conical sections as staging in the modelling of these definitions in the GeoGebra software. So, Pablo is expected to know that the behaviour that occurs in the two-dimensional plane through the rotation of the tangent line at any point on the circumference forms a right triangle, as well as that this line intersects only and exclusively a single point on the circumference.

Features of the KPM subdomain

Pablo presents a characteristic of the KPM subdomain, identified from its corresponding indicator. This teacher has partial knowledge of the practices of mathematical work (modelling of real situations of mathematical contents) identified from the indicator IP1 (Table 3) since he knows that to teach the tangent line to a circumference, he can do so through the modelling of any point on the circumference through which one and only one tangent line passes in GeoGebra and that, in turn, this process leads to a right triangle being formed each time the rotation of the tangent line occurs with the respective point in the Cartesian plane. However, we consider that the identification of knowledge of the teacher is partial because it does not comply with everything proposed in IP1 since the modelling process that Pablo carries out does so in a particular

content of mathematics, such as circumference, however, does not apply modelling to capture situations of daily life where the content corresponding to the circumference is applied.

Features of the KMT subdomain

Pablo presents two characteristics of the KoT subdomain, each identified from the corresponding indicators. This teacher is aware of the potential of the virtual resource and/or material in the teaching of the contents identified from the IE3 indicator (Table 2), this is evidenced when Pablo proposes to trace a tangent line to a point on the circumference, so that the students will observe that it cuts it at a single point, therefore, given the functionality of the resource (GeoGebra) to select any point on the circumference and thus trace the line, Pablo makes use of it, to select a point called G that was the reference of the application of the definition of the tangent line to a circumference.

Likewise, Pablo knows that the GeoGebra software allows the graphs to have movement and dynamism, so he asks them to apply the same procedure of plotting the tangent line to another point on the circumference called D, which shows that he knows that, through the GeoGebra movement command, students can show that at any point that is selected from the graph, one and only one tangent line will pass to the conic. As Paul observes in the following statement: *“Yes, look, I’m going to place it at point D, that is to say that the tangent line passes through that point on the circumference, so now it begins to move [pause] and every time it passes through the quadrants, what is marked? [Pauses the GeoGebra software and shows the representation], a tangent line,”* which frames his knowledge about the commands offered by the resource for teaching the content related to the conical sections.

On the other hand, Pablo evidences knowledge of strategies and examples that contribute to the teaching of the mathematical contents identified from indicator IE4 of Table 2: *“Knowing a variety of activities, exercises, examples and tasks to be performed with ICT tools that contribute to the construction of mathematical models related to the sections”* this because he proposed more than one example that would capture the graph of a tangent line to a point on the circumference, using several points (point G and point D) in whatever happened what the teacher wanted to prove. As well as the strategy of dynamising the movement of points in the plane, to generalise the definition of tangent line to a point of the circumference in the Cartesian plane.

Relationships between subdomain characteristics

Within the results, we found different relationships between subdomains of the same knowledge domain and different domains and proposed knowledge indicators. Within these relationships, it is evident that Pablo knows how to apply school mathematics procedures in mathematical modelling (KoT), and, particularly, knowledge of the observation and experimentation phase as proposed by Villa-Ochoa (2007) is highlighted, since he knows that to model the tangent line in the plane of the GeoGebra software, the movement of the radius of the circumference with the axis y must first be observed and analysed, i.e., a right triangle will be formed to make the modelling, which is partially related to the IP1 indicator of the KPM “*knowledge of the practices of mathematical work (modelling of real situations of mathematical contents)*”, since Pablo knows of the ways of modelling the behaviour of the rotation of the radius of a circumference and the right triangle that is formed in the movement of the radius itself to a point G through which a tangent line to the circumference passes; however it does not transcend in the modelling of that situation to a real problem, as proposed Villa-Ochoa (2007) in the modelling phases, exactly at the time of evaluation and validation, which is related to this category in Table 1.

In addition, the knowledge indicator IC3 has a potential relationship in this episode with the indicator IE2, “*knowledge of the potentiality of the virtual and/or material resource*”, and both are a sample of the teacher’s specialised knowledge to have their mathematical and didactic-pedagogical knowledge in teaching. Since it is necessary to use pictorial and graphic representation registers as key elements in the modelling of the line tangent to a point of the circumference that forms a right triangle, it is also necessary to know that the GeoGebra software allows inserting elements such as the point (G) [pictorial register] and the tangent line (black colour) [graphic register] that are observed in Figure 2. This, in turn, is related, as mentioned above, to the timing of strategy selection (Villa-Ochoa, 2007), and that the mathematics teacher must take into account to carry out modelling processes. We assume that Pablo knows that through GeoGebra, he can involve the student to know the various representations necessary to capture the situation intended to be modelled (Acevedo-Rincón & Flórez-Pabón, 2020).

However, this is considered only as an approach of knowledge, since Villa-Ochoa proposes these phases of modelling applied to situations related to everyday life where elements of mathematics intervene.

Similarly, it should be noted that for the teacher to know representation registers applied to the contents of school mathematics that they teach requires knowing the formal definitions of mathematics. Thus, to graph the circumference, its radius, the tangent line and other parts that compose it, they must first know what the radius is, what the tangent line is, and what implications its (tangent line) construction has. This is framed in the IC1 indicator, which is described as: “*knowing the definitions of trigonometric functions and conical sections in the construction of mathematical models using ICT (GeoGebra)*”, which relates to the IC3 indicator, referring to the representation registers, category of the MTSK proposed by Flores-Medrano et al. (2014) as part of the same subdomain of knowledge of the KoT.

Now some of the knowledge of the definitions of the circumference are explicitly evidenced by Pablo in this episode (2 Cor. S2 -L53-57). Through questions such as: How many points does the tangent line intersect? What if the line did not intersect the circumference at a single point? This is the intuited formula so that students, when observing the graph in GeoGebra (Figure 2), interpret this and know the definition of a tangent line to a circumference, with respect to that it intersects the conic at a single point. This denotes Pablo’s knowledge about this definition, to later translate it dynamically into GeoGebra (KMT).

DISCUSSION AND CONCLUSIONS

The construction of knowledge indicators of the KoT, KPM, and KMT allowed us to understand and interpret the teacher’s specialised knowledge when teaching conics and, in particular, the circumference, through mathematical modelling with the specialised GeoGebra software. Due to this, we identified the need to relate the didactic and mathematical knowledge of the teacher with the teaching of this content to use ICT tools in the classroom effectively. Consequently, the technological competencies in a teacher who teaches mathematics contents are related to the knowledge and use of technological resources for didactic and pedagogical purposes for teaching (KMT). Moreover, we found that the disciplinary competencies (mathematics) that a teacher must possess are linked to the knowledge of the foundations and development structure of mathematics, a knowledge that is crucial for the promotion of mathematical modelling when using ICTs. This is related to the research carried out by Santana and Climent (2015), who studied the mathematics teacher specialised knowledge for the teaching of the bisector of a triangle, the incentre, and a circumference inscribed to a triangle through the GeoGebra software,

which is the only research before this one, in which a mathematics teacher's specialised knowledge was studied considering the use of a technological resource (GeoGebra).

In this sense, as in this study, indicators of the different subdomains studied of the MTSK model were provided that described the necessary knowledge of the teacher for the teaching of the content in question, from technological, disciplinary, and didactic pedagogical knowledge and their respective relationships. This is interesting and new in this model as it specifically describes the knowledge required for a teacher to teach specific math content using ICTs. In future research, it will be convenient to transcend the three subdomains addressed here, further exploring the subdomains not addressed in this study: KSM, KFLM, and KMLS, and the teacher's beliefs, to obtain a more complete view of the specialised knowledge of the mathematics teacher who teaches conics through ICTs.

On the other hand, as a projection towards future research, it is essential that within the mathematics teachers' knowledge and education, teachers know how to model the contents they teach using ICTs in a practical way and, based on their mathematical knowledge, know how to apply it in various situations of daily life corresponding to the same mathematics or other areas of knowledge related to mathematics, such as economics, physics, engineering, statistics, administration, and chemistry, among others. Thus, teachers can apply, within the study of the modelling of this type of problem, areas that are transversal to mathematics, and the phases of modelling referring to observation and experimentation, problem delimitation, strategies selection, evaluation and validation, and connection with other models.

In the study of modelling, the use of specialised tools and/or software of mathematics (for example, GeoGebra, GeoTic, Matlab, or Cabri) takes relevance given its potential to carry out modelling processes. This implies that among the characteristics of the mathematics teacher should be the ability to relate the sciences of knowledge in an interdisciplinary way, based on their experiences and transversal knowledge between the sciences and mathematics. Thus, the teacher can put into practice the validation and connection between models that correspond to a contextual situation with similar situations explored from different perspectives, using aspects of modelling.

AUTHORSHIP CONTRIBUTION STATEMENT

IAPE and JPAR conceived the presented idea. IAPE developed the theory, JPAR adapted the methodology to this context, and MAMN created the models, carried out the activities and collected the data. IAPE and JPAR analysed the data. All the authors actively participated in the discussion of the results, and reviewed and approved the final version of the paper.

DATA AVAILABILITY STATEMENT

The data supporting the results of this research will be made available by all authors (IAPE, JPAR and MAMN) upon reasonable request.

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