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Reverse Mathematical Modelling

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ABSTRACT

Background: Research in mathematics education shows that mathematical modelling is a practice that seeks to "translate" problem situations into possible mathematical models without, however, explaining the complexity involved in the reverse formulation, starting from the mathematical model to delimit the type of situation. **Objective**: To highlight the problem of reverse mathematical modelling, in the sense of a reverse formulation that goes from the mathematical model to the situation. **Design**: For this, a course of study and research was carried out guided by the investigative cycle of mathematical modelling that is based on theoretical-methodological resources of the anthropological theory of the didactic. **Setting and Participants**: Pre-service teachers of a teaching degree at a public institution were faced with a problem in an unusual context about the decimal number system and, more broadly, the positional number system. **Data collection and analysis**: We present an empirical approach based on research carried out by Ferreira (2020) with teachers in initial training. **Results**: The empirical results observed confirm the hypothesis of the existence of the problem of reverse mathematical modelling, even in the face of normative models, in the sense that they can faithfully describe a real situation. **Conclusions**: Ultimately, the study of a type of problem in an unusual context, in addition to highlighting the encounter of teachers with different objects of knowledge, revealed the remarkable difficulty in delimiting the type of quantification situation that can be associated with the mathematical model of the number and as stimulate future research on the teaching of reverse mathematical modelling.

Keywords: Teacher education; Mathematical modelling investigative cycle; Numeral; Anthropological theory of didactics.

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RESUMO

Contexto: Pesquisas na Educação Matemática evidenciam a modelagem matemática como uma prática que busca “traduzir” situações-problemas em possíveis modelos matemáticos, sem entretanto, explicitar a complexidade que envolve a formulação reversa, partindo-se do modelo matemático para delimitar o tipo de situação. **Objetivo**: Evidenciar a problemática da modelagem matemática reversa, no sentido de uma formulação reversa que vai do modelo matemático à situação. **Design**: Para isso, foi realizado um percurso de estudo e pesquisa orientado pelo ciclo investigativo de modelagem matemática que se assenta sobre recursos teórico-metodológicos da teoria antropológica do didático. **Ambiente e participantes**: Professores em formação de um curso de licenciatura de uma instituição pública a partir do enfrentamento de um problema em contexto não usual para eles sobre o Sistema de Numeração Decimal, e mais amplamente do Sistema de Numeração Posicional. **Coleta e análise de dados**: Apresentamos um recorte empírico a partir da pesquisa realizada por Ferreira (2020) com professores em formação inicial. **Resultados**: Os resultados observados da empiria confirmam a hipótese de existência da problemática da modelagem matemática reversa, mesmo diante de modelos do tipo normativo, no sentido de que podem descrever uma situação real fielmente. **Conclusões**: Em última análise, o estudo de um tipo de problema em contexto não usual além de evidenciar o encontro dos professores com diferentes objetos de saberes, revelou a notável dificuldade para delimitar o tipo de situação de quantificação que pode estar associada ao modelo matemático do numeral, bem como estimulam pesquisas futuras sobre o ensino da modelagem matemática reversa.

Palavras-chave: Formação de Professores; Ciclo Investigativo de Modelagem Matemática; Numeral; Teoria Antropológica do Didático.

INTRODUCTION: THE PROBLEM OF THE REVERSE MATHEMATICAL MODELLING

Our theoretical-methodological assumptions are based on the notion of the mathematical modelling investigative cycle (Sodré, 2019), henceforth MMIC, and, more broadly, from notions of the anthropological theory of the didactic, henceforth ATD, which postulates that all situated human activity can be modelled through praxeological organisations that, as such, assume the notion of situation as a strong hypothesis of the very definition of mathematical knowledge from the theory of didactic situations (Brousseau, 1995), i.e., “knowledge is a situation"[1] (Bosch & Chevallard, 1999, p. 3, our translation).

In this context, the interest in mathematical modelling, or simply MM, is notorious. In recent decades, the subject has become an important research topic focused on the teaching and learning of mathematical objects in the classroom, as discussed by different researchers, such as Barquero (2020) and Florensa, Garcia, and Sala (2020).

Regardless of the theoretical current adopted on MM, there is a pertinent question to be answered: How can we teach modelling?[2] (Schukajlow, Kaiser, & Stillman, 2018, p. 11, our translation). Frejd and Bergsten (2018) call for empirical studies independent of the theoretical current adopted on MM to clarify this issue, considered, according to Blum (2011), as the most important one discussed at the International Conferences on Teaching Mathematical Modelling and Applications or simply ICTMA.

Here, we consider this issue in the light of ATD, as it allows us to see it as one of the types of problems experienced by teachers in the exercise of their profession. Specifically, the type of problem denoted by $\mathbf{P_0}$, put like this: What to teach about an object and how to teach it to a given class or study community? (Barquero, Bosch, & Gascón, 2011).

In this theoretical context, Florensa, Garcia, and Sala (2020), based on García, Gascón, Ruíz-Higueras and Bosch (2006), highlight that this issue, seen as a problem of the type $\mathbf{P_0}$,, gains new formulations among them and more frequent, “How to teach mathematical modelling? And how to teach mathematics through modelling?”[3] (Florensa, Garcia & Sala, 2020, p. 22, our translation).

Research on MM teaching from the theoretical approach of ATD admits the consensus of researchers from another area, including Borromeo Ferri (2006), Blum and Borromeo Ferri (2009), Perrenet and Zwaneveld (2012), Blum (2015), Greefrath and Vorhölter (2016), Vorhölter (2019), and Barquero and Jessen (2020), on the didactic technique of MM cycles, in order

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[1] Text fragment: *une connaissanceest une situation*.

[2] Text fragment: *How can we teach modelling?*

[3] Text fragment: *las dos formulaciones más frecuentes del problema son: ¿cómo enseñar modelización matemática? y ¿cómo enseñar matemáticas a través de la modelización?*

to minimise the complexity existing in the MM process in the classroom, which includes, among others, aspects of building mathematical models about situations in concrete contexts.

However, from the TAD perspective, the need to question the practices carried out within institutions is assumed and, in this sense, the MM cycle is questioned (García, Gascón, Ruiz Higueras & Bosch, 2006; Bosch, García, Gascón & Ruiz Higueras, 2006), not to criticise it, but to provide it with solid theoretical frameworks that allow a better understanding of it and, if possible, make it more accessible for teaching and, with that, for the learning of the MM.

Following this line, Sodré and Guerra (2018) and Sodré (2019) proposed the MMIC "as a methodology for the development and analysis of mathematical models of situations in concrete contexts" (Sodré & Guerra, 2018, p. 253) which must "be understood always as relative and provisional, open to questioning and revisions, in addition to being relevant insofar as it is rich for the identification of didactic phenomena and the formulation of didactic problems"[4] (García, Barquero, Florensa, & Bosch, 2019, p. 78, our translation).

The MMIC (Sodré & Guerra, 2018; Sodré, 2019) was proposed from the three genres of genuine tasks of mathematical activity, i.e., *to use familiar math*, *learn (and teach) mathematics*, and *create a new math*, announced by Chevallard, Bosch, and Gáscon (2001), which we rewrite from Sodré (2019) as follows:

G_1 - Use socially legitimate mathematical models for situations in social contexts to answer questions about them, highlighting the associative relationship between situations in contexts and mathematical models;

G_2 - Study a mathematical model before different situations and contexts, and study a situation in a concrete context before different mathematical models;

G_3 - Create a mathematical model associated with a new situation from the study of situations and their associated mathematical models, taking into account the analogies or homologies between those situations and the new situation.

[4] Text fragment: *Ser entendidos siempre como relativos y provisionales, abiertos a cuestionamiento y revisión, y pertinentes en la medida en que sean fértiles para la identificación de fenómenos didácticos y la formulación de problemas didácticos.*

Otherwise, the MMIC is supported by those *task genres* (Chevallard, 1999) that give it reasons about its practices, starting with the presentation of models as customisations of mathematical objects taught to situations in concrete contexts, such as the simple change of “letters” in an algebraic expression, for example, and goes on to the reverse way, starting from a model in search of associated situations and, finally, in an advanced way, the construction of models as articulations of studied situations and models.

Thus, these types of tasks are seen as guidance for teaching MM practices that seek to avoid the routine teaching of mathematics that is usually limited to making the students do what they saw the teacher do. This teaching, which achieves relative success in certain types of tasks in the strict field of mathematics, equipped with algorithmic techniques that can be performed by machines, encounters difficulties in MM teaching and learning.

In MM, Guerra e Silva (2009) and Sodré (2021) say that the situation and the mathematical model are interdependent and, therefore, it makes no sense to speak in the strict field of mathematical tasks. In this case, they are praxeological organisations with mathematics, those involving mathematical tasks and non-mathematical tasks, for example, the definition of a variable that is not reduced only to numbers, as mathematics considers, as they are endowed with senses and meanings not achieved by the mathematical tasks and, not least, by the complexity of articulations between different types of knowledge for the development of the MM process. This involves relations between the studied mathematical practices and the non-mathematical practices belonging to a situation in context that is almost, if not always, ignored in the teaching of mathematics and, as an extension, in the teaching of MM.

Under this thinking, the functionality of the MMIC for teaching was developed from the three genres of tasks as a praxeological organisation, consisting of six types of tasks, which may not all be necessary given a specific situation in context. Concretely, the six types of tasks that define the MMIC are as follows:

Task T_0: Build an Initial Reference Situation for the problem in context

An initial reference situation is understood as the first abstraction about the problem in the real context considered. The technique of this task consists of considering problems in context of the same type with all the known data, including the data to be found, which allow finding possible relationships between them, not necessarily mathematical.

Task T_1: Investigate the mathematical models that live in the school institution regarding the problem in context

Here, it is necessary to take into account the mathematical complexity of the mathematical models for the assumed situation. This is a vital question for the study, as a greater demand for mathematical knowledge may lead to the rejection or inappropriate adoption of a mathematical model about the studied reality.

Mathematical knowledge of a given *study community*[5] is always limited by the school through curricula and programmes and this can constitute a restrictive condition for the search for existing mathematical models in school literature, even those available outside the school walls, such as the internet.

Situations not initially imagined in data can be revealed with the use of the model, mainly from mathematical models that govern social situations, such as, for example, mathematical models on funding.

Following the understanding adopted by the ATD, we assume that the greater the knowledge of a person in situations with mathematics, i.e., situations that admit praxeologies with mathematics, the greater the availability of knowledge to find situations with associated mathematical praxeologies. This is the purpose of the next task.

Task T_2: Finding situations that can be associated with a mathematical model

The technique is to analyse the mathematical model against the considered initial reference situation. This includes deconstructing the mathematical model to build situations and vice versa.

> It must also be noted that a person may be confronted with a situation in which they see only a certain aspect, and that the construction of a model may oblige them to take a sharper look

[5]In this text, the expression *community of study* is the students and the teacher in the classroom.

at the situation and discover characteristics which, at the beginning, they had not noticed. Knowing more can help us see more[6] (Revuz, 1971, p. 50, our translation).

The analysis of the mathematical model forwards the encounter with the situations, as a reverse formulation, in the sense that it goes from the model to the situation with which the model formulation can be associated.

Task T_3: Evaluate mathematical models

The evaluation is carried out assuming as criteria the suitability and multivalence of those models. Both are evaluated against situations. The technique is forwarded by the following interconnected subtasks:

Subtask S_{T31}: Evaluate the adequacy of the reconstructed situations faced with the problem

The adequacy of the mathematical model is not a mathematical question; however, it is a vital question for the study of the reality that deals with the type of problem in context since, if someone uses an inadequate model because of its convenience, simplicity, for example, without observing its inadequacy before the situation, one must be aware of the dangers of drawing conclusions about the reality from the study of such a model (Revuz, 1971).

Subtask S_{T32}: The multivalence of the mathematical model and associated situations against the type of problem

Mathematical models that account for different types of situations and, consequently, different questions about the reality considered are preferable, rather than those that only account for a specific particular situation.

Task T_4: Develop a mathematical model

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[6]Text fragment: *One must also remark that a man can be confronted with a situation of which he sees only certain aspects, and that the building of a model may compel him to throw a more acute look at the situation and discover features of which, at the beginning, he was not aware. To know more may help to see more.*

The technique for this task is a product of the previous tasks, so it can result in: one of the studied mathematical models; modifications, including customisation, of one of the models or articulation and integration of studied models.

The experience of the MM study community is one of the conditions that help develop a model, in addition to the levels of didactic co-determination, which are not always clear, such as those imposed by the subject syllabuses and the resources available and used, such as calculators and/or computers, which usually limit or enhance the mathematical activities of the study community in the execution of MM *superstructural* and *infrastructural* tasks (Chevallard, 2019).

Task T_5: Disseminate and defend the mathematical model

The technique for this task consists of two phases:

I - Use the mathematical model developed to cope with the situations chosen as a reference, if possible, with different data sets but considering the data of interest to the problem under study as unknown. The demonstration of the consistency of the responses obtained by the model with the known response legitimises the mathematical model for the reference situation;

II - Use the mathematical model developed against the problem in a real context with the original data. The reasonableness of the answers obtained with the repeated use of the model with different possible data for the problem context encourages the legitimacy of the mathematical model for the type of problem under study.

The success achieved simultaneously in both phases temporarily legitimises the mathematical model as an answer to the type of problem under study. The failure of one of them shows the failure of the model, which requires reformulating the reference situation, leading to the beginning of a new cycle.

THEORETICAL-METHODOLOGICAL RESOURCES

In this investigation, we assume theoretical-methodological resources of the MMIC understood here as a *Guided Study and Research Path* (Sodré, 2019), henceforth GSRP, and more broadly from notions of the ATD whose MMIC emerges, specifically, based on elements of the didactic-methodological device called a study and research path (Chevallard, 2005, 2013), from now on SRP, which takes its concrete form through the articulation of academic and non-academic knowledge.

In the wake of this construction, the MMIC materialises, articulated by the functionality of the six types of tasks that integrate it, forwarded under a deeper look of the study advisor or researcher who, before the dyad, situation, and mathematical praxeology, asks: How does this relationship exist and why? This allows other questions about knowledge ecology, specifically: Why is a given observed knowledge considered and the other is not?

The GSRP (Sodré, 2019) works as a methodological research device and, at the same time, as a teacher education path, a way to meet teachers' needs to base their school didactic organisations on the DNS-PNS (decimal number system- positional number system) in a functional epistemology, in the sense that knowledge appears as "machines" that produce useful knowledge so that they can create answers to their different questions, as Bosch and Gascón (2010) defend.

From this point of view, we assume the GSRP is a methodological device for teacher education, keeping in mind that its realisation demands a change in the *topos* (Chevallard, 2009a) of the students or pre-service teachers since they can elaborate a personal answer, as they classically do when producing their solution to a problem given by the study director to investigate, as well as they can propose and introduce in the studies any work that they wish, through available and/or suggested literature.

Under this understanding of the GSRP, the praxeological organization of the MMIC:

> [...] contradicts the development of MM as an activity exclusive to mathematics when it directs the didactics of GSRP since, under the paradigm of questioning the world, it takes its concrete form, calling for itself the infrastructural, mathematical, and non-mathematical praxeologies, which includes all academic and non-academic knowledge required for the study of a domain of reality (Sodré, 2019, p. 128).

This complexity of relationships between mathematical and non-mathematical knowledge is present in the development of MMIC tasks and may prove to be problematic for teachers and students when they think of MM in the strict field of a single knowledge, such as mathematics, for example.

Concretely, the execution of task $\mathbf{T_2}$ constitutes what we call a process of *reverse mathematical modelling*, i.e., *find a situation that can be associated with a given mathematical model.*

This type of task involves the deconstruction and reconstruction of a mathematical model with constructions/reconstructions of situations. In general, this can prove to be problematic when students or professors face unknown or unusual problems in real contexts, as observed by Guerra and Silva (2009) about a group of professors who, despite having a master's degree in mathematics, could not model in unfamiliar contexts.

THE RESEARCH QUESTION AND OBJECTIVE

Our gazes take task $\mathbf{T_2}$ of the MMCI as a focus without losing sight of the other tasks, as it cannot be tackled in isolation from the others. Thus, methodologically, from the perspective of the ATD and the GSRP device considered here, it is necessary to create conditions to establish the MMIC for this purpose, i.e., to find answers to questions such as:

> *What conditions are necessary to develop the MMIC in a given concrete context?*

There are no ready answers to this question, although a condition is pointed out as determinant by academic mathematics, specifically, the one that assumes the indispensability of solid mathematical knowledge for success in MM. However, studies by mathematics educators on MM, such as Guerra e Silva (2009), Iversen and Larson (2006), Guerra e Silva (2009), Greefrath and Vorholter (2016), and Vorhölter, Greefrath, Borromeo Ferri, Leiß, & Schukajlow (2019), among others, show that mathematical experience is not a sufficient condition for learning MM.

Otherwise, mathematical knowledge, although necessary, is not decisive for the success of students and teachers in MM. Thus, here we aim to highlight the encounter of the *reverse mathematical modelling* as a problem of the MMIC as a didactic device for MM teaching and learning.

MMIC EMPIRIA WITH TEACHER EDUCATION RELATED TO THE DECIMAL NUMBER SYSTEM

Initially, we point out that the choice of the decimal number system, henceforth DNS, and more broadly about the positional number system, henceforth PNS, owes to the interest of different institutions, including basic schools and teaching institutions and, not least importantly, from mathematics education researchers, including Terigi and Wolman (2007), Itzcovitch (2008), Sadovsky (2010), Cenci, Becker, and Mackedanz (2015), Ferreira, Guerra, and Nunes (2019), and Ferreira and Guerra (2020), when they highlight the relevance of these themes in teacher education.

To meet our objective, we consider praxeological organisations carried out in a GSRP by a group of twenty-five teachers in initial education[7] as part of a subject of a teaching degree course in sciences and mathematics for the early years of elementary school from a public institution, referred by Ferreira (2020) from a problem in an unfamiliar context involving notions about the DNS in the context of a PNS.

We consider the GSRP in the light of the MMIC notion, with an interest in highlighting the teachers' encounter with the issue of *reverse mathematical modelling* through one of the trajectories taken by them during their education, considering that the problem in the context they faced led, in some way, to the encounter of teachers with the practice of quantification from the PNS.

The proposed problem for GSRP deals with an unfamiliar context with the purpose of preventing it from being reduced to a routine situation endowed with a prompt response, in the sense of moving *habitus*[8] associated with this

[7]The empirical data obtained from the teachers' manifestation in initial education took place in the context of a curricular subject at a public institution of higher education. In this sense, the professors' registers highlighted here do not give their identities, images, and voices, thus ensuring the dignity and due protection of the participants in scientific research. For this reason, no prior permission was requested by the appropriate councils of the research project from which the work arises. Thus, we assume and exempt the Acta Scientiae from any consequences arising therefrom, including full assistance and possible compensation for any damage resulting from any of the research participants, as directed by Resolution No. 510, of April 7, 2016, of the National Health Council of Brazil.

[8] The notion of habitus is highlighted here as “a system of durable and transposable dispositions that, integrating all past experiences, functions at every moment as a matrix of perceptions, appreciations, and actions and enables the fulfilment of

routine situation, according to studies by Bourdieu (2002) because, if that were the case, it would not meet the foundations of the methodology of a study and research path.

Specifically, given our objective, we take the following excerpt from the problem initially described by Ferreira and Guerra (2020):

> I belong to a people that are similar to humans. I have I mouth, V eyes and Z limbs, like them. But I differ by having only A, that is, Z minus I, fingers on each of those limbs, and I do not have hair, i.e., O hair on the whole body. On my planet, we grow grains and tubers like Earthlings. In particular, in our last solar year AIOOO, which numerically corresponds to the Earth's Christian solar year of 2000, we obtained the following output:
>
> Table I - Representation of grains or tubers
>
> | PRODUCTS | PRODUCTION |
> |---|---|
> | Beans | AZOIO |
> | Rice | ZVAII |
> | Manioc | ZZAAV |
>
> On my planet, we only use the V, A, Z, I, and O representation registers to represent quantities. According to the information described in the text, answer the following question: $\mathbf{Q_2}$ - How probably did ETs arrive at the representation of quantities in the way presented in the text? (Ferreira & Guerra, 2020, p. 10).

The analyses presented below focus on task $\mathbf{T_5}$ of the MMIC the teachers perform, organized into five groups, represented here by $\boldsymbol{FI_1}$, $\boldsymbol{FI_2}$, $\boldsymbol{FI_3}$, $\boldsymbol{FI_4}$, and $\boldsymbol{FI_5}$, taking into account the synchrony of this task with the other tasks and, of our interest, task $\mathbf{T_2}$.

ANALYSIS OF THE RESULTS

The task $\mathbf{T_5}$ demands the diffusion and defense of the models, so each FI_k group presented and defended their model and situation before the class

infinitely different tasks thanks to the analogical transfer of schemes acquired in a previous practice” (Bourdieu, 2002 [1972], p. 261).

[FI, D], where FI represents the set of pre-service teachers or all groups, and D represents the direction of study or teacher educator.

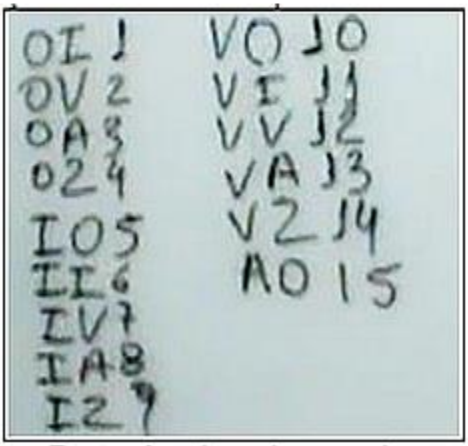
The produced situations St_k and M_k models, after the performance of tasks T_0, T_1, T_2, T_3 and T_4 within each group, were placed by each group for class evaluation [FI, D], but we focus on the defences of situations St_k and M_k models that we consider of greater relevance for the construction of the final answer approved by the class, and for containing the defence of elements of answers of the task T_2, which serves the purpose of seeking to highlight the problem of *reverse mathematical modelling*.

The St_1 situation and the M_1 model were referred to by the *FI_1* group, as follows:

> **FI_1** – *I was thinking a lot about those relationships, and I came to almost the same conclusion about the representation that she did, I made the same relationship here... But group 1 presented, the representation of the ETS, but when it comes to 10, it passes to our reality, IO, that would be represented by 10 for us... but I believe... actually... I think that it would be more or less like this... the OI = 1, OV = 2, OA = 3 and OZ = 4, then since theirs is quinary, it only goes up to 4... so when it gets here [...] it gets more or less like FI_1 did... the result is:*

Figure 1

Register of the relationship between letters and numbers. (Ferreira, 2020)



> *This means that we count 1,2,3,4, but when we get to 5, it is as if we had got to 9, so it becomes IO = 5, we move to 10, and automatically 1 goes forward, and 0 zero is left behind...*
>
> *Let me explain... when does it get to 99?*
>
> *It goes to 100, it's like the last ten, 44 is the last, got it?*

We deduce from the dissemination that:

> **St_1**: It can be reduced to carrying out the task "relating decimal numbers, as names of quantities, with quinary numbers";
>
> **M_1**: It was defined as a one-to-one correspondence between decimal numerals, as the name of quantities, and quinary numerals, in which they follow the same intuitive rule of writing registers of decimal numerals, i.e., they are made up of positions occupied by digits, in the case O (zero), I (one), V (two), A (three) and Z (four) and each position, when occupied by the digit that corresponds to the maximum value, in this case, Z, must be restarted from the O (zero), taking the successor of the digit in the next left position.

Put to the test before the class [**FI, D**], a limitation of the model was revealed when D asked the group $\boldsymbol{FI_1}$:

D - *If you were to do it, would you reach the relationship between 2000 with AIOOO?*

FI_1 - *Oh! the teacher would take too long! The process is long...*

The difficulty lies in the effort required to enumerate all the decimal numerals up to the decimal number 2000, since it is not possible to find in isolation a quinary (decimal) number corresponding to a given decimal (quinary) number, in particular, when that given quinary (decimal) number is a large decimal (quinary) number.

Otherwise, the validation of the mathematical model M_1 presented by the group of teachers $\boldsymbol{FI_1}$ remained pending, perhaps because it is an extremely difficult stage of the MM process, as highlighted by Frejd and Bergsten (2018, p.123, our translation): "validating the model is extremely difficult [...] without human negotiation, trusting too much on mathematical models as a basis, things can go wrong".

In this context of doubts, the group $\boldsymbol{FI_2}$, facing task $\mathbf{T_1}$ of the MMIC, to investigate the mathematical models that live in the school institution, found a mathematical model that can be described as $\mathbf{M_2}$:$\boldsymbol{d = r_N\beta^N + \cdots + r_1\beta^1 + r_0\beta^0 = \sum_{k=0}^{N} r_k\beta^k}$,, according to the basic positional representation β (Ripoll, Rangel, & Giraldo, 2016, p. 25):

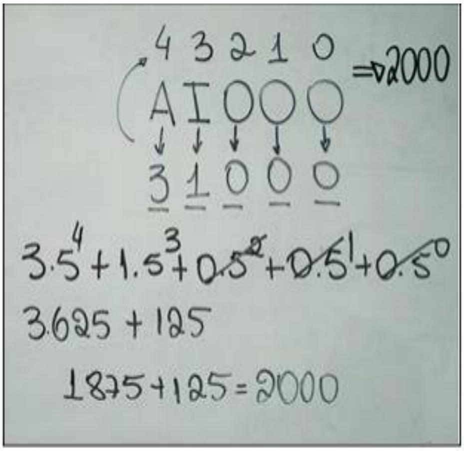
FI_2 $-$*I'll show you what I researched, can I be a teacher? Everything I researched here was taken from some articles like https://pt.wikipedia.org/wiki/Codifica%C3%A7%C3%A3o, and youtuber: https://www.youtube.com/watch?v=2pGkFn4Sgao.*

I could relate to what the colleague did, that I researched, like... we take the ET's system as a principle, which is the quinary, and from my point of view, it is indisputable, okay, so we have: The →0, I → 1, V → 2, A → 3, Z → 4. We observe that it is a sequence, they are the natural numbers, I searched the internet on the site: http://producao.virtual.ufpb.br/books/camyle/introducao-a-computacao-livro/livro/livro.chunked/ch0d3s03.html.I found out that there is a formula, I'm going to apply this formula here and you're going to see if it works or not, okay then?

> *I did it to see if it was logical. It was right, because just when I inserted the formula, in this AIOOO, I wanted to know if the formula to which that was related, it would give 2000. Then I hit it. The ET was normal, it had two eyes, lol...*

Figure 2

Register of use of the mathematical model. (Ferreira, 2020)



The insertion of the mathematical model $\boldsymbol{M_2}$ created a condition for teachers to build an answer on one of the problems raised by them, that is, validating the relationship of the quinary number 31000 with the decimal number 2000.

This model allowed ratifying results found with the use of the mathematical model $\boldsymbol{M_1}$. However, the problem teachers face of representing a decimal number in a quinary number remained open. As a condition to face this, study direction D recommended the abacus, so the teachers built the new *situation* $\boldsymbol{St_2}$ *for the model* $\boldsymbol{M_2}$. So, the confrontation of the $\mathbf{T_2}$ task, the search for a situation for a given model, begins explicitly.

Situation St_2 and model M_2 were forwarded by the group *FI_3* as follows:

> **FI_3 -** *Teacher, can I try to show you how I did it? I'll try to show it on the abacus, let's start with the unit, if our system were the quinary, right?! Let's count 1, 2, 3, 4, it passes to the next pin, when it's 5 then, it will reset the units, won't it? Let's count again: 1, 2, 3, 4, so it will become 4 and I will use a ball of another colour, filled it up again and we take it out and start again, and move to the next pin, every time the people count from 4 to 4, at 5 it resets and starts again.* (Figure 3)

Figure 3

Representation on the abacus. (Ferreira, 2020)

> **FI_5** – *If we take the abacus, this is interesting, taking three marbles, how many will fit in the unit of higher-order 2? how many fit? What is the maximum this place takes? Up to 4, right? So, I'm going to put 4 balls, let's start from scratch, how many do you need to fill the place?*
>
> **FI_1** – *4*
>
> **FI_5 –** *Right*
>
> **FI_3** – *Mine is on the right...*

FI_5 - *No! you can't! It keeps the same principle, from left to right... so here it only fits 4, if it were in the decimal system, it would fit 9.*

FI_5 – *How many can I put, then...?*

FI_1 – *4*

FI_5 – *Can I put another ball?*

FI_4 - *No!!!*

FI_5 – *So, it goes against the system because it doesn't support it, the house only supports 4, so I take the unit of order 2 and go to the unit of order 3, which means that the unit of order 2 arrived at 5, so I take, and I go to the next one, in this case, the unit of order 3 and I represent it with another ball, with another colour, in this case, a red one, I put one more ball in the unit of order 3, how much is it?*

FI_2 – *I understand that this question is clear in our minds, but when it comes to teaching children using concrete material... we can't...*

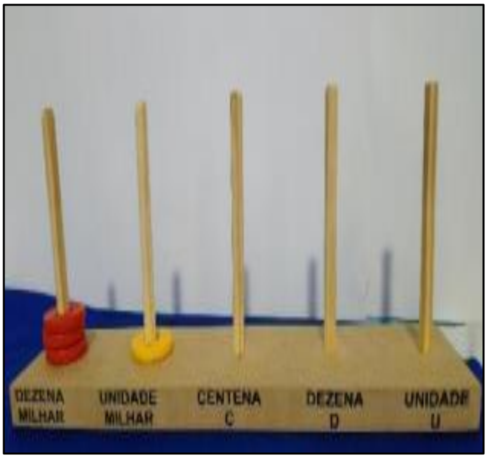
FI_2 – *I just want to conclude... a way to represent the potencies...*

FI_2 – *625 + 625 + 625 how much is it?...1875*

FI_2 – *1875 plus 1, the little yellow one (125) how much is it?...2000*. (Figure 4)

Figure 4

Representation by FI_2 of the quinary number 31000 for the decimal number 2000. (Ferreira, 2020)



> **FI_5** – *Observing the quinary system, I did it with the power for the decimal system, and it works great, it resets, really!! It's a cool tip to get back to zero!*
>
> **FI_5** – *Next tie $5^4 = 625$ and so on, until each one's needs.*

We observed the following situational praxeological organization through the teachers' manifestation faced with their practices:

> **St_2:** Representing a number on an abacus and vice versa, interpreting the abacus positions as the constant powers in the mathematical model **M_2** and vice versa, that is, the powers constant in the model as abacus positions.

The situation **St_2** and the mathematical model **M_2**, however, do not answer the question **Q_2**, and, with that, they invalidate this situation and the model, although the model **M_2** answer the posed situation **St_2**. This was observed in the presentation of the group ***FI_4*** subject to meeting the following condition forwarded by the direction of study ***D***.

> **D** - *Now we need to build the practice because this is at our level. Now we need to do as we will teach it to the children. We need to understand how this can work with children. In the next*

class (session) we need to clarify what each thing is, the basics...

FI₄ – *Now let's work on the quinary system, where the base is not 10. So, it's: 1 cap, 2 caps, 3 caps, 4 caps, and 5 caps and we close a tie... [...],*

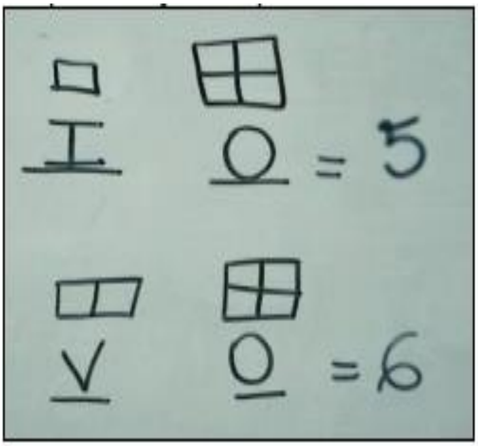
Figure 5

Group counting bottle caps. (Ferreira, 2020)

FI₄ – *[...] so we have a little tie, made up of five units with two more units I made the seven, different from there... which is another system, but I also formed little ties. From there, I skipped the house, it is positional, this question of being positional is crucial, because it needs to take the position of the houses and it is necessary to see the position of the houses, for example....*

Figure 6

Representation of the number of letters in base 5. (Ferreira, 2020)



FI₅ – *IO, how much is this worth here?*

FI₄: *10*

FI₅ – *Nooooo*

FI₁ – *4*

FI₅ - *Nooooooooo...*

FI₃: *it's 5...*

FI₅ – *5... and it will hit the formula that group FI_2 presented. If we do it and rummage around and make calculations... we'll hit there .... there in the formula that the other group presented.*

FI₅ – *And that's it... and we'll continue in the next... and I'll leave it for the next class, there are other things, but I'll leave it for the next class, I'll bring the golden material to present as if it were for the children, and then we develop and follow the numbers... I got to five, I'm going to stop.... because after that I found it difficult.... I couldn't see...*

FI₅ – *Now I need help... do you agree with me that the IO equals 5? What is the 5... and the 6? would it be V and absence? Yes* *or no? I'm going to write this representation VO = 6 here on the board.*

FI_1 - *No!*

FI_5 - *No? Why not?*

FI_1 – *Because it would be absence with the next*

FI_5 – *What would be next?*

FI_1 - *The "I"*

FI_5 – *The "I" is here*

FI_1 – *Yeah, but you didn't replace absence with 1. That would not be the case... Without listening*

FI_5 – *Here I represent two, which for me is the V.... because it is here.... let's start the disagreement. It's a problem... that's why I said it would be better to stop here... because there will be disagreement.*

The manifestation of the class [FI, D] showed a confrontation between the situations **St_1** and **St_2** with the explanation of **VO=6** *(Figure VI),* obtained based on **M_1** and **M_2**, but contradicting the model **M_1** which highlights the correspondence of **II = 6.** In another way**,** the teachers could not represent numbers on the abacus by quantification.

In that sense, the model **M_2** associated with the situation **St_2** did not answer question **Q_2**, which requires representing a physical quantity in a number. A situation to answer **Q_2** was not found, although it could be derived from the model **M_2**.

The condition of using concrete materials to be quantified without using the abacus, introduced by study director D, led teachers **FI** to face a new situation **$S._3$** before the model **M_2**. This situation ***St_3*** was forwarded by the group **FI_5** as follows:

FI_5 – *There are 7 caps, I'll add them together, or rather group them in twos because we're doing it on base 2.*

FI_5 –*The one that's left, is it the Single Unit?!*

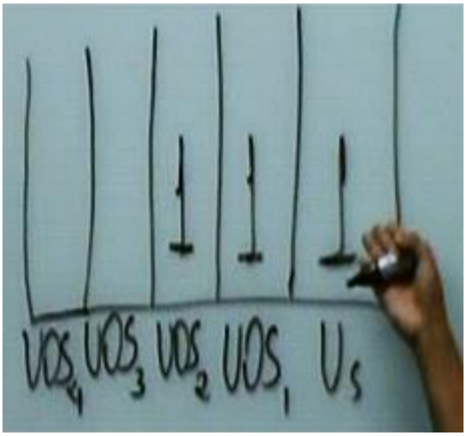
FI_5 – *Yeah*

FI_5 – *Okay, now, what? Do we regroup, teacher?*

> **FI_5** –*So, will it stay like this? [doubt], do I put them together again? Two out of two, 1 left over again, so there's 1 cluster of 4 and 1 cluster of 2, so there's 1 on US, 1 on UOS1, and 1 on UOS2.*
>
> **FI_5** – *Then, it registers in the written abacus the quantities by types of obtained clusters.*

Figure 7

Number registration in the binary system. (Ferreira, 2020)



The model **M_2** can then be associated with the following situation **St_3** of construction of a positional number, including quinary, from the quantification of a cluster of discrete units without using a physical abacus:

> **$St._3$:** Representing with a number from the quantification of a given cluster of physical units, considering different types of cluster of those units.

The dyad [**$St._3$**; **M_2**] allowed teachers to see the limitations of dyads [**St_1**; **M_1**] and [**St_2**; **M_2**]. This was revealed by manifestations of the class [FI, D] in appropriating the algorithmic practice of organising a cluster of units to be quantified by clusters.

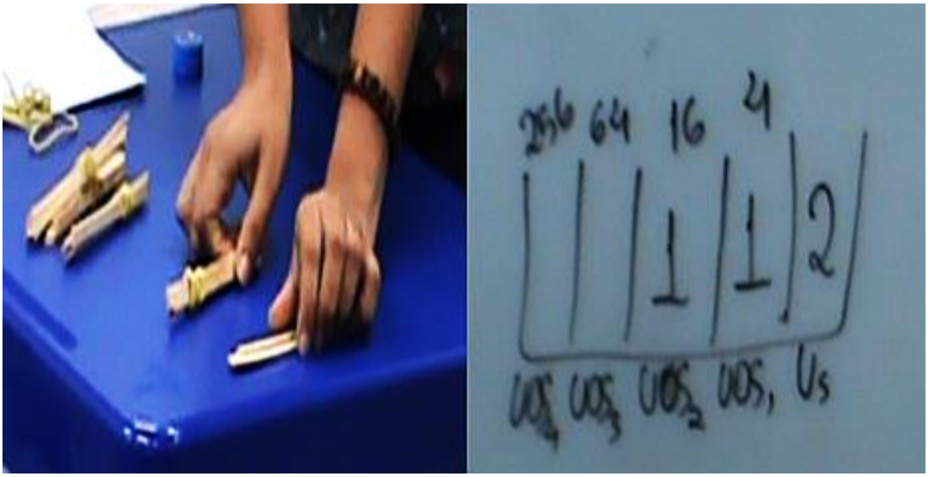
> ***FI_1**: I got it!!!!! I want to do it again!*

FI_4: Pal, change the base to see if it improves, do it on base 4.

FI_1: The count is now in fours and the number is 22. Let's separate and tie, so there are five groups of 4, remaining 2, which is the single unit... clustering them, we have one group in fours, remaining one group of 4. This leaves one single unit, one unit of higher-order 1, and one unit of higher-order 2. (Figure VIII)

Figure 8

Cluster 4 of 4 sticks and their representation on the written abacus. (Ferreira, 2020)



Furthermore, and no less important, it is worth noting that the situation ***St_3*** sent an answer to question **Q_2**, insofar as it revealed a possible way to arrive at a number by quantifying discrete quantities.

The answer to question **Q_2** found by the class can be described through four tasks divided into two steps:

- ✓ ***1st Step: Definition:***

  t_1 – Set the β limit, or base, of the unit count;

- ✓ ***2nd Step: Iterative process -*** Starting with the single units:

t2 – If possible, build clusters of a higher order than the existing ones, forming new clusters of β units of clusters of immediately lower order.

t3 – Register the number of cluster units that were being clustered in the previous step.

t4 – If the number of new higher-order clusters formed is less than the count limit β, register this amount in the corresponding position and finish by highlighting the written number. Otherwise, return to task t2.

In summary, the MMIC included a complex dynamic of mobilising tasks for its performance and, among them, the task $\mathbf{T_2}$ in *reverse mathematical modelling* became evident as a problem for pre-service teachers. This task promoted the "demagification" of numbers as they are good "machines" for producing knowledge (Bosch, Chevallard, & Gascón, 2006) related to the domain of reality that they may be associated with.

DIRECTIONS AND PERSPECTIVES OF REVERSE MATHEMATICAL MODELLING PROBLEM

This article met one of the recommendations of Niss (2015) (apud Stillman, 2019) on the necessary development of theoretical and empirical research in prescriptive or normative modelling, given that the structuring of quantification of physical quantities of numbers governs or translates "faithfully" domains of realities and revealed evidence on the problem of *reverse mathematics modelling* in the sense of the reverse formulation that starts from the model for the knowledge or construction of a situation in context that can be associated with the model.

The results of this research showed that overcoming the issue of *reverse mathematical modelling* that integrates the MM process may require that the teacher or study director introduce conditions that allow the study individuals to find possible answers to this problem.

Here, the conditions introduced considered that the mathematical model for structuring positional numbers can be seen as of the normative type (Greefrath & Vorhölter, 2016), in the sense that it faithfully describes a reality, in this case, of quantification of physical quantities. Hence the referrals from the director of studies to the teachers, so that they could use clusters of discrete units to be quantified. While this condition was not met, the pre-service teachers could not find the sought situation.

It became evident what the ATD recommends about changing the quality of relationships with knowledge (Chevallard, 2005), in this case, of teachers with positional numbers, and this characterises that the MMIC carried out is a possible answer to the problem of teacher education, relating to the positional number system, which includes the DNS.

The role of the director of studies was decisive for the advancement of the investigation since he is “who must continuously 'surprise' to fulfil his function [...], that is to say, the minimum condition” (Chevallard, 2005, p. 81, our translation) that contributed to the encounter of teachers with the situation of quantification of physical quantities of the number structuring.

In addition, it seemed clear to us that it is not enough to know or not to know mathematical knowledge to reach the necessary understanding to face the *reverse mathematical modelling* that requires the functioning of a social practice, such as quantification, for example, to associate it with a given mathematical model, even if this model can function as the norm for this practice. This confirms that “between knowledge and practice, there is a distance that is never entirely abolished”[9] (Chevallard, 2005, p. 171, our translation).

Finally, the practice of school MM must consider, above all, a wide variety of contexts of situations that allow students and/or teachers to recognise situations against a type of problem in context, bearing in mind that “we cannot expect any mystic transference from one example or context to another”[10] (Blum, 2015, p. 84, our translation). In this line and taking into account the results found here, we are encouraged to carry out future research on possible problems that may be revealed from the tasks of the MMIC and, in particular, on the teaching of *reverse mathematical modelling*.

[9] Text fragment: *que entre un saber y una práctica hay una distancia nunca enteramente abolida.*

[10] Text fragment: *cannot expect any mystical transfer from one example context to another.*

AUTHORSHIP CONTRIBUTION STATEMENT

GJMS and RSRF forwarded the submitted investigation. RBG oversaw the planning and execution and delimited the use of the mathematical modelling investigative cycle. GJMS, RSRF, and RBG planned the investigation question, the objective, the theoretical-methodological resources, the analysis of the empirical data and results found, and the investigation referrals and future perspectives.

DATA AVAILABILITY STATEMENT

The data supporting this study and investigation will be made available by the corresponding author (GJMS), upon prior request.

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