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Decision-Making in Situations of Uncertainty as School Mathematical Knowledge

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ABSTRACT

Background: One of the explicit objectives of school mathematics is to prepare students to make decisions. However, decision-making itself is not usually considered curriculum content. Recently, the Chilean school mathematics curriculum incorporated decision-making in contexts of uncertainty as a teaching object in differentiated secondary education. **Objective:** Discuss from a socioepistemological perspective how the teaching of decision-making in uncertainty is proposed in the Chilean curriculum, considering both the study programme and the official school text. **Design**: Based on a qualitative methodology, thematic analysis is used to identify historical-epistemological criteria and content analysis to review the activities of the curriculum texts. **Setting and participants:** Three textual corpora are analysed: a historical work, the syllabus, and the official school text. **Data collection and analysis:** By applying interpretative content analysis, the activities proposed for teaching decision-making in contexts of uncertainty are analysed, considering as a criterion the epistemological distinction between single-case decisions and decision-making processes, as well as the use of a priori and a posteriori probabilities. **Results:** The textbook provides more tasks or questions involving the student in decision-making than the syllabus, but not always explicitly, emphasizing calculation and formulas. Based on the defined analysis criteria, it was possible to classify all the activities except one related to the Monty Hall problem. **Conclusions:** The activities proposed in both the textbook and the syllabus almost exclusively promote single-case decisions, with a predominance of a posteriori probabilites, which generates a conceptual overlap between chance and randomness.

Keywords: Decision-making; Random; School curriculum; Randomness; Uncertainty.

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Toma de decisiones en situaciones de incerteza como un saber matemático escolar

RESUMEN

Contexto: Uno de los objetivos explícitos de las matemáticas escolares es preparar a los estudiantes para la toma de decisiones. Sin embargo, la toma de decisiones por sí misma no suele ser considerada un contenido curricular. Recientemente el currículo escolar chileno de matemáticas incorpora la toma de decisiones en contextos de incerteza como un objeto de enseñanza en la formación diferenciada de educación secundaria. **Objetivo:** Discutir desde una perspectiva socioempistemológica cómo se propone la enseñanza de la toma de decisiones en contexto de incerteza en el currículum chileno, teniendo en cuenta tanto el programa de estudio como el texto escolar oficial. **Diseño**: A partir de una metodología cualitativa, se usa análisis temático para identificar criterios históricos-epistemológicos y análisis de contenido para la revisión de las actividades de los textos curriculares. **Entorno y participantes:** Se analizan 3 corpus textuales: una obra histórica, el programa de estudio y el texto escolar oficial. **Recopilación y análisis de datos:** Aplicando el análisis de contenido interpretativo, se analizan las actividades propuestas para la enseñanza de la toma de decisiones en contextos de incerteza, considerando como criterio la distinción epistemológica entre decisiones de un solo caso y procesos de toma de decisiones, así como también el uso de probabilidades *a priori* y *a posteriori*. **Resultados:** El texto escolar provee una mayor proporción de tareas o preguntas para involucrar al estudiante en la toma de decisiones que el programa de estudio, pero no siempre de manera explícita; con énfasis en lo calculatorio y uso de fórmulas. Con base en los criterios de análisis definidos, fue posible clasificar todas las actividades excepto una, relacionada con el problema Monty Hall. **Conclusiones:** Las actividades propuestas tanto en el texto escolar como en el programa de estudio promueven casi exclusivamente decisiones de un solo caso, con predominio de probabilidades *a posteriori*, lo que genera una superposición conceptual entre azar y aleatoriedad.

Palabras clave: Toma de decisiones; Azar; Currículum Escolar; Aleatoriedad; Incertidumbre.

INTRODUCTION

The global health contingency triggered by the pandemic, among its multiple human, social, economic, and philosophical consequences, clearly showed how vital mathematical and statistical skills are for decision-making. Overall, the challenges of the 21st century demand that data and knowledge, widely available in a global era, be used to improve decision-making at all levels with a sense of governance (Lopez-Claros et al., 2020). Decision-making is intrinsic to everyday life and becomes critical, especially in uncertainty. Uncertainty may be due to the nature of the phenomenon itself or the lack of ability or knowledge of the individuals interacting with the phenomenon (Helton, 1997). According to van der Bles et al. (2019), the knowledge on which decisions are based is always surrounded by different types and degrees of uncertainty. Hence its ubiquitous nature.

In education, international standards have highlighted that school mathematics must prepare future citizens to face the problems and challenges of daily life (Organization for Economic Co-operation and Development [OECD], 2010, 2019). In this sense, it is reasonable to expect that such preparation includes the development of decision-making skills in contexts of uncertainty. Although notions such as probability, randomness, data, chance, uncertainty, and risk are combined in probability and statistics teaching (Sriraman & Chernoff, 2020), they are not necessarily equivalent concepts. For example, when probability estimates or calculations are not available, we speak of "decision-making under uncertainty", rather than "decision-making under risk" (Knight, 1921). These, like other differences around decision-making, have social and epistemological bases that require attention.

Research has shown links between the development of probabilistic thinking and the ability to reason, make inferences and make decisions under uncertainty in preschool (Denison & Xu, 2014), primary (Malaspina & Malaspina, 2020), and secondary school students (Vergara-Gómez et al., 2020), as well as in mathematics teachers (Elbehary, 2021). Likewise, from statistics, the importance of its teaching at the school level is in preparing citizens who are capable of making real decisions based on data (Lajoie, 1998; Shaughnessy, 2019). In this way, making appropriate decisions when facing uncertainty is a formative need, which is usually addressed by school education by including probability and statistics in the curricula, from primary to higher education (Batanero, 2020).

Regarding the role of teaching and learning uncertainty, Pratt and Kazak (2017) conducted a review of the literature, finding three axes of development: heuristics and biases, conceptual and experiential commitment to uncertainty, and a modeling perspective on probability, the first axis being related to the study of decision-making processes mainly. Although the presence of uncertainty is permanent in different everyday contexts, its mathematization emerged late, thanks to the development of probability (Greer & Mukhopadhy, 2005). Hence, uncertainty and probability are approached in a close relationship.

Decisions in situations of uncertainty are essentially bets whose results are determined both by the choices made by the people involved and by the specificity of the associated random procedure (Cortada, 2008). At the same time, uncertain situations are characterized by the impossibility of calculating the probabilities of all cases and, as such, by decision-making processes that cannot be completely deductive or inductive but rather tend to be heuristic (Mousavi & Gigerenzer, 2017). A heuristic "is a strategy that ignores some information in favor of quicker, more frugal, and/or accurate decisions than more complex methods" (Gigerenzer & Gaissmaier, 2011, p. 454). In short, the strategies we use and the justifications we develop for decision-making in contexts of uncertainty require broader explanatory frameworks.

Nowadays, given the prominence that decision-making has acquired in contexts of uncertainty, we consider it essential to reflect on this issue from mathematics and statistics education. Some research suggests that judgments based on simple heuristics, common sense, or intuition are often erroneous in making decisions (Garfield & Ahlgren, 1988; Shaughnessy et al., 1996). However, intuitive ideas are persistent, even when they are recognized to be false and have been subjected to corrective teaching processes (Garfield & Ben-Zvi, 2007). Thus, the relationship between decision-making in contexts of uncertainty and the use of intuition is a recurring topic of study. In fact, the discussion about the conflicts that arise between intuition and reason when decisions are made in contexts of uncertainty dates back to the work of Kahneman and Tversky (1973), i.e., it has been developing for almost half a century. In these early investigations, intuitive estimates were considered a detrimental resource for drawing inferences. However, more recent research (Arkes et al., 2016; Kubricht et al., 2017) has shown that to solve problems of great complexity and uncertainty, intuition and heuristic strategies are valid resources that facilitate the drawing of inferences.

In particular, considering learning situations at the school level, the relationship between decision-making processes, both under risk and uncertainty, and the development of probabilistic thinking has been analysed (for example, Brovcnik & Kapadia, 2011; Martignon, 2014; Bennett, 2014). In general terms, these studies agree that, although decision-making in situations of uncertainty requires probabilistic reasoning, other elements not based on formal probability calculations but on simple strategies and intuitive strategies prevail. Indeed, intuition has been one of the most complex aspects to deal with during the construction of probabilistic concepts (Fischbein, 1975; Gandhi, 2018). Thus, although several studies address decision-making under uncertainty in teaching and learning situations at the level of mathematics or school statistics (Shi, 2000; Borovcnik, 2015; Serradó-Bayés, 2018; Ingram, 2022), we still lack investigations inquiring into their historical and epistemological bases of decision-making and its eventual didactic scope.

For decades, Chile had not considered decision-making processes in contexts of uncertainty in the national curriculum. In 2020, this topic was incorporated into the differentiated scientific-humanistic mathematics education plan for the last two years of compulsory schooling. Noticing this background, we took an interest in understanding the types of decision-making in contexts of uncertainty, along with the possible related probabilistic knowledge, starting from a historical epistemological perspective. From this conceptual base, we reviewed and classified all the activities proposed in the school textbook and syllabus, which are the documents that support the implementation of the new subject proposed by the Ministry of Education of Chile [MINEDUC]. We identified the activities that used contexts or phenomena associated with situations of uncertainty and how they incorporated authentic questions and/or tasks to encourage decision-making.

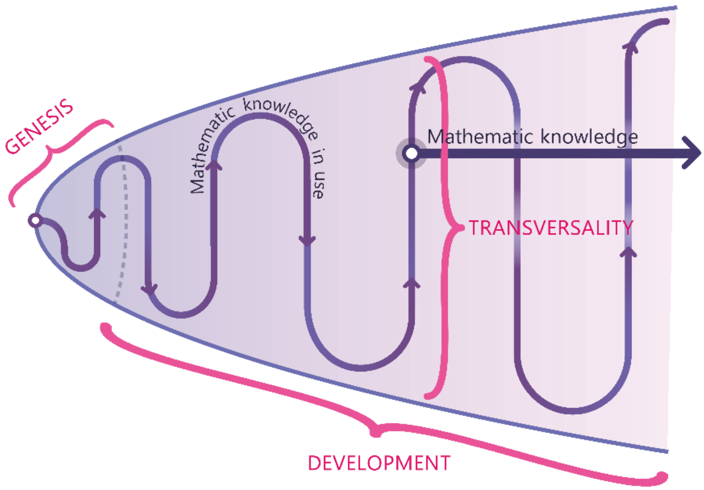
THEORETICAL FRAMEWORK

The socioepistemological theory of mathematics education (STEM) addresses "the phenomena of production and dissemination of knowledge from a multiple perspective of the dimensions of knowledge in use, through the study of the interaction between epistemology, sociocultural dimension (emphasis on the value of use), associated cognitive processes, and institutionalization mechanisms through teaching (cultural heritage)" (Cantoral, 2019, p. 791). From this perspective, the present study considers decision-making processes as mathematical-statistical knowledge in use. The STEM proposes to problematize those deliberate processes that allow the construction, exchange, and use of mathematical knowledge. In this way, knowledge in use "is constructed, reconstructed, signified and resignified; it is found in time and space; it is explored from the point of view of those who learn, those who invent, those who use" (Cantoral, 2013, p. 97), hence its analysis is not limited to the borders of school mathematical knowledge. Emphasis is placed on understanding decision-making as a human activity, exploring it in its social, cultural, and historical dimensions. As such, decision-making constitutes an important part of everyday life, which operates at both individual, collective, and institutional layers and puts into use different knowledge and skills from various disciplines. Given the object of study, we focus specifically on the statistical-mathematical knowledge involved in school mathematics.

One way to carry out a didactic analysis in STEM is to study this statistical-mathematical knowledge in use by analyzing its *historicisation* and *dialectisation* (Cantoral, 2013). When historicising, knowledge in use is located in time and space, explored from the point of view of those who invent, learn, and use it, assuming a historical, cultural, and institutional perspective (Cantoral, 2013). When conceiving the existence of knowledge in use in the continuous construction process, three fundamental moments of *historicisation* were identified: *genesis*, *development*, and *transversality* (see Figure 1).

Figure 1

Scheme of the theoretical model to study the constitution of knowledge in use. (Espinoza et al., 2018, p.252)



In the genesis, aspects of the *historicisation* related to the production of knowledge in use and its germinal meanings are explored; in *development*, its historical trajectory is analysed over time, and in the *transversality*, we study how knowledge is used in different human practices (Espinoza et al., 2018).

In this research, we carry out *historicisation* by specifically analysing the germinal moment of knowledge in use, in which the interest is to get situated in the contexts, intentions, and specific activities of the human being that accompanied and promoted knowledge production (Espinoza et al., 2018). The analysis of germinal moments is crucial, given that we can explore the constitutive and essential meaning of mathematical knowledge in use. Along these lines, STEM understands resignification as the progressive appropriation of meanings in specific contexts (Cantoral, 2013).

We recognise germinal meanings as fundamental pieces to understanding knowledge in use (Espinoza et al., 2018). However, these are often diffuse or invisible in school mathematics and statistics today (Cantoral, 2013). For this reason, *dialectisation* is proposed as a didactic analysis process in which the results of *historicisation* contrast with the way in which specific pieces of mathematical-statistical knowledge are conceived, organised, and taught in schools in search of generating didactic innovations.

METHODOLOGY

The research approach is qualitative, with a descriptive scope. Regarding the methods in the phase of *historicisation*, through a documentary search in the French digital library Gallica, we identified a book that is a precursor to the study of decision-making under uncertainty. The work, titled *Exposition de la théorie des chances et des probabilités* [Exposition of the theory of chances and probabilities], was published in 1843 by the mathematician and economist Antoine Augustin Cournot (1801-1877).

This work not only sums up Cournot's work on probability theory but also exposes an unprecedented epistemology to explain the relationship between the theories that constitute scientific knowledge and empirical reality (Martin, 2007). It also provides a philosophical view on how to apply probabilities and statistics to understand better those problems that became increasingly relevant in the 19th century: demography, valuation of life insurance premiums, financial market behavior, games of chance, and decision-making in civil courts, among others.

We scrutinised the book under the hybrid approach to thematic analysis (Boyatzis, 1998), articulating an anatomy of the work and its production context. Coding and the formulation of themes were carried out using the ATLAS.ti 8 software. The hybrid approach to thematic analysis considers deductive steps (literature-driven) and inductive steps (data-driven), described in Table 1.

Table 1

Stages and steps for the hybrid approach to thematic analysis of the work Exposition de la théorie des chances et des probabilités (1843). Adapted from Boyatzis (1998).

Stages of the hybrid approach to thematic analysis	Steps	Type of analysis
I. Preliminary review (primitive coding)	1. Delimitation of semantic units. 2. Selection of samples for preliminary inductive coding. 3. Complete preliminary inductive coding.	Inductive
II. Theme and Code Development (Thematic Coding)	1. Reduction of raw information. 2. Identification of relationships between primitive codes.	Inductive
	3. Refine and validate groups of primitive codes. 4. Theme through relationships	Deductive
III. Theme evaluation	1. Internal qualitative evaluation of the themes. 2. Qualitative validation of the themes, through expert analysis.	Deductive

From this analysis, we obtained 3,000 codes organized into 45 groups, which gave rise to six themes through the relationship and network functions of ATLAS.ti 8. The six themes were *randomness, variability, distribution, contexts, indeterministic laws,* and *decision*. Each of these themes reported characteristic meanings of the beginnings of the mathematisation of decision-making processes in contexts of uncertainty. For this research, we combine the germinal meanings provided by the themes of *randomness* and *decision*, which allow the construction of a conceptual basis that recognises authentic random situations that demand decision-making processes.

In relation to the dialectisation phase, we do an interpretative content analysis (Drisko & Maschi, 2016) to analyse the subject Probabilities and Descriptive and Inferential Statistics proposed by the Chilean curriculum for differentiated humanistic, scientific education, corresponding to the levels of 3rd and 4th grades of secondary education (16 and 17 years old). This subject addresses reasoning and decision-making in situations of uncertainty. The analysis considers the review and classification of all the activities contained in the syllabus and the decision-making unit of the school textbook. Since interpretive content analysis addresses both the manifest and the latent, it begins with an emergent coding that flexibly contrasts with the categories defined a priori (Drisko & Maschi, 2016). In this way, a preliminary review of the data is carried out, allowing us to identify the common structure between both corpora (programme and learning unit of the school text). Then, we apply the classification criteria defined deductively from the theory.

As a classification criterion, we considered the presence or the absence of two aspects: 1. Contexts or phenomena of uncertainty, 2. Questions or tasks that invite or require students to make decisions. To identify the second point, we used the meanings and characteristics of the combination of the themes decision and randomness, resulting from the previously reported thematic analysis.

To organise the recounting and visualisation, we contemplated the stages and steps described in Table 2.

Table 2.

Stages and steps used for the content analysis of the curriculum resources associated with the component Probabilities and Descriptive and Inferential Statistics.

Analysis stages	Steps
I. Identification	1. Thematic units. 2. Specific learning objectives per thematic unit.

		3. Lessons by thematic unit. 4. Mathematical-statistical activities per lesson.
II.	**Description**	1. Mathematical-statistical activities per lesson. 2. Contexts or phenomena of uncertainty that frame each activity. 3. Questions or tasks contained in each activity.
III.	**Classification**	1. According to the presence or absence of contexts or phenomena of uncertainty. 2. According to the presence/absence of questions or mathematical-statistical tasks that involve the student in decision-making.

To classify the questions or tasks contained in the activities, we ask the following: Is this a question or task that authentically involves the student in decision-making in contexts of uncertainty? We consider that the answer is affirmative if the activity meets the following conditions:

1. It challenges the student to make a decision in a situation or context of uncertainty.
2. The decision is required to solve a problem or guide the solution of a problem.
3. It requires the student to provide a justification or rationale for the decision.
4. It cannot be directly replaced by a calculation question.

RESULTS AND ANALYSIS

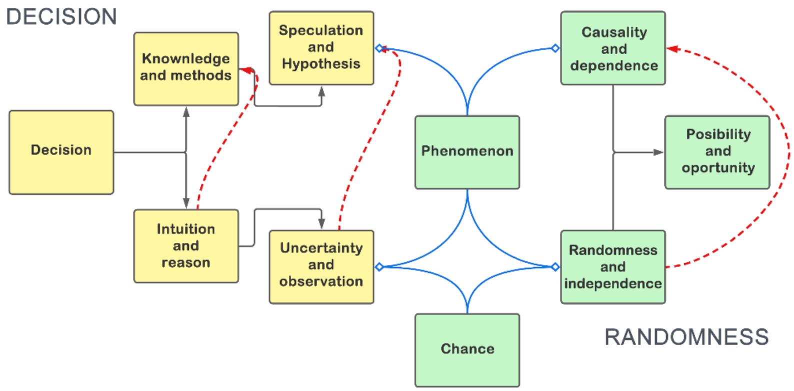
First, we will present the results associated with the process of *historicisation* and then the process of *dialectisation*, following what was stated in the theoretical framework.

Regarding *historicisation*

Regarding the germinal meanings obtained from the socioepistemological analysis, the combination of *decision* and *randomness* can be seen in Figure 2.

Figure 2

Thematic network connecting the characteristic meanings of randomness and decision.



The decision brought together five groups of codes (in yellow) and the randomness another five groups of codes (in green). According to the analysis, making a decision mobilises known knowledge and methods while requiring the coordination of intuition and reason. The latter is specifically due to the presence of uncertainty. Likewise, making a decision leads us to observe and understand the phenomenon so we can speculate and develop hypotheses, which could eventually be consolidated into new knowledge and methods. Systematic observation of the phenomenon allows us to distinguish randomness from causality and thus evaluate which possibilities of occurrence could result in favourable opportunities. On the contrary, given the impossibility of conducting a systematic, or at least extensive, observation of the phenomenon, we can only perceive chance and act intuitively.

Difference between chance and randomness

The work *Exposition de la théorie des chances et des probabilités* analyses different types of problems, which are solved through the study of the possibilities of error or risk in decisions under uncertainty. Cournot's work provides us with a conceptual difference between chance and randomness, based mainly on the notion of independence of a series of occurrences or causes. On the one hand, Cournot states that chance is not the absence of cause or a state of ignorance about causes; instead, it is the multiplicity of causes without dependence or a traceable relationship between them that appear in the occurrence of a specific event (Cournot, 1843). On the other hand, for the mathematician, randomness is a more complex notion; it is related to how we perceive and observe phenomena to analyse them.

Cournot gives us an example. The trajectory of a projectile can be modeled by a parabolic curve, but this does not necessarily mean that the behaviour of the projectile results perfectly in a quadratic function. Similarly, random phenomena can be explained through models, in which the degree of correspondence can be improved by increasing the number of data and the quality of the sample. In fact, we start to notice certain regularities when we have large data sets or enough processes of actions/events over time, properly collected and systematised. In this way, randomness is the regular expression that arises from the systematic registration of chance events, which occur under similar conditions of the same phenomenon (Cournot, 1843). Therefore, randomness can be made explicit through laws or properties, and its distribution takes on a form as the phenomenon is studied more completely.

This distinction between chance and randomness turned out to be essential in the germinal mathematisation of decision-making processes in contexts of uncertainty since it allowed the objective study of different random phenomena based on laws such as the law of error or the law of large numbers (Vergara-Gómez, 2020). Specifically, it was necessary to separate the fortuitous occurrence of a specific event in time —difficult to anticipate or measure— from the regular expression acquired by the registration of a large number of fortuitous occurrences under equal phenomenological conditions

—susceptible to being subjected to measurements and mathematical and/or statistical analysis.

Difference between single-case decisions and decision-making processes

Although Cournot never mentions the term model in *Exposition de la théorie des chances et des probabilités* nor in his other works (Walliser, 2007), every time he applies theoretical ideas to study randomness, he uses some simple examples, from which he builds a more general analytical principle or expression. In this way, it simplifies the phenomenon, assuming that events are repeated many times under similar circumstances. The phenomenon, then, is treated as a collection of independent experiences, where each one, although unpredictable, determines the same type of possible outcomes. The range of possibilities could certainly be broader, but the observer can focus their interest on one of them, simplifying the analysis. Cournot begins with the assumption that events $A', A'', \ldots, A^n$ are repetitions of event A and that events $B', B'', \ldots, B^n$ are repetitions of the complementary event B.

The value of the probabilities of these events can be defined, as the mathematician explains, as either a priori or a posteriori. In both cases, the probability measure is obtained through the ratio between the number of favourable possibilities and the total number of opportunities. However, in the a priori probability, all abstractly possible combinations are considered, while in the a posteriori probability, the frequencies of known outcomes of random events are contemplated (Cournot, 1843). The mathematician explains that these probabilities are approximated only when the value of the number of trials is large enough. Hence the importance of having many trials for the randomised experiment one wants to analyse.

From the a priori perspective, if one assumes that the probabilities p and q of the events A and B, respectively, remain significantly invariant during the process, then, for m attempts, the product $(p + q)(p' + q')(p'' + q'')\ldots$ is converted to $(p+q)^m$, whose general analysis can be carried out using Newton's binomial. This idea allowed Cournot not only to calculate specific probabilities but also to determine a minimum or maximum number of experiments to reach a previously established specific probability indicator. A simple and representative example of how this idea establishes the basis for a distinction between decision-making processes and single-case decisions is given using a game of dice. Cournot proposed a game with two dice in which one wins by rolling a double 6. Instead of asking, "What is the probability of winning?" which implicitly assumes a random trial, he asked: "How many minimum trials are necessary to ensure a 1/2 probability of obtaining the event of interest at least once?" Note that a random process of many actions is needed to answer the question. In response, Cournot proposed the equation that can be seen in Figure 3.

Figure 3

Proposed logarithmic equation to determine a minimum number of attempts. Cournot (1843, p. 48).

$$q^m = \frac{1}{2}, \quad p = \frac{1}{36}, \quad q = 1 - p = \frac{35}{36};$$

et l'équation précédente donnerait

$$m = \frac{\log 2}{\log 36 - \log 35} = 24{,}6\ldots$$

Based on the equation expressed in Figure 3, Cournot concludes that at least 24 attempts are necessary to obtain at least one successful event. Round the result to 24 due to the discrete nature of the experiment. We must keep in mind that the cumulative probability symmetrically around the mean, as Cournot proposes for this case, is only 0.5. For example, if one wanted to ensure a higher cumulative probability, say 0.75, the number of trials would increase to 49. This cumulative probability is based on sampling logic. A probability P of 0.5 means that, for a large number of samples, for each of 24 trials or events, approximately half would present at least one "double six" event. This type of analysis promotes the understanding of probability in the scenario of randomness, not chance. Consequently, Cournot says it is possible to make reliable a priori probabilistic estimates only when one can decide the number of trials.

Cournot uses a posteriori perspective mainly in those random events whose possible combinations cannot be determined arithmetically, such as those that derive from natural or social phenomena. In these cases, the multiple variables that provide probability laws for the values that the random magnitude takes are almost always of an unknown nature (Cournot, 1843). At that time, it was practically impossible to subject the occurrence of such events to theoretical calculation. The calculation of a posteriori probabilities makes sense when a sufficiently large amount of data has been systematically collected. Currently, this would be equivalent to the appropriate use of sampling and recording techniques. Knowing the a posteriori probabilities facilitates decision-making, as they reveal possible trends or shapes in the distribution of frequencies, which, in turn, allows evaluating possible readjustments in precision, increasing or improving the registration of events. One example is the decision-making processes of civil justice courts in France in Cournot's time. The French justice system kept a systematic record of rulings and appeals. A group of judges handed down many sentences over several years, which required a predetermined majority of votes.

Thus, the calculation of the a priori or a posteriori probability is relevant, depending on the nature of the situation, with the decision-making processes being those that give meaning to choosing one or another perspective. Thus, on the one hand, the a priori probabilities allow us to know the minimum number of trials required to make a favourable decision, and, on the other hand, the a posteriori probabilities inform about the frequencies of events, favouring decision-making regarding a set of future events. At the origins of the mathematisation of decision-making, decision-making processes stand out over single-case decisions. The processes involve several trials or recordings of events, in which the calculation of probabilities arises to estimate the convenient number of trials or events. Unlike single-case decisions, where probabilities are calculated to estimate the possibility of a specific event occurring, decision-making processes take a more global look at the number of attempts and the distribution of results as the number of events increases.

Regarding *dialectisation*

The school textbook, edited especially for the Ministry of Education and for free distribution countrywide, is structured in four units, the first of which is called "Decision-making in situations of uncertainty", and comprises six lessons. In total, we reviewed 49 activities. The Probability and Descriptive and Inferential Statistics syllabus (MINEDUC, 2021) is structured in four units (see Table 3), each with four main lessons and between five and 12 evaluation lessons. In total, 193 activities were reviewed. Table 3 reports the results of the syllabus review, and Table 4 reports the results of the students' text review.

Table 3

Description of the units, learning objectives, and lessons of the syllabus, together with the classification of activities by lesson.

Unit	Learning objective (LO)	Lessons	A	B	C
1. What do the graphs say? Critical analysis of information	Argue and communicate decisions based on the critical analysis of information present in histograms, frequency polygons, cumulative frequency, box diagrams, and point clouds, including digital tools (LO1).	1.1 Critically analyse information in the context of vital statistics (life expectancy).	16	1	16
		1.2 How to statistically represent data and phenomena? (histograms in different contexts).	13	1	13
		1.3 Make decisions from box plots (box plots in different contexts).	11	0	11
		1.4 Scattered or related data? (point clouds in different contexts).	9	0	9
		1.5 Evaluation.	5	0	5

		Total	54	2	54
2. Understand the sample mean, dispersion measures and correlation	Solve problems that involve the concepts of sample mean, standard deviation, variance, coefficient of variation, and sample correlation between two variables, both in handwriting and using digital technological tools (LO2)	2.1 Analyse graphic information in different contexts.	11	0	9
		2.2 The sample mean and the population mean in different contexts.	13	0	13
		2.3 Use sample correlation in social science contexts.	8	1	8
		2.4 Apply the linear correlation model in population censuses.	12	0	12
		2.5 Evaluation	12	1	9
		Total	56	2	51
3. Modelling of phenomena using the probabilities of binomial or normal distributions	Model everyday phenomena or situations in the scientific and social fields, which require the calculation of probabilities and the application of binomial and	3.1 Random experiments with Bernoulli and Binomial models.	10	0	6
		3.2 Understand the normal probability model.	5	0	2

	normal distributions (LO3)	3.3 Apply the normal model in the transportation of people.	13	1	13
		3.4 Approximate the binomial distribution by the normal distribution	9	1	7
		3.5. Evaluation	12	1	11
		Total	49	3	39
4. Make statistical inference	Argue inferences about parameters (mean and variance) or characteristics of a population, based on data from a random sample, under the assumption of normality and applying procedures based on confidence intervals or hypothesis tests (LO4)	4.1 Make inferences about the mean of a population using confidence intervals.	4	0	1
		4.2 Inferences in different contexts using confidence intervals.	5	0	5
		4.3 Develop a hypothesis and verify or reject it in different contexts.	9	0	5
		4.4 Develop and verify or reject a hypothesis.	7	0	7
		4.5.	9	1	8

	Evaluation			
	Total	34	1	26
	Overall total	193	8	170

Note: A indicates the number of total activities in the lesson; B indicates the number of activities that involve the student in decision-making; C indicates the number of activities that are based on contexts or phenomena of uncertainty.

Table 4

Description of the units, learning objectives, and lessons of the student text, along with the classification of activities.

Unit	Learning objective (LO)	Lessons	A	B	C
1. Decision-making in situations of uncertainty	Decision-making applying data dispersion measures.	1.1 Height of the Chilean soccer team players and measurement of central tendency.	1	1	1
		1.2 Calculate and interpret measures of central tendency, quartiles. Counting techniques. Classical probability calculus.	6	0	5
		1.3 Dispersion measures.	8	3	8
		1.4 Comparison of data sets.	8	5	7
	Decision-making applying conditional probabilities.	1.5 Conditional probability.	6	2	6
		1.6 Total probability.	15	3	8

	1.7 Assessment.	5	2	5
	Total	49	16	40

Note: A indicates the number of total activities in the lesson; B indicates the number of activities that involve the student in decision-making; C indicates the number of activities that are based on contexts or phenomena of uncertainty.

As shown in Table 3, the total of activities in the syllabus that invite students to make a decision is approximately 4.63% of the proposed activities (9 of 194), while in the textbook, they correspond to approximately 32.65% of the proposed activities (16 of 49). On the other hand, 88% and 82% of the activities in the syllabus and the textbook, respectively, evoke contexts or phenomena of uncertainty, showing that the context of uncertainty by itself is not enough to trigger decision-making. In total, we identified 24 activities that foster decision-making in contexts of uncertainty. Several make explicit reference to the notions of chance and randomness without a clear conceptual distinction. For example, concepts are used as synonyms in problems associated with sample selection. Furthermore, although the textbook provides a greater proportion of tasks or questions to involve the student in decision-making, many of these are not explicit and are presented after the tasks of calculating statistical or probabilistic measures, i.e., the activities associated with making decisions do not motivate the search for strategies, but rather emphasise calculations and formulas. It should be noted that the units and learning objectives refer to both statistics and probability. Hence, not all decision-making tasks appeal to probability calculus; however, several statistical tasks could be addressed by considering analysis of frequency distributions and, therefore, frequentist probabilities. In this way, it is possible to organize the activities by crossing criteria, as can be seen in Table 5.

Table 5

Classification of activities that involve the student in decision-making in the syllabus and the textbook.

	A priori probabilities		**A posteriori probabilities**		**Total**
	Statistics	Probability	Statistics	Probability	

Syllabus	**Single case decisions**	0	3	4	0	7
	Decision-making processes	0	1	0	0	1
Textbook	**Single case decisions**	0	3	12	0	15
	Decision-making processes	0	0	0	0	0
	Total	0	7	16	0	23

Note: A indicates the number of total activities in the lesson; B indicates the number of activities that involve the student in decision-making; C indicates the number of activities that are based on contexts or phenomena of uncertainty.

Table 5 displays the predominance of activities associated with the use of a priori probabilities in matters of probability and a posteriori probabilities in statistics issues, the latter being more than double that of the former. Activities on statistical issues refer exclusively to decisions in a single case. Something similar happens with activities on probabilistic topics, except for a single activity. The activity belongs to lesson 3.4 of the syllabus and is based on the question, “On what basis would you decide from what value of *n* it is worth using the normal approximation of the binomial distribution? Argue. Guess a rule of thumb to determine from what value of *n* it is convenient to use the normal approximation of the binomial”. Although the activity does not explicitly make multiple attempts and does not offer a context of meaning, this is the only activity that has the potential to promote searching by varying the number of attempts or the size of the samples.

Of the 24 activities that offer students the opportunity to make decisions, only 23 could be classified. The unclassified activity is from lesson 1.5 “Conditional probability” of the textbook, and refers to the Monty Hall problem. The reason why it was not possible to assign a category is that the activity presents a single-case decision, but the instructions for students to analyse the situation are presented as a decision-making process. Furthermore, the first question expects students to use an a priori measure of probability; but, to verify whether their decision is correct, they must simulate the process, recording the data and using the frequencies to obtain a posteriori probabilities.

The activity proposes that students analyse the problem in pairs, observing the image in Figure 4, where the contestant is Leonardo.

Figure 4

Illustration of the textbook. MINEDUC (2020, p. 22).

In this activity, students must observe the following actions:

> "a) If you were Leonardo, what would you choose: change the door or stick to it? Why? Argue and discuss your answer with your classmates.
>
> b) Before Leonardo chooses a door, what is the probability that he chooses the door that has the car behind it? And that he chooses the one with the goat?"

Next, to help students see that the probabilistically correct option is to change the decision, they are asked to use concrete materials to make cards that simulate the door options and use them to "reproduce the situation several times, counting the result and completing the information in a table" (MINEDUC, 2020, p.22). That is, while in the context of the well-known problem, the contestant can make a single choice, the explanation of the theoretically optimal decision is based on the assumption of a long series of games or choices.

Although this activity involves simulating the decision several times using cards, in the actual context of the contest, the possibility of changing the decision is presented to the contestant as a unique opportunity. In this sense, analysing the possible results of the decision after several attempts is not consistent with the spirit of the contest. On the other hand, it is unclear how many attempts would be enough to evaluate the appropriateness of the decision. For example, if students, using the card mechanism, play 10 or 15 times, the empirical variability is high, and, therefore, they could conclude, based on their data, that it is more convenient to maintain the first decision. Furthermore, the opportunity to change or preserve the decision occurs as an isolated event, not as a series of events. Thus, the presenter's offer, of a peremptory nature, requires a quick response, which does not allow mechanisms for anticipation, comparison, or measurement. In the eyes of the decision maker, the outcome of their decision is fortuitous. And, in fact, for each isolated decision, not even the most precise of conditional probability calculations guarantees a correct decision; the outcome is genuinely determined by chance.

Needless to say, given the nature of the contest, the decision corresponds to a single-case decision. If a student answers before doing the simulation –that it is more favourable not to change the decision–, there are no physical means to verify the supposed error of that decision other than showing what is behind the selected door. In single-case situations, there is no direct verification or success evaluation criterion for the estimated probability (Borovcnik, 2016). In this regard, we pose the following question: Why should we consider a decision as incorrect or suboptimal if, in factual terms, it is not possible to verify its effectiveness? The situation would be different if decisions could be evaluated in a decision-making process since, as Cournot proposes, in a process, one can analyse the minimum number of attempts necessary to ensure a specific cumulative probability. This cumulative probability must be estimated beforehand to overcome simple "luck."

DISCUSSION AND CONCLUSIONS

There is a germinal moment for the mathematisation of decision-making processes in contexts of uncertainty through the historical and epistemological analysis of Cournot's (1843) work. This analysis reveals two conceptual distinctions of interest: the difference between chance and randomness and the difference between single-case decisions and decision-making processes. These distinctions are interconnected and allow us to highlight the importance of discussing the epistemic foundation of decision-making under uncertainty. Single-case decisions refer primarily to chance, while decision-making processes require randomness as a broader conceptual basis.

From the analysis, the absence of an adequate epistemological basis to conceptually support the proposal of the Chilean school curriculum is evident, which translates into a predominance of activities associated with single-case decisions. Given that single-case decisions do not provide an adequate understanding of randomness, it is worth questioning the relevance of the treatment of these concepts in the Chilean school curriculum. Moore (1990) explains that "phenomena that have uncertain individual results, but a regular pattern of results over many repetitions, are called random. Random is not synonymous with chance: […] probability is the branch of mathematics that describes randomness" (p. 98). Similarly, Yates et al. (1998) define a random phenomenon as one in which "individual outcomes are uncertain, but there is nevertheless a regular distribution of outcomes over a large number of repetitions" (p. 314). These definitions, on the one hand, explain the statistical distinction between chance and randomness and, on the other hand, maintain epistemic correspondence with what Cournot postulated at the beginning of the systematisation of decision-making processes.

Regarding the difference between single-case decisions and decision-making processes, Baumann (2008) denies the normative force of probabilistic arguments for decisions made for a single case. In this same sense, Borovcnik (2015) explains that "there is a big difference in the success of the strategy used if one has a single decision or decides similar cases repeatedly. What is good in a single decision may be bad for the repeated decision" (p. 127). Thus, in the decision-making field, the need to think probabilistically arises when decisions are presented as processes, the results of which help to connect theoretical or a priori probability with the frequentist or a posteriori probability. In this regard, Gigerenzer and Todd (1999) indicate that in contexts of uncertainty in which we are required to make a particular decision, with little time to evaluate the possibilities and little reliable information quantitatively, intuition can be a valuable tool, especially if the calculation strategies are complex or time-consuming. This explains why, in single-case decisions, intuition compensates for effort and adapts more quickly to the context.

It is noteworthy to say that the didactic assumption behind the activity associated with the Monty Hall problem presented in the Chilean textbook is not far from what has been proposed in previous research. For example, Saenen et al. (2018) propose the use of processes that simulate the random experiment to repeat the choice many times in order to overcome students' difficulties in understanding the Monty Hall dilemma. In general, the research consulted uses the problem as input to analyse how people react to the dilemma, trying to find explanations for the prevalence of suboptimal decisions (Batanero et al., 2009; DiBattista, 2011; Elicer & Carrasco, 2017). However, from this socioepistemological analysis, we can argue that the contradictions appear because the context of the contest requires a single-case decision framed by chance. These meanings generate persistently intuitive and non-probabilistic answers.

AUTHORSHIP CONTRIBUTION STATEMENT

AVG developed the idea presented, adjusted the theoretical framework, adapted the methodology to this context, created the analysis categories, and collected and analysed the data.

DATA AVAILABILITY STATEMENT

Data supporting the results of this study will be made available by the corresponding author, AVG, upon reasonable request.

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REFERENCES

Arkes, H., Gigerenzer, G., y Hertwig, R. (2016). How bad is incoherence?. *Decision*, *31*(1), 20-39. https://doi.org/10.1037/dec0000043

Batanero, C. (2020). Probability teaching and learning. In: Lerman S. (eds.) *Encyclopedia of Mathematics Education* (pp. 682-686). Springer.

Batanero, C., Fernández, J.F., y Contreras, J.M. (2009). Un análisis semiótico del problema Monty Hall e implicaciones didácticas. *Suma: Revista sobre Enseñanza y Aprendizaje de las Matemáticas,* (63), 11-18.

Baumann, P. (2008). Single-case probabilities and the case of Monty Hall: Levy's view. *Synthese*, *162*(2), 265-273. https://doi.org/10.1007/s11229-007-9185-6

Bennett, D. (2014). Sticking to your guns: a flawed heuristic for probabilistic decision-making. In E.J. Chernoff, & B. Sriraman. (eds.), *Probabilistic Thinking* (pp. 261-281). Springer. https://doi.org/10.1007/978-94-007-7155-0_14

Borovcnik, M. (2015). Risk and decision making: The "logic" of probability. *The Mathematics Enthusiast, 12*(1), 113-139. https://doi.org/10.54870/1551-3440.1339

Borovcnik, M. (2016). Probabilistic thinking and probability literacy in the context of risk. *Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática*, *18*(3), 1491-1516.

Borovcnik, M., y Kapadia, R. (2011). Modelling in probability and statistics. En J. Maasz, & J. O'Donoghue, J. (eds.), *Real-World Problems for Secondary School Mathematics Students* (pp. 1-43). Sense Publishers. https://doi.org/10.1007/978-94-6091-543-7_1

Boyatzis, R. (1998). *Transforming qualitative information: Thematic analysis and code development.* Sage Publications.

Cantoral R. (2019) Socioepistemology in Mathematics Education. In: Lerman S. (eds.) *Encyclopedia of Mathematics Education* (pp. 790-797). Springer, Cham. https://doi.org/10.1007/978-3-319-77487-9_100041-1

Cantoral, R. (2013). *Teoría Socioepistemológica de la Matemática Educativa. Estudios sobre construcción social del conocimiento*. Editorial Gedisa SA.

Cortada de Kohan, N. (2008). Los sesgos cognitivos en la toma de decisiones. *International Journal of Psychological Research, 1*(1), 68–73. https://doi.org/10.21500/20112084.968

Cournot, A.A. (1843). *Exposition de la théorie des chances et des probabilités*. Hachette.

Denison, S., & Xu, F. (2014). The origins of probabilistic inference in human infants. *Cognition*, *130*(3), 335-347. https://doi.org/10.1016/j.cognition.2013.12.001

Drisko, J. W., & Maschi, T. (2016). *Content analysis*. Pocket Guide to Social Work Research Methods.

diBattista, D. (2011). Evaluación de un objeto de aprendizaje digital para el dilema de Monty Hall. *Enseñanza de la Psicología*, *38*(1), 53-59. https://doi.org/10.1177/0098628310390916

Elbehary, S. G. (2021). Reasoning under uncertainty within the context of probability education: A case study of preservice mathematics teachers. *Pythagoras, 42*(1), 630. https://doi.org/10.4102/pythagoras.v42i1.630

Elicer, R., y Carrasco, E. (2017). Conditional probability as a decision-making tool: A didactic sequence. In T. Dooley, & G. Gueudet (eds.), *CERME 10* (pp. 748-755). https://hal.archives-ouvertes.fr/hal-01927857

Espinoza-Ramírez, L., Vergara-Gómez, A., & Valenzuela-Zúñiga, D. (2018). Geometría en la práctica cotidiana: la medición de distancias inaccesibles en una obra del siglo XVI. *Revista latinoamericana de investigación en matemática educativa*, *21*(3), 247-274. https://doi.org/10.12802/relime.18.2131

Fischbein, E. (1975). *The intuitive sources of probability thinking in children*. Reidel

Gandhi, H. (2018). Understanding Children's Meanings of Randomness in Relation to Random Generators. In C. Batanero, & E. J. Chernoff (eds.), *Teaching and Learning Stochastics* (pp.181-200). Springer. https://doi.org/10.1007/978-3-319-72871-1_11

Garfield, J., y Ahlgren, A. (1988). Difficulties in learning basic concepts in probability and statistics: Implications for research. *Journal for research in Mathematics Education, 19*(1), 44-63. https://doi.org/10.2307/749110

Garfield, J., y Ben-Zvi. D. (2007). How students learn statistics revisited: A current review of research on teaching and learning statistics. *International statistical review, 75*(3), 372-396. https://doi.org/10.1111/j.1751-5823.2007.00029.x

Gigerenzer, G., y Todd, P.M. (1999). Fast and frugal heuristics: The adaptive toolbox. In G. Gigerenzer & P.M. Todd (eds.), *Simple heuristics that make us smart* (pp. 3-34). Oxford University Press. https://doi.org/10.1002/acp.793

Gigerenzer, G., y Gaissmaier, W. (2011). Heuristic decision making. *Annual review of psychology*, *62*, 451-482. https://doi.org/10.1146/annurev-psych-120709-145346

Helton, J. C. (1997). Uncertainty and sensitivity analysis in the presence of stochastic and subjective uncertainty. *Journal of Statistical Computation and Simulation*, *57*(1-4), 3-76. https://doi.org/10.1080/00949659708811803

Ingram, J. (2022): Randomness and probability: exploring student teachers' conceptions. *Mathematical Thinking and Learning*, 1-19. https://doi.org/10.1080/10986065.2021.2016029

Kahneman, D., y Tversky, A. (1973). On the psychology of prediction. *Psychological review*, *80*(4), 237. https://doi.org/10.1037/h0034747

Knight, F. H. (1921). *Risk, uncertainty and profit*. New York: Hart, Schaffner and Marx.

Kubricht, J. R., Holyoak, K. J., y Lu, H. (2017). Intuitive physics: Current research and controversies. *Trends in cognitive sciences, 21*(10), 749-759. https://doi.org/10.1016/j.tics.2017.06.002

Lajoie, S. P. (1998). Reflections on statistics: Learning, teaching, and assessment in grades K-12 (studies in mathematical thinking and learning series). Lawrence Erlbaum Associates.

Lopez-Claros, A., Dahl, A., & Groff, M. (2020). The Challenges of the 21st Century. In *Global Governance and the Emergence of Global*

Institutions for the 21st Century (pp. 3-29). Cambridge University Press.

Malaspina, M., & Malaspina, U. (2020). Game invention as means to stimulate probabilistic thinking. *Statistics Education Research Journal, 19*(1), 57-72. https://doi.org/10.52041/serj.v19i1.119

Martignon, L. (2014). Fostering children's probabilistic reasoning and first elements of risk evaluation. In E. J. Chernoff, & B. Sriraman (eds.), *Probabilistic Thinking* (pp. 149-160). Springer. https://doi.org/10.1007/978-94-007-7155-0_9

Ministerio de Educación de Chile. (2020). *Programa de Estudio 3° o 4° Medio. Formación Matemática Diferenciada. Probabilidades y Estadísticas Descriptiva e Inferencial*. UCE, Ministerio de Educación. https://www.curriculumnacional.cl/614/articles-140145_programa.pdf

Ministerio de Educación de Chile. (2021). *Texto del Estudiante, Matemática 3° y 4°*. Edición especial SM para el Ministerio de Educación. https://curriculumnacional.mineduc.cl/614/articles-140074_recurso_1.pdf

Moore, D.S. (1990). Uncertainty. In L. A. Steen (ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 95-146). National Academy Press. https://doi.org/10.7326/0003-4819-113-11-902_1

Mousavi, S., y Gigerenzer, G. (2017). Heuristics are tools for uncertainty. *Homo Oeconomicus, 34*(4), 361-379. https://doi.org/10.1007/s41412-017-0058-z

OECD (2019). *Assessment and Analytical Framework PISA 2018, PISA.* OECD Publishing. https://doi.org/10.1787/b25efab8-en

OECD (2010). *Learning Mathematics for Life: A Perspective from PISA.* PISA, OECD Publishing. https://doi.org/10.1787/9789264075009-en

Saenen, L., Heyvaert, M., Van Dooren, W., Schaeken, W., y Onghena, P. (2018). Why humans fail in solving the Monty Hall dilemma: A systematic review. *Psychologica Belgica, 58*(1), 128-158. https://doi.org/10.5334/pb.274

Serradó Bayés, A. (2018). Reasoning in Decision Making Under Uncertainty and Decisions of Risk in a Game of Chance. *ICME-13 Monographs*, 201–221. https://doi.org/10.1007/978-3-319-72871-1_12

Shaughnessy, J. M. (2019). Recommendations about the big ideas in statistics education: A retrospective from curriculum and research. *Cuadernos, 18*, 44-58. https://doi.org/10.22533/at.ed.2952206043

Shaughnessy, J.M., Garfield, J., y Greer, B. (1996). Data handling. In A. Bishop, M. Clements, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (eds.), *International handbook of mathematics education* (pp. 205-237). Springer. https://doi.org/10.1007/978-94-009-1465-0_8

Shi, Y. (2000). The game PIG: Making decisions based on mathematical thinking. *Teaching Mathematics and Its Applications: International Journal of the IMA, 19*(1), 30-34. https://doi.org/10.1093/teamat/19.1.30

Sriraman, B., & Chernoff, E. J. (2020). Probabilistic and statistical thinking. In S. Lerman (ed.), Encyclopedia of mathematics education, 675-681. https://doi.org/10.1007/978-3-030-15789-0_100003

van Der Bles, A. M., van Der Linden, S., Freeman, A. L., Mitchell, J., Galvao, A. B., Zaval, L., & Spiegelhalter, D. J. (2019). Communicating uncertainty about facts, numbers and science. *Royal Society open science*, *6*(5), 181870. https://doi.org/10.1098/rsos.181870

Vergara-Gómez, A. (2020). *Estudio socioepistemológico de los procesos de toma de decisiones en contextos de incertidumbre; una mirada desde la práctica cotidiana hacia la matemática escolar.* Tesis de Doctorado. Pontificia Universidad Católica de Valparaíso, Chile. https://catalogo.pucv.cl/cgi-bin/koha/opac-detail.pl?biblionumber=433540

Vergara-Gómez, A., Estrella, S., & Vidal-Szabó, P. (2020). Relaciones entre pensamiento proporcional y pensamiento probabilístico en situaciones de toma de decisiones. *Revista latinoamericana de investigación en matemática educativa, 23*(1), 7-36. https://doi.org/10.12802/relime.20.2311

Walliser, B. (2007). The Functions of Economic Models. In 1st edition. M. Touffut (ed.), *Augustin Cournot: modelling economics* (pp. 41-54). Edward Elgar Publishing Limited. https://doi.org/10.4337/9781847208866.00013

Yates, D., Moore, M. D., y McCabe, G. (1999). *The Practice of Statistics.* W.H. Freeman.