ACTA SCIENTIAE

ISSN: 2178-7727
DOI: 10.17648/acta.scientiae.7806

Development of Algebraic Thinking in Elementary School: An Analysis from Design-Based Research

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Received for publication 16 Jun. 2023. Accepted after review 22 Sep. 2023
Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Background: A concern originating from the professional experience of the researcher as a teacher in basic education, motivated by difficulties students face in developing concepts related to algebra, in addition to the analysis of external evaluations and academic products related to the subject. **Objective**: Analyse the development of algebraic thinking in 8th and 9th year students of fundamental education. **Design**: Based on theoretical-methodological assumptions of research based on design, using the phases proposed by Reeves (2000). Setting and participants: The pedagogical intervention involved 22 students from two classes of 8th and 9th grade of fundamental education, where the researcher was class regent, in a public school in Cerro Branco, Rio Grande do Sul, Brazil. **Data collection and analysis**: The data were obtained by participant observation, field diary and documentary analysis. **Results**: It is possible to verify that students have difficulties interpreting quests in explaining the reasoning used in the solutions as mathematical concepts. In all of these questions, they are mobilised, even partially, except for the capacity to think algebraically, therefore, a partial mobilisation of the capacity to establish relationships and comparisons, fundamental for the structure of algebraic thinking, can compromise the mobilisation of other capacities. **Conclusions**: The results reveal that it is necessary to offer students a teaching that mobilises these capacities through the stimuli provided by the teachers.

Keywords: algebraic thinking. teaching and learning algebra. It is fundamental. design-based research.

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Desenvolvimento do pensamento algébrico no ensino fundamental: Uma análise a partir da pesquisa baseada em design

RESUMO

Contexto: A inquietação oriunda da experiência profissional do pesquisador como professor na educação básica, motivada por dificuldades apresentadas pelos estudantes em desenvolver conceitos relacionados a álgebra, além da análise de avaliações externas e produções acadêmicas relacionadas à temática. **Objetivo**: Analisar o desenvolvimento do pensamento algébrico em estudantes de turmas do 8º ano e 9º ano do ensino fundamental. **Design**: A partir pressupostos teóricos-metodológicos da pesquisa baseada em design, foram utilizadas as fases propostas por Reeves (2000). **Ambiente e participantes**: A intervenção pedagógica envolveu 22 estudantes de duas turmas de 8º e 9º ano do ensino fundamental, nas quais o pesquisador era regente de classe, em uma escola pública no município de Cerro Branco, Rio Grande do Sul, Brasil. **Coleta e análise de dados**: Os dados foram obtidos por meio da observação participante, diário de campo e análise documental. **Resultados**: Foi possível constatar que os alunos tiveram dificuldades na interpretação das questões ao explicar o raciocínio utilizado nas resoluções bem como os conceitos matemáticos. Em todas as questões foram mobilizadas, mesmo que de forma parcial, pelo menos uma capacidade do pensar algebricamente, porém, a mobilização parcial da capacidade de estabelecer relações e comparações, fundamental para a estruturação do pensamento algébrico, pode comprometer a mobilização das demais capacidades. **Conclusões**: A partir dos resultados, pode-se concluir que é necessário oferecer aos alunos um ensino que mobilize essas capacidades por meio de estímulos proporcionados pelos professores.

Palavras-chave: pensamento algébrico. ensino e aprendizagem de álgebra. ensino fundamental. pesquisa baseada em design.

INTRODUCTION

The choice of the theme results from a concern arising from the researcher's professional experience as a basic education teacher, motivated by difficulties students presented in developing concepts related to Algebra, besides the analysis of external assessment and academic productions related to the theme. We realise that students can operate arithmetic reasonably but end up not relating such operations and their properties to algebraic operations, so they often use rules or algorithms to solve meaningless problems.

In this process, there is no contribution to the successful construction of algebra-related cognitive capabilities, reducing algebraic thinking to a symbolic language formed by abstract and meaningless symbols. Therefore, it is necessary to explore skills such as interpreting situations, solving problems, and generalising mathematical relationships to develop this type of thinking. An approach that provides this relationship between theory and practice is design-based research.

Van den Akker (1999) emphasises that the interrelationship between theory and practice present in design-based research is complex and dynamic and that the direct application of theory is often insufficient to solve some practice-related problems. He further states that "without the cooperative involvement of researchers and professionals, it is not possible to obtain clarity about the problems arising from implementation and generate effective measures to reduce them" (Van den Akker, 1999, p.9)

In this context, this work describes the outline of a study using the design-based research proposed by Reeves (2000) to analyse the development of algebraic thinking in students from 8th-grade and 9th-grade classes at a public school in Cerro Branco, RS.

THEORETICAL-METHODOLOGICAL ASSUMPTIONS

1. Design-based research proposed by Reeves (2000)

In this work, we consider the term *design-based research* (DBR) used by Wang and Hannafin (2005).

Wang and Hannafin (2005, p.6) ensure that theory is both the basis and the result. They define DBR "as a systematic, but flexible methodology, which aims to improve educational practices, through iterative analysis, design, development and implementation, based on collaboration between researchers and professionals, in a real-world scenario". Therefore, it can be indicated to contribute to the solution or reduction of systemic problems related to the teaching and learning process.

Mckenney and Reeves (2012) highlight five characteristics of DBR: theoretically oriented, interventionist, collaborative, fundamentally responsive, and interactive.

It is theoretically oriented, as theories must be the starting and ending point of the investigation. They are the design and modelling principles for the solutions required. The theoretical proposal must be the foundation for the construction of the educational design; therefore, it is the basis for the construction of the practical proposal. However, it must also be studied, improved, and understood, depending on the results obtained.

Using the chosen theoretical foundation, being in dialogue with an implementation context, the research intervenes in the field of pedagogical praxis, producing educational products such as teaching materials; pedagogical processes such as teaching recommendations, new teaching proposals; educational programs such as curricula, courses, and professional development for teachers; or educational policies such as teacher or student assessment protocols and options for the relationship between the school and the community. In fact, DBR begins with the identification of a problem that requires intervention and a practical development result, which is only possible to obtain through applied scientific investigation.

The DBR is always developed through collaboration. Developing and searching for an application that solves identified problems requires the collaboration of all involved. Therefore, it is necessary to consider everyone as part of the research team. One recommendation is that the problem be defined in a shared way with those who experience this difficulty so that they can get involved and delve into the study and understanding of the context to be researched, gaining the ability to dialogue and engage in solving the problem and in the partner community.

The methodology is fundamentally responsive, as it is shaped by the dialogue between the participants' diverse knowledge, theoretical knowledge, and various tests and validations carried out during the process. Advances, whether theoretical or practical, together with potential adjustments to the intervention developed, will prosper in dialogue and validation due to the complexity of the application context. Knowledge is developed in close dialogue in interactions in practice.

In DBR, each development is the result of a stage, a process and will necessarily be the beginning of the next moment of improvement. This approach is based on cycles of study, analysis, projection, application, and results, which are then revisited when necessary or possible. It is intended to be an iterative approach and refinement of the practical solution found. Iteration is perhaps the most striking characteristic, giving it the formative character that is identified with it.

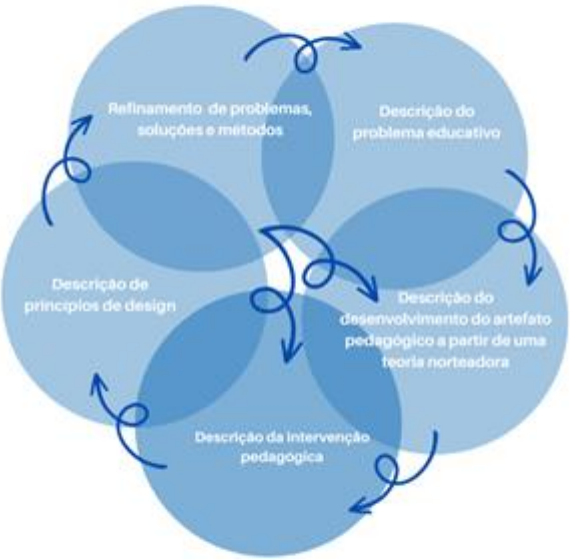
In this sense, Mckenney and Reeves (2012) state that DBR is committed to developing theoretical insights and practical solutions simultaneously, in real contexts, involving interested parties. Reeves (2000) states that the methodology is not defined by the chosen methods but by its fundamental objectives, which are the development of a product or process, accompanied by the construction of usable knowledge or theory, and professional development.

DBR is also strongly connected with cooperative work[1], where several individuals are linked to the problem in question, valuing the research contexts. Thus, researchers learn from practitioners, and vice versa, by adapting interventions that meet the same objectives differently than those initially conceived.

Reeves (2000) identifies four phases to understand design-based research better. These phases are described in Figure 1.

Figure 1

*Phases of the Design-Based Research***.** (Adapted from Reeves (2000))



[1] We understand the term *cooperative work* as work that involves action and involvement by a group of individuals aiming for the same purpose.

The initial phase of the DBR –description of the educational problem– is a decision-making process in which the involved individuals seek to align objectives, needs or opportunities with challenges and limitations found in pedagogical contexts (Edelson, 2002).

According to Herrington et al. (2007), identifying and exploring a significant educational problem is the first step. To the authors, this will be the target for the research, and the creation and validation of a potential solution to this problem will be the focus of the entire study.

Mckenney and Reeves (2012) state that the first phase is constantly reviewed in DBR, being based on the analysis and the creation of perspectives. Researchers carry out a literature review, problem definition, context, and needs analysis while visiting the site and having professional meetings. This encourages cooperative work. The main results of this phase are a better understanding of the problem, exploration of possible changes, addressing the problem, and a partial design.

At this stage, it is necessary to establish and strengthen partnerships between the researcher and collaborators, being an essentially dialogic phase, as they will be in constant communication. Guiding learning theories are also defined to substantiate the understanding of problems and guide the design, construction, and research of pedagogical interventions.

In the second phase –describing the development of the pedagogical artefact– we seek to explore and analyse, based on the educational problem and the theoretical guidelines for developing a pedagogical artefact. This design phase involves social interactions and findings from the literature review. Therefore, cooperative work is required to generate, connect, and refine the design and work.

Thus, we develop some solutions, such as a plan, text, teacher guide, technological tool, teaching resource, programmes, teaching-learning strategies, materials, systems, and products (PLOMP et al., 2009).

The third phase of the DBR –describing the pedagogical intervention– involves its application in the educational environment, seeking to understand and evaluate how the pedagogical artefact contributes to solving academic problems. Therefore, the intervention must be implemented, collecting and analysing the information arising from the application, promoting refinement for a new cycle.

The fourth phase –describing design principles– involves producing recommendations for improving the intervention, that is, a retrospective analysis for reflection and betterment in implementing the solution and the refinement of the pedagogical artefact.

This entire process is documented and evaluated, and the knowledge generated through this evaluation allows us to reflect on the process and, based on this reflection, to design and plan new actions. These characteristics highlight pedagogical problems, guiding theories and interventions as key elements of the DBR process, therefore configuring the focus of our analysis.

2. Our characterisation of algebraic thinking

We can see from the concepts presented by Lins (1992), Radford (2009, 2011), Blanton and Kaput (2005), Kieran (1992,2004), Kieran, Pang, and Schifter (2016), Fiorentini, Miorim, and Miguel (1993), and Fiorentini, Fernandes and Cristóvão (2006) that characterising algebraic thinking is somewhat complex, probably due to the extensive field and different mathematical objects in which this way of thinking is inserted: algebra. Therefore, defining a characterisation of this form of thought was necessary.

Given the above, the cooperation group[2] believes that algebraic thinking is related to constructing meanings for mathematical objects and algebraic symbolic language based on the ability to establish relationships/comparisons, model, generalise, and represent/operate with the unknown.

The ability to establish relationships/comparisons can be defined in reading, understanding, writing, and operating with ordinary symbols, besides translating information from other forms of representation and vice versa.

In the modelling skill, the student expands the initial ability to relate, seeking to identify patterns to deduce an algebraic symbolic expression for the problem. The construction of this model can initially take place in a natural language and later in an algebraic language.

[2] The group was formed by six professionals, in addition to the researcher, to develop algebraic thinking in the final years based on DBR. The name attributes the meaning of helping and collaborating mutually from the same perspective to achieve a common objective.

Concurrent with the modelling process, the generalisation capacity reveals the understanding of a presented situation, in which a synthesis of existing relationships is carried out and described in a genuinely algebraic language. In this case, the idea of the unknown or variable arises.

With the emergence of the unknown through generalisation, we seek to work as if it were known since it results from different mathematical properties, whether arithmetic, geometric or probabilistic. Therefore, it is necessary to represent and operate with the unknown.

In this process of thinking algebraically, the ability to establish relationships/comparisons is the first characteristic of algebraic thinking to be developed and manifested by the subject, followed by the others simultaneously. In Figure 2, we present a scheme demonstrating how those abilities articulate and interrelate with each other for the development of algebraic thinking proposed by the cooperative group.

Figure 2

Scheme of algebraic thinking abilities.

Combining these capabilities leads to constructing meanings for objects and symbolic language, implementing algebraic thinking.

According to Fiorentini, Fernandes, and Cristóvão (2006), algebraic thinking can be developed gradually, even before the existence of a symbolic language, highlighting the first characteristic of algebraic thinking presented, also proposed by BNCC:

> Working with algebra at the beginning of schooling helps students develop a specific type of reasoning called algebraic thinking. This idea, currently considered, differs from the idea of school algebra as a process of manipulating symbols. From this perspective, some dimensions of working with algebra are present in the teaching and learning processes from the early years, such as notions of regularity, generalisation, and equivalence. (BRASIL 2018, p. 278)

Furthermore, the authors determine aspects to be developed that characterise algebraic thinking: establishing relationships/comparisons between numerical expressions or geometric patterns; perceiving and trying to express the arithmetic structures of a problem situation; producing more than one arithmetic model for the same problem situation; creating multiple meanings for a numerical expression; interpreting equality as an equivalence between two quantities or between two numerical expressions; transforming an arithmetic expression into a simpler one; developing generalisation processes, perceiving and trying to express regularities or invariances, and developing/creating a more concise language when expressing oneself mathematically.

Most of those characteristics are evidenced throughout the National Common Curricular Base in the skills established in each object of knowledge to develop algebraic thinking, which proves to be essential for using mathematical models in the understanding, representation, and analysis of quantitative relations of quantities and also situations and mathematical structures, making use of letters and other symbols. This is the purpose of algebra in elementary school.

RESULTS AND DISCUSSION

The first pedagogical intervention was conducted in a school with two classes, an 8th-grade and a 9th-grade class, involving 22 participating students. The researcher and the collaborating teacher implemented a didactic sequence to foster algebraic thinking.

The predominantly descriptive data were collected through the following instruments: participant observation, field diary, and documentary analysis.

The cooperative group discussed the students' records seeking to identify the development of algebraic thinking based on the construction of meanings for mathematical objects and algebraic symbolic language through the abilities of establishing relationships/comparisons, modelling, generalising, and representing/operating with the unknown, articulated with the skills recommended by BNCC for this way of thinking.

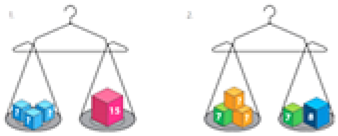
This article selected seven questions from the didactic sequence for analysis, enabling students to mobilise all the highlighted algebraic thinking capabilities. Students will be assigned the aliases E-1 through E-22 in the analysis.

The first question involves the properties of equality and algebraic language: variable, unknown, and polynomial equations of the 1st degree, fostering abilities EF06MA14, EF07MA13, and EF07MA18, as shown in Figure 3.

Figure 3

First question.

1. Em duas balanças, há certos objetos e alguns deles estão identificados e são expressos em quilogramas.



Perguntas investigativas:

a) Considerando que as balanças estão em equilíbrio. Qual o valor da massa marcado pelas balanças?

b) Qual o valor das peças com o ponto de interrogação (laranjas e azuis)?

c) Acrescentando dois blocos laranjas na primeira balança em um dos lados, a balança estará em equilíbrio?

d) Que quantidade deve ser acrescentada para que esteja em equilíbrio?

e) Na segunda balança, o que ocorre se retirarmos o bloco de 7kg de cada lado?

f) Construa uma sentença algébrica para expressar a relação entre os blocos em cada balança.

g) Quais características você identifica nessa sentença algébrica?

The students could establish the balance on the scales as equality. According to a record of the dialogue between students E-10 and E-12:

> *E-10: If the scales are balanced, they read the same weight.*
>
> *E-12: But if they read the same weight, then it's easy to find the value of each piece.*
>
> *E-10: Yes, initially, both scales read 30kg each. The smaller pieces are 5kg each, as on the other side there is a 15kg piece. On the other scale, each tiny piece weighs 4kg, as there is already 7kg on the scale, which should total 15kg as well.*
>
> *E-12: If we add two blocks to one side, the scale goes out of balance, and there is no equality, right?*
>
> *E-10: Ahem, then, you would have to put 8kg on the other side to balance it.*
>
> *E-12: The same does not happen if we remove a 7kg block from each side, as it remains balanced, but with less weight.*

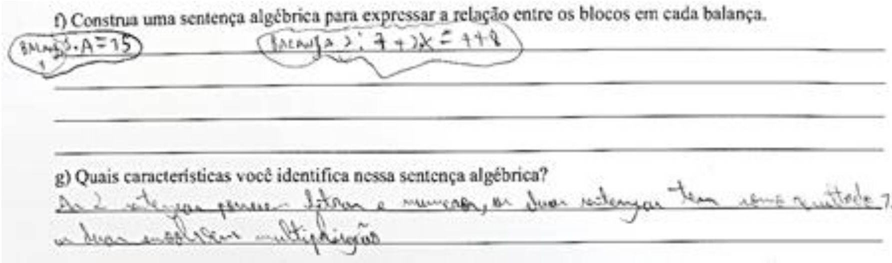
The construction of the algebraic sentence to express the relationship between the blocks in each scale was satisfactory.

> *E-10: On the first scale, the three blocks with a question mark have the same weight, so we can use a letter to represent this unknown value, I will use the letter 'a'. I can write then* $3.a = 15$.
>
> *E-12: On the second scale, then, can I write* $7 + 2.a = 7 + 8$*?*
>
> *E-10: I would put another letter because the teacher said that equal letters represent equal values, and the piece on the first scale is 5kg, and, on the second scale, it is 4kg.*
>
> *E-12: They are very similar, the expressions.*

No student used the term *equation* to identify the characteristics of the algebraic sentences constructed for each scale. In Figure 4, student E-11's record exemplifies the development of most students for items *f* and *g* of the question.

Figure 4

E-11's record for the first question.


f) Construa uma sentença algébrica para expressar a relação entre os blocos em cada balança.

g) Quais características você identifica nessa sentença algébrica?

Students demonstrated the ability to establish relationships/comparisons through understanding, writing, and operating with common symbols to translate information from other forms of representation. They expanded the initial capacity to deduce an algebraic symbolic expression for the problem, firstly from a natural language to an algebraic language, emerging the idea of the unknown. In this way, they mobilised the ability to model and operate with the unknown using the properties of equality and the construction of 1st-degree polynomial equations to represent the situation, even without using the correct term.

The second and third questions addressed the learning objects that work problems involving directly proportional quantities and inversely proportional quantities through abilities EF07MA16, EF07MA17, EF08MA12, EF08MA13, and EF09MA08. Figure 5 presents the second question.

The students managed to develop the question satisfactorily. In item *b*, they did not use the term *inversely proportional* to describe the relationship between the quantities but explained the same relationship in other words.

> *E-15: If the standard in the table is maintained, the purchase is interest-free, right?*
>
> *E-13: I think so, the amount that will be paid for the refrigerator is always 2400.*
>
> *E-15: So, he will pay 240 for each instalment if paid in 10 partial payments; and 160 for each instalment in 15 parts. It is a simple division.*

E-13: In letter b*, the relationship is that the more instalments the total is divided into, the lower the monthly amount to be paid.*

E-15: And the algebraic sentence?

E-13: To answer the letter a*, we divided 2400 by 10 and then by 15, the result was the value of the instalment. The sentence must be 2400 divided by a number, which is equal to the value of the instalment.*

E-15: As we don't know the number, we can indicate it by x *and the value of the instalment by* p*. We will only know the value of the instalment if we know how many times it will be divided.*

E-13: So it could be $v = 2400: q$.

Figure 5

Second question.

2. O professor decidiu comprar uma geladeira. Para isso, iniciou pesquisas em lojas e sites buscando pelo modelo preferido, valor acessível e formas de pagamento. O quadro a seguir apresenta algumas das opções de pagamento e parcelamento.

Quantidade de Parcelas	Valor das parcelas (R$)
2	1200,00
3	800,00
4	600,00

Perguntas investigativas:

a) Mantendo o padrão identificado na composição e número de parcelas, qual seria o valor de cada parcela se a compra fosse efetuada em 10 parcelas? E em 15 parcelas?

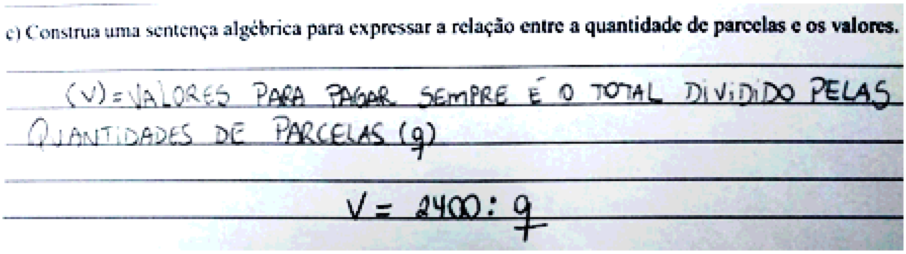
b) Qual relação existe entre as grandezas quantidade de parcelas e os valores respectivos?

c) Construa uma sentença algébrica para expressar a relação entre a quantidade de parcelas e os valores.

In Figure 6, student E-16's record for item *c* of the question.

Figure 6

E-16 record for item c *of the second question.*



c) Construa uma sentença algébrica para expressar a relação entre a quantidade de parcelas e os valores.

(V) = VALORES PARA PAGAR SEMPRE É O TOTAL DIVIDIDO PELAS QUANTIDADES DE PARCELAS (q)

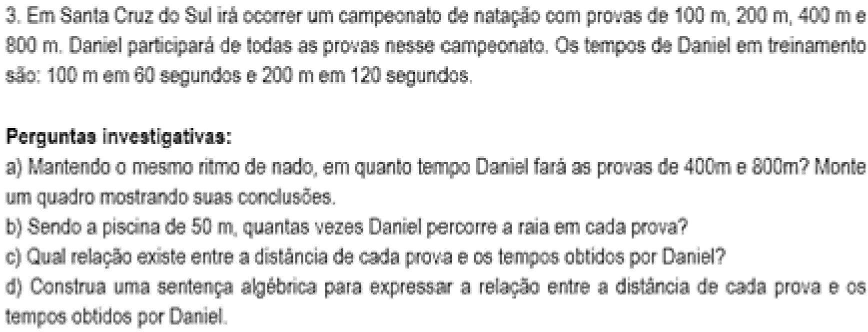
V = 2400 : q

Given the above, students demonstrated the ability to establish relationships/comparisons through understanding, writing, operating with ordinary symbols, and translating information from other forms of representation. By seeking to identify the pattern, they expanded their ability to relate, to deduce, an algebraic symbolic expression for the question, i.e., a model initiated by natural language, with a synthesis of the relationships involving the inversely proportional quantities to operate with the unknown, enabling the use of symbolic language in representing the model.

Figure 7 presents the third question. In this activity, students could correctly solve most of the items.

Figure 7

Third question.



3. Em Santa Cruz do Sul irá ocorrer um campeonato de natação com provas de 100 m, 200 m, 400 m e 800 m. Daniel participará de todas as provas nesse campeonato. Os tempos de Daniel em treinamento são: 100 m em 60 segundos e 200 m em 120 segundos.

Perguntas investigativas:

a) Mantendo o mesmo ritmo de nado, em quanto tempo Daniel fará as provas de 400m e 800m? Monte um quadro mostrando suas conclusões.

b) Sendo a piscina de 50 m, quantas vezes Daniel percorre a raia em cada prova?

c) Qual relação existe entre a distância de cada prova e os tempos obtidos por Daniel?

d) Construa uma sentença algébrica para expressar a relação entre a distância de cada prova e os tempos obtidos por Daniel.

When identifying the relationship between the distance of each race and the times, they stated that the time doubled as the distance also doubled, without mentioning the direct proportional relationship.

E-14: If he keeps the pace, he swims 100m in one minute. So, in 400m it will take 4min, in 800m, it will take 8min. This way, the distance doubles and the time also doubles.

E-12: To determine how many times he swims a 50m pool, just divide the distance of each race by 50m. This is very easy!

E-14: Really, quite easy. In the 100m race, you will swim the pool twice. In the 200m... 4 times, in the 400m...8 times and in the 800m...16 times.

In item d, the teacher's help was needed to construct the algebraic sentence. For this, the table assembled in item a was used as a resource. According to the dialogue between the researcher (Pq) and the students,

Pq: Look at the chart assembled on the letter 'a'. What happens to distance as time increases?

E-14: Both double.

Pq: They double in relation to what?

E-14: Always to the previous term.

Pq: For example, if the race is 300m, how long will it take?

E-14: It will take 3 minutes.

Pq: But, does the time double compared to the previous term, as you said?

E-14: No...

Pq: So we must analyse the relationship between distance and time, i.e., speed. What is Daniel's speed?

E-14: 100m per minute.

E-12: If he always swims at this pace, just multiply 100 by the time, right?

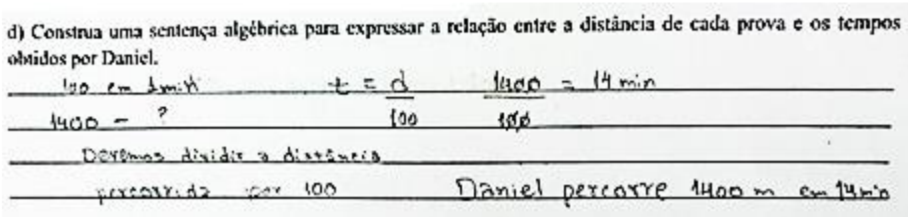
Pq: Exactly. What would the algebraic sentence look like?

E-12: $y = 100.x$, *being x the time travelled. Replacing the values closes it.*

Figure 8 presents the record of student E-3, a model similar to that built by other colleagues, but the time variable is highlighted and can be determined directly.

Figure 8

E-3's register for the third question.



d) Construa uma sentença algébrica para expressar a relação entre a distância de cada prova e os tempos obtidos por Daniel.

100 em 1min $t = \frac{d}{100}$ $\frac{1400}{100} = 14$ min

1400 — ?

Devemos dividir a distância percorrida por 100 Daniel percorre 1400 m em 14 min

The students demonstrated the ability to establish relationships/comparisons based on the skills of understanding, operating with ordinary symbols and other forms of representation, such as in the construction of the table. They expanded this capacity by identifying the pattern involved in the situation linked to directly proportional quantities to deduce the algebraic sentence. In constructing the model and the generalisation capacity revealed, the students demonstrated partial mobilisation, as they needed the teacher's help.

In the following questions, the fourth and the fifth, they sought to develop the learning objects related to the association of a 1st degree linear equation with a straight line in the Cartesian plane and the system of 1st degree polynomial equations: algebraic resolution and representation in the Cartesian plane, promoting the EF08MA08 and EF08MA07 skills.

Figure 9 presents the fourth question, in which the students had difficulty solving most items.

Figure 9

Fourth question.

4. Charles imprimiu uma receita de sabonete líquido da internet.

	Sabonete líquido: ### ml
Água	110 ml
Base perolada	65 ml
Essência	### ml

Perguntas investigativas:

a) Como você escreveria uma equação que descrevesse as quantidades dos componentes desse sabonete líquido? Utilize a linguagem algébrica.

b) Alguns números apagaram durante a impressão. Quais são esses números? Como você pensou para descobrir?

c) Charles sabe que a quantidade total do sabonete líquido é entre 200 ml e 240 ml. Quais são as quantidades de essência nesses dois casos?

d) Poderíamos representar essa relação entre a quantidade de essência e quantidade total do sabonete líquido no plano cartesiano? Coloque no plano cartesiano os pares ordenados para (quantidade de essência, quantidade sabonete). Trace um segmento de reta que ligue os dois pontos no plano cartesiano e escolha um ponto qualquer no segmento de reta. Substitua os valores do par ordenado na equação. Repita este procedimento com mais pares ordenados. O que você pode concluir?

In item *a*, students could construct the equation representing the situation.

> *E-18: What was deleted in the recipe, we can represent by a letter.*
>
> *E-16: So, it would be* $110 + 65 + x = a$.
>
> *E-18: How will we find out the numbers that were erased in the printing?*
>
> *E-16: I think it has to be greater than 175 because adding 110+65 gives 175.*

The students did not relate that the volume of the liquid soap was a function of the volume of the essence; therefore, they did not associate the equation with the possibility of infinite solutions.

E-16: To know the amount of essence in 200ml and 240ml of liquid soap?

E-18: I think we must subtract, as there is already 175ml of other things.

E-16: So, we just reduce it. It will be 25ml for the 200ml soap and 65ml for the 240ml soap.

In determining the amount of essence in the volume of liquid soap, some students made a mistake in the subtraction, pointing out that there would be 75 ml of the essence for the volume of 240 ml of liquid soap. At that moment, the teacher questioned what would actually happen if the error occurred.

Pq: Guys, what would happen to the soap if it was manufactured with this error?

E-19: It would look different than expected.

E-20: It could change colour, consistency and become thicker or more liquid.

E-19: It wouldn't bond.

In item *d*, assistance from the teacher was required for the resolution process. As this was the first issue involving the use of the Cartesian plane, a review was conducted.

Pq: Could we represent this relationship between the amount of essence and the total amount of soap in the Cartesian plane?

E-16: I think so, one being ox and the other being y. But I don't know what is what.

Pq: Let's associate the x-axis with the amount of essence and the y-axis with the amount of soap. What now?

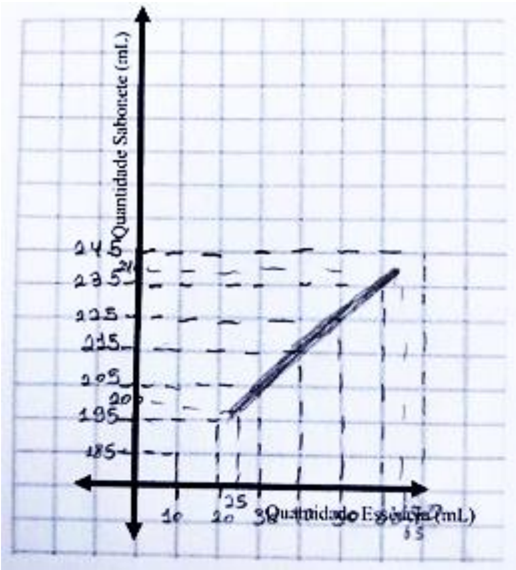
E-16: We can mark the values from the previous question, 65ml for 240 and 25ml for 200ml.

Pq: Perfect. Build it. Then resume working on the question.

The rest of the item asked them to choose any point on the straight line, determine the ordered pair in the equation and see if it belonged to the solution set and if the equation had infinite solutions. In this context, the students only built the graphical representation without checking the ordered pair in the equation, as seen in student E-14's record in Figure 10.

Figure 10

E-14's register for question four.



Despite the issue providing the mobilisation of all capabilities of thinking algebraically, students eventually presented only the ability to partially establish relationships/comparisons, as understanding, operating with usual symbols, and translating information from other forms of representation and vice versa compromised the possibility of achieving the additional capabilities, being restricted to the first characteristic of algebraic thinking.

In the fifth question, illustrated in Figure 11, no student managed to develop it correctly.

Figure 11

Fifth question.

5. Marcia vai estender as roupas que utiliza aos fins de semana para jogar futebol. Dentre essas roupas existem pares de meias e calções. Ela utiliza um único pregador para cada par de meia e dois pregadores para cada calção. Ela utilizou 20 pregadores pendurando 25 peças no varal.

Perguntas investigativas:

a) Como poderíamos representar essa situação utilizando expressões algébricas?
b) Essas expressões possuem relações? Podem ser caracterizadas como equações?
c) Quantas incógnitas possuem as expressões algébricas?
d) Trace a representação geométrica para essas expressões algébricas no plano cartesiano.
e) Existe algum ponto em comum entre as representações geométricas? Qual ou quais?
f) Qual a diferença entre essa situação e a anterior?
g) Quantos pares de meia e calções Marcia estendeu no varal? Compare com a representação algébrica.

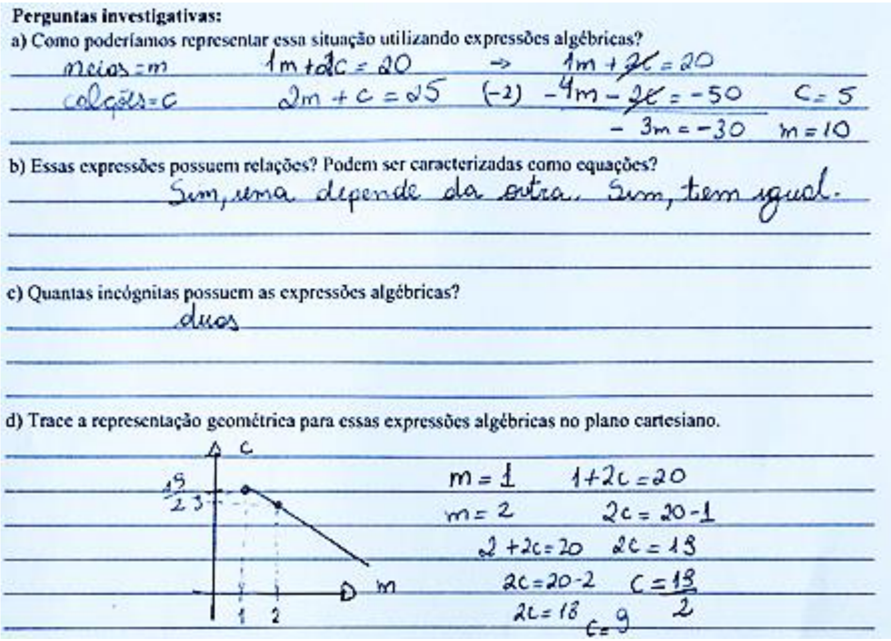
One fact that we needed to clarify was that a pair of socks is equivalent to two pieces of clothing.

> *E-19: Teacher, I used* m *for socks and* c *for shorts to organise the algebraic expressions, so that* $1m + 2c = 20$ *and 1m+1c=25. But when I try to solve it, the result is a negative number and it can't be so!*
>
> *Pq: Why did you do 1m+2c =20?*
>
> *E-19: There are 20 pegs in total. One is used to fasten the socks, and two are used to fasten the shorts.*
>
> *Pq: Perfect. And why did you do* $1m + 1c = 25$*?*
>
> *E-19: I did because there are only 25 pieces on the clothesline, and there are socks and shorts.*
>
> *Pq: That's the problem, how many pieces are there in a pair of socks?*
>
> *E-19: Two. So it would be* $2m + 1c = 25$*?*
>
> *Pq: That's it.*
>
> *E-19: This information could be clearer in the question, right?*

Most of them could represent the algebraic expressions for the situation, identifying them as equations with two unknowns that had a dependency relationship. Still, they could not represent the algebraic expressions in the Cartesian plane, as exemplified by student E-16's register in Figure 12.

Figure 12

E-16's register for the fifth question.



The wrong representation of algebraic expressions in the Cartesian plane meant that students could not compare the representations between the lines, seeking to identify a common point. Only two students could determine the number of pieces involved in the situation, with E-16 being the only one to use the addition method in a system of equations to determine the number of pairs of socks and shorts.

The issue envisaged the mobilisation of all algebraic thinking skills, but the students presented, partially, only the ability to establish relationships/comparisons. The skills of understanding, operating with usual symbols, and translating information from other forms of representation proved ineffective, as they could not represent the situation presented using algebraic expressions geometrically. In this way, they compromised the possibility of achieving other skills, hampering the first characteristic of algebraic thinking. A similar fact occurred in the resolution of the previous question.

In the last questions, the learning objects relate the functions: numerical, algebraic, and graphic representations and the ratio between different quantities, fostering the EF09MA06 and EF09MA07 skills.

Question six, shown in Figure 13, was applied only in the 9th grade of elementary school since the skills related to this content are only planned for that school level.

Figure 13

Sixth question.

6. A variação da temperatura num dia de inverno é descrita na tabela abaixo.

Medidas	Temperatura
1ª	**18ºC**
2ª	**15ºC**
3ª	**12ºC**
4ª	**9ºC**
5ª	**6ºC**

Perguntas investigativas

a) Se a temperatura continuar a variar desse modo, qual será a 6ª medida? E a 9ª?
b) Construa a sequência dos 10 primeiros termos dessa sequência.
c) Que padrão de regularidade você observou nos valores tabulados?
d) Construa um gráfico relacionando as medidas com a temperatura.
e) O gráfico que você obteve representa uma função?
f) Qual é o domínio? E a imagem dessa função?
g) Qual é a variável dependente e a independente?
h) Escreva uma expressão algébrica que represente a situação.

The students identified the pattern involved in the situation, represented the situation graphically, established a relationship with the concept of function, identified the domain, image, independent and dependent variable, and generalised the presented situation.

E-19: For the 6th measurement, the temperature will be 3ºC, and for the 9th, it will be -6ºC.

Pq: How do you conclude this?

E-19: Piece of cake, teacher... with each measurement, the temperature reduces by 3.

Pq: How could we represent the relationship between measurements and temperature on a graph?

E-19: It will be a descending straight line.

Pq: Why?

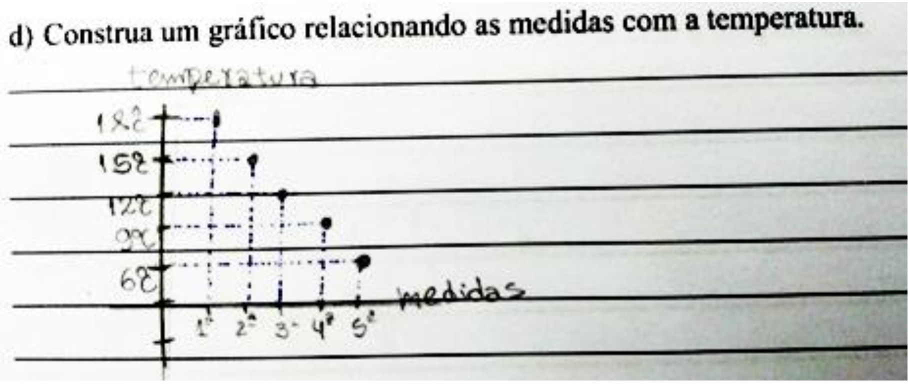
E-19: The temperature decreases as measurements are taken.

Pq: Let's check by building the graph.

Figure 14 depicts student E-20's register of the graph relating the measurements to temperature.

Figure 14

E-20's register of item 'd' of the sixth question.



We can see in the graph that the student did not connect the dots, which occurred with most students. This fact made it possible to identify that they understood the concept of function, as can also be evidenced in the following dialogue.

Pq: Why didn't you join the dots on the graph?

E-19: Because you can't. If I join them, it will look like the measure is 1.7 and it cannot be so. Each measurement is an integer, it cannot be negative either. Or you measure -9m?

Pq: And is this situation a function?

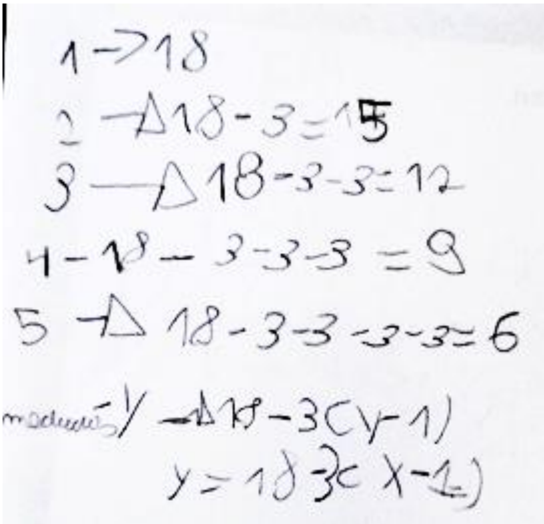
E-19: Yes, because each measurement has a unique temperature.

Pq: Which would be the independent and dependent variable?

E-19: I think the temperature is dependent because if you don't measure it, you don't know the temperature.

Figure 15

E-21's register of item h *of the sixth question.*



For writing an algebraic expression representing the situation, the students used the table and the standard for reducing the temperature in each measurement as a resource. This stage was developed jointly by the students in the class. In Figure 15, the construction process is exemplified through E-21’s register.

Students could mobilise the ability to establish relationships/comparisons through understanding, writing, operating with ordinary symbols, and translating information from other forms of representation and vice versa. They expanded the initial ability to relate, identifying the regularity pattern to deduce the algebraic symbolic expression for the problem.

This process initially occurred in a natural language and later in an algebraic language. The students revealed their generalisation ability because they built the model in a genuinely algebraic language, making it possible to represent and operate with the unknown. Given this, they presented all the capabilities of thinking algebraically. Figure 16 displays the seventh question.

Figure 16

Seventh question.

7. Uma corda de um circo estava enfeitada com esferas e cilindros como mostra abaixo:

Perguntas investigativas

a) Qual é grupo que se repete?

b) Se forem utilizadas 60 esferas, quantos cilindros existirão? E quantos grupos?

c) Preencha a tabela abaixo. Explique o raciocínio usado.

Nº de grupos	Nº de esferas	Nº de cilindros	Nº total de objetos
1	2	2	4
		4	
	6		
4			

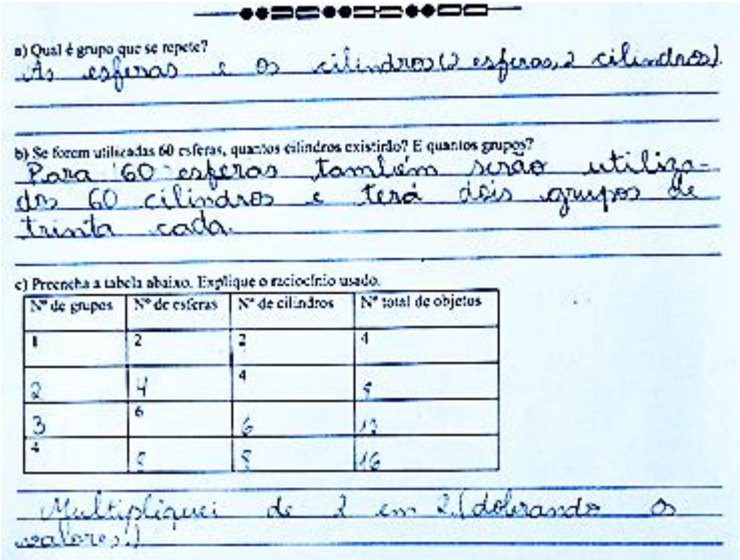
d) Estabeleça uma lei que possa ser encontrado o número de esferas e de cilindros.

e) Faça o mesmo com o número total de objetos.

In this question, students had an easy time with the first items of the activity, as in Figure 17, which illustrates E-18’s register.

Figure 17

E-18's register of the first items of the seventh question.


a) Qual é grupo que se repete?

As esferas e os cilindros (2 esferas, 2 cilindros).

b) Se forem utilizadas 60 esferas, quantos cilindros existirão? E quantos grupos?

Para 60 esferas também serão utiliza-dos 60 cilindros e terá dois grupos de trinta cada.

c) Preencha a tabela abaixo. Explique o raciocínio usado.

Nº de grupos	Nº de esferas	Nº de cilindros	Nº total de objetos
1	2	2	4
2	4	4	8
3	6	6	12
4	8	8	16

Multipliquei de 2 em 2 (dobrando os valores!)

Establishing a law to determine the number of spheres, cylinders, and the total number of objects in relation to the repeating group in the string was challenging. Only two students managed to solve items *d* and *e*

> *E-9: I can't relate it, because there are two different things together. I know it will always double, but I can't represent it.*
>
> *E-10: Should you relate the number of spheres and cylinders to the group?*
>
> *Pq: Exactly. And then the total number of objects.*
>
> *E-9: But then we can represent by* $y = 2x$*, as it always doubles.*
>
> *E-10: This will be for both spheres and cylinders.*
>
> *E-9: And the one in the group will be* $y = 2x + 2x$*?*
>
> *E-10: I think the letters have to be different, as they are spheres, cylinders and the total.*

E-9: Well thought, then it will be $g = 2e + 2c$.

The question had the potential to develop all the abilities to think algebraically, but the students were able to mobilise the ability to establish relationships/comparisons through the skills of understanding, writing, and operating with usual symbols, as well as translating information from other forms of representation and vice versa. However, they did not expand the initial ability to relate; even though they sought to identify a pattern, they could not deduce an algebraic expression for the situation in most cases.

CONCLUSIONS

Algebraic thinking can be defined as an approach involving situations that emphasise general aspects related to tools that are not necessarily symbolic language but that can ultimately be used as cognitive support.

Regarding the intervention, the students had difficulties interpreting the questions, explaining their reasoning, and dealing with the mathematical concepts involving area, perimeter, factorisation, operations involving decimal numbers, Cartesian plane, and systems of equations the teacher had reviewed.

Considering that seven questions predicted the mobilisation of all algebraic thinking skills, the students manifested these skills in their entirety in three of these questions. The biggest problems identified were the connection between the algebraic and geometric representation through the Cartesian plane and structuring an algebraic sentence to determine the situation involved, i.e., the algebraic language is not used as cognitive support for solving problems.

Fiorentini, Miorim, and Miguel (1993, p. 88) highlight that "algebraic thinking is a special type of thinking that can manifest itself not only in different mathematics paths but also in other areas of knowledge." Therefore, it is necessary to provide students with various contexts and situations in the proposed activities.

Even though the didactic sequence provided different contexts, at least the ability to think algebraically was mobilised in all questions, even if partially. However, in this process, establishing relationships/comparisons is the first skill to be manifested to structure algebraic thinking. As a result, when the student presents difficulty interpreting and understanding various mathematical concepts, they will end up having setbacks in developing this way of thinking.

Another relevant aspect of algebraic thinking presented by Lins (1994) is that this type of thinking does not develop spontaneously; it only develops through an intentional act, i.e., stimuli are necessary in the teaching process.

The analysis of the students' registers and data from SAEB in Cerro Branco, Rio Grande do Sul, revealed that both agree about the development of algebraic thinking skills. Therefore, teachers must stimulate students so that they can mobilise these skills.

Kieran, Pang, and Schifter (2016) emphasise that curricula can have a small impact on what happens in the classroom since the development of algebraic thinking requires professional development.

Based on the algebraic thinking skills identified during the pedagogical intervention, we noted that the didactic sequence allowed students to demonstrate skills related to the development of algebraic thinking and that teachers could develop planning to resolve the difficulties identified.

ACKNOWLEDGMENTS

This work was carried out with the support of the Coordination for the Improvement of Higher Education Personnel - Brazil (CAPES) - Financing Code 001 for the author Charles Bruno da Silva Melo's doctorate studies.

AUTHORSHIP CONTRIBUTION STATEMENT

This article was prepared and organised by both authors. CBSM developed the theoretical-methodological assumptions and applied and collected the data. CBSM and EB analysed the data and worked on the general construction of the article.

DATA AVAILABILITY STATEMENT

The authors agree that the data supporting the results of this study are available upon reasonable request at the authors' discretion.

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