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The use of a didactic model as a source of reflection: The identification of new categories of problem situations of the composition type involving fractional numbers

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ABSTRACT

Background: The attempt and the impasses in employing an existing didactic model are here an invitation to various epistemological reflections on numbers in problem situations involving arithmetic operations. **Objectives**: From these reflections, we propose a didactic model for composition problem situations covering fractional and non-fractional numbers. **Design**: This is an expansion of one of the classes of the additive conceptual field proposed by Gérard Vergnaud within the scope of the theory of conceptual fields. Considering fractional numbers, the didactic model presented here consists of eight categories, six more than those the author describes. Initially, the model was constructed based on an essentially epistemological analysis. It was compared to empirical data from a protocol carried out with Brazilian elementary school students. **Setting and participants**: 987 Brazilian students from the 5th to the 9th grades of elementary school. **Data collection and analysis**: The students solved nine problem situations individually. Data analysis took place through students' written productions. **Results**: The creation of a teaching model that includes eight problem situations involving the composition of measurements, six of which involve fractional numbers. Problem situations involving only fractions have a significantly higher hit rate than those that involve the fraction of a measurement. **Conclusions**: Three difficulty levels were identified among the composition problem situations for fractional and non-fractional numbers. The didactic model and the levels of complexity presented here can serve as an instrument to ensure the diversity of problem situations proposed in the classroom and support curriculum choices throughout schooling.

Keywords: Additive conceptual field; Multiplicative conceptual field; Didactics of mathematics; Composition situations.

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O uso de um modelo didático como fonte de reflexão: A identificação de novas categorias de situações-problema do tipo composição envolvendo números fracionários

RESUMO

Contexto: A tentativa e os impasses em empregar um modelo didático existente são aqui um convite a diversas reflexões de natureza epistemológica sobre os números em situações-problema envolvendo operações aritméticas. **Objetivos**: Dessas reflexões, propõe-se um modelo didático para situações-problema de composição abrangendo números fracionários e não fracionários. **Design**: Trata-se de uma expansão de uma das classes do Campo Conceitual Aditivo, propostas por Gérard Vergnaud, no âmbito da Teoria dos Campos Conceituais. Considerando os números fracionários, o modelo didático aqui apresentado é composto de oito categorias, seis a mais do que aquelas descritas pelo autor. Em um primeiro momento, o modelo foi construído a partir de uma análise essencialmente epistemológica. Ele foi confronta do a dados empíricos oriundos de um protocolo realizado com estudantes brasileiros do Ensino Fundamental. **Ambiente e participantes**: 987 estudantes brasileiros, do 5º ao 9º ano do Ensino Fundamental. **Coleta e análise de dados**: Os estudantes resolveram nove situações-problema, individualmente. A análise dos dados ocorreu por meio das produções escritas dos estudantes. **Resultados**: A constituição de um modelo didático que contempla oito situações-problema de composição de medidas, sendo seis delas envolvendo números fracionários. As situações-problema envolvendo apenas frações possuem uma maior taxa de acerto significativamente superior que àqueles que envolvem a fração de uma medida. **Conclusões**: Três níveis de dificuldades foram identificados dentre as situações-problema de composição para números fracionários e não fracionários. O modelo didático e os níveis de complexidade aqui apresentados podem servir como instrumento para garantir a diversidade das situações-problema propostas em sala de aula e amparar escolhas curriculares ao longo da escolarização.

Palavras-chave: Campo conceitual aditivo; Campo conceitual multiplicativo; Didática da matemática; Situações de composição.

INTRODUCTION

The development of models based on empirical observations is a standard scientific exercise in the practice of researchers in different areas. These models aim to describe and produce information about what is being studied, among other objectives, shedding light on some aspects and inevitably casting shadows on others. When a given model proves insufficient when applied to a specific set of data, i.e., when the shadows it produces camouflage aspects considered essential to the researcher, the evolution of this model becomes a research objective. This article illustrates this scientific process in the field of didactics of mathematics, specifically in relation to problem situations involving fractional numbers.

In didactics of mathematics, we use *didactic models* to describe situations and phenomena typical of the diffusion (and non-diffusion) of mathematical knowings in society. These models allow us to describe elements of mathematical activity practised by a given institution or group of people. To this end, epistemological, cognitive, and institutional approaches are sometimes combined.

Within the scope of the theory of conceptual fields (TCF), two didactic models were (and are) used in different studies interested in the teaching and learning of the four arithmetic operations: the additive conceptual field and the multiplicative conceptual field (Vergnaud, 1990). According to Magina et al. (2001), convincing evidence of the influence of these models can be found in the elaboration of the National Curricular Parameters (Parâmetros Curriculares Nacionais - PCN) (1997). Kaspary (2020) also shows that changes in the most recent Brazilian textbooks compared to those from the 1990s reveal a search for conformity with the classes of these models. Therefore, they went beyond the research scope and became instruments for choosing and organising the curriculum.

When particularly interested in problem situations involving fractional numbers, we first seek to transcend the abovementioned two models. In this exercise, we were led to recognise the existence of *new* categories. The available models did not allow us to *focus* on some characteristics of the situations that we believe deserve to be taken into account. Nunes et al. (2007) had already observed this need in some way:

> The impact of situations on the development of mathematical concepts in Vergnaud's theory (1997) has inspired much research. It is now known that the size of numbers alters not only the rate of correct answers but also the strategies for solving the problem (Carpenter & Moser, 1982; Vergnaud, 1982; 1983b). Surprisingly, no similar systematic comparison work has been developed in the field of rational numbers.

> (Nunes et al., p. 255, 2007)[1].

In this text, we present the movement of identification of new categories focusing on the class of problem situations of the additive field measure composition type. An epistemological analysis of the structures of problem situations and a research protocol with 987 Brazilian students are the pillars of the methodological approach used.

When we are interested in teaching and learning fractions, we are led by the literature to consider the *senses of fractions* (such as part-whole, quotient, operating coefficient, ratio, abstract number*)*. This model has been widely revisited in different works: Kieren (1976), Behr et al. (1983), Vergnaud (1983), Allard (2015), and Nunes et al. (2007), among others. However, we share Thompson and Saldanha's (2003) point of view regarding the discomfort in considering such a model in place of an epistemological analysis of rational numbers and more specifically, of fractional numbers:

> Kieren (1988; 1993a; 1993b) and the rational number project (Behr et al., 1992, 1993; Lesh et al., 1987) have given the most extensive analyses of rational number meanings. Their approach was to break the concept of rational number into subconstructs—part-whole, quotient, ratio number, operator, and measure—and then describe rational number as an integration of those subconstructs. Our feeling is that their attempt to map systems of complementary meanings into the formal mathematical system of rational numbers will necessarily be unsatisfactory in regard to designing instruction for an integrative understanding of fractions. Each subconstruct is portrayed as a body of meanings, or interpretations, of the "big idea" of rational numbers. Mathematical motivations for developing the rational numbers as a mathematical system, however, did not emerge from meanings or subconstructs. Rather, it emerged from the larger endeavor of arithmetizing the calculus. So, to focus on subconstructs or meanings of the mathematical system of

[1] L'impact des situations sur le développement des concepts mathématiques mis en avant dans la théorie de Vergnaud (1997) a inspiré de nombreuses recherches. On sait aujourd'hui que la taille des nombres fait varier non seulement le taux de réponses correctes mais également les stratégies pour résoudre le problème (Carpenter & Moser, 1982; Vergnaud, 1982; 1983b). Étonnamment, aucun travail similaire de comparaison systématique n'a été développé dans le domaine des nombres rationnels. (Nunes et al., p. 255, 2007)

> rational numbers ultimately runs the risk of asking students to develop meanings for a big idea that they do not have. Our approach will be to place fraction reasoning squarely within multiplicative reasoning as a core set of conceptual operations. (p. 14)

As the authors explain, the mathematical motivations for creating rational numbers as a numerical system are not based on the didactic model of the meanings of fractions. Intending to place the epistemology of numbers at the centre of our reflection, we assume the definition given by Bezout, dating from the 18th century, and recalled by Neyret (1995), according to which a number expresses the number of units or parts of units that make up a quantity. The interest of this definition for us is that it places rational numbers on a continuum with natural numbers, allowing us to also establish this continuity between the situations described by Vergnaud (1990) and those identified by us. These and other reflections will be detailed later in this text.

ELEMENTS OF THE THEORY OF CONCEPTUAL FIELDS

For the TCF, problem situations can be organised into *classes* $\hat{\mathcal{C}}$ *and categories* $\mathcal{C}$ ($\mathcal{C} \subset \hat{\mathcal{C}}$), for which an *invariant organisation of behaviour* of the subject is expected when confronted with situations of the same class and, more specifically, of the same category. An invariant organisation is built over a relatively long time and, according to Vergnaud (2009), results from the subject's *adaptations* when experiencing situations he considers similar.

Recognising classes and categories of problem situations can, in itself, be an object of research. When identified, they become a working hypothesis to predict and interpret the subjects' behaviour. From being an object of study, they acquire the status of an instrument for the analysis and interpretation of phenomena.

Class identification is generally done through a discursive and contextual analysis of problem situations that may lead the subject to behave in a certain way. For example, in the sentences "the urn has two blue balls and three red balls" and "in the urn that contained two blue balls, three red balls were placed," the images we construct of these situations can impact the way the subject models each. If we are interested in the number of balls in the urn, the strategy used can be equally impacted in each case.

The epistemological analysis of problem situations plays a fundamental role in identifying the categories of a given class. In this sense, when two situations of the same class can be modelled by two different mathematical models, we consider them as belonging to two distinct categories. We then assume as a working hypothesis that the subject's behaviour is, once again, impacted when he is confronted with situations originating from two different categories. This working hypothesis is based on the argument that the epistemological nature of a situation affects the effective strategies that allow it to be resolved.

Specific classes of situations, when combined, can determine what we call a *conceptual field*, in which we can observe a strong connection between different concepts, properties, theorems, and representations. Among the various classes of problem situations addressed by Vergnaud (1990), we direct our study to those linked to the four arithmetic operations, for which the author establishes the constitution of two conceptual fields: additive and multiplicative.

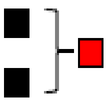
The additive conceptual field encompasses six classes of problem situations. In this text, we will focus on the first: measurement composition. In this class, we find problem situations in which two or more measurements are combined, resulting in a third. The notion of *parts of the whole* is intrinsic to these situations, and, therefore, the contexts that illustrate them allow us to attribute a static (non-temporal) dimension to the measurements of the magnitudes in play.

When problem situations of the *composition of a measurement* involve discrete quantities, they can be modelled using the following set-type reasoning: if the two sets A and B, of the same nature, have no elements in common (i.e., $A \cap B = \emptyset$), we have that $n(A) + n(B) = n(A \cup B)$, where $n(A)$ and $n(B)$ represent the number of elements of the sets A and B, respectively.

Vergnaud (1990) identifies two categories relating to this class. The first consists of situations in which we know the *parts of the whole* and seek to calculate the *whole*.

Figure 1

Category 1, with M_1, M_2, and M_3 non-fractional positive real numbers (authors of this research)



$M_1 + M_2 = M_3$, where M_3 is unknown

- **Example 1:** In Professor Lucas' class, 22 students do some sporting activity and 7 students do not do any sporting activity. How many students are there in teacher Lucas' class?

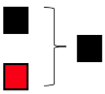
  According to Magina et al. (2001),

> These are problems that most very young children (children aged 6 or even 5) no longer have difficulty solving because the procedure required –putting the parts together to find the whole– is precisely the first addition situation a child understands, i.e., the first representation of addition that he or she forms, and its resolution, in general, is associated with the counting process (p. 34).

The second category Vergnaud (1990) identified consists of finding *a part of the whole*, knowing *one of the parts* and the *whole*.

Figure 2

Category 2, with M_1, M_2 and M_3 non-fractional positive real numbers (authors of this research)



$M_1 + M_2 = M_3$, where M_2 is unknown

- **Example 2:** There are 28 students in teacher Maria's class. We know that 21 students do some sporting activity and the rest do not do any sporting activity. How many students in teacher Maria's class do not practice sports?

When transposing these categories to describe situations of composition of measurements involving fractional numbers, we confronted some reflections that we will seek to present below. This exercise led us to recognise six other categories, which are our main object of study.

SITUATIONS OF COMPOSITION OF MEASUREMENTS INVOLVING FRACTIONAL NUMBERS

> The measurement of quantities and the quantification of relationships between quantities have constituted for centuries a foundation and a driving force behind the advancement of mathematics and the construction of numbers. Mathematics today is based directly on sets and numbers without reference to quantities (Perrin-Glorian, 2002, p. 299)

Neyret (1995) reminds us that Bezout proposed the distinction between concrete and abstract numbers. The first is associated with a specific unit of measurement, while the second lacks this characteristic.

Problem situations in the field of arithmetic, when calling upon an extra-mathematical context, are essentially based on concrete numbers, establishing the inescapable relationship between the *measurements* and their *measured quantities*: the numbers in the statement of such a situation are *measurements* of a specific *quantity*, even though the latter is not made explicit in this way, nor is its *unit of measurement*.

> Even if there is "evaporation" at a given moment of the name of the unit, to use an expression by Guy Brousseau, which allows us to speak of an abstract number, the implicit reference will always be the comparison between a magnitude and a unit. (Neyret, 1995, p. 67)[2]

When we announce the measurement of a quantity, we do so by considering a specific quantity of that greatness as a *unit of measurement*. Such *unit of measurement* is, therefore, a *unitary greatness* considered as a *reference* to measure all other quantities of the same nature as the one used to measure.

Measuring is, therefore, an act of comparing. Chambris (2007) explains the relationship between these concepts as follows:

> Let us consider objects all endowed with a given quality. Two objects that are "equal" from the point of view of this quality have the same magnitude. The equivalence class of objects of

[2] Même s'il y a "évaporation" à un moment donné du nom de l'unité, pour reprendre une expression de Guy Brousseau, ce qui permet de parler de nombre abstrait, la référence implicite sera toujours la comparaison d'une grandeur à une unité (Neyret, 1995, p. 67).

> the same magnitude is then a magnitude. We call the measurement of an object or magnitude a number (characterised differently according to the theories). For a given quality, an object has several measurements while having only one magnitude. (p. 11)[3]

In both examples discussed above, the unit of measurement in question is ‘one student’. With this unit of measurement, we assign numbers that allow us to indicate the measurement of the quantities’ teacher Maria’s class,’ ‘teacher Lucas’s class,’ ‘students who practice a sporting activity,’ ‘students who do not practice a sporting activity,’ etc. With this same unit of measurement, however, we cannot measure the number of “members of teacher Maria’s family,” for example. This unit of measurement, therefore, allows measuring quantities whose quality of interest is the *quantity of students.* For this reason, teachers are not counted as part of the *whole* in examples 1 and 2.

> To conceive of a measured quantity is to imagine the measured attribute as segmented (Minskaya, 1975; Steffe, 1991b) or in terms of a coordination of segmented quantities (Piaget, 1970; Schwartz, 1988; P. W. Thompson, 1994). The idea of ratio is at the heart of measurement. To conceive of an object as measured means to conceive of some attribute of it as segmented, and that segmentation is in comparison to some standard amount of that attribute. (Thompson & Saldanha, 2003, p. 15)

Using example 2 presented previously[4], let us state that “$\frac{3}{4}$ students from teacher Maria’s class do sports activities.” How do we consider, conceptually, the number $\frac{3}{4}$ in this sentence? Would it also be a measurement

[3] Considérons des objets tous dotés d’une qualité donnée. Deux objets « égaux » du point de vue de cette qualité ont même grandeur. La classe d’équivalence des objets de même grandeur est alors une grandeur. Nous appelons mesure d’un objet ou d’une grandeur, un nombre (caractérisé diversement selon les théories). Pour une qualité donnée, un objet a plusieurs mesures alors qu’il n’a qu’une seule grandeur (p. 11).

4 Example 2: There are 28 students in teacher Maria's class. We know that 21 students do some sporting activity and the rest do not do any sporting activity. How many students in teacher Maria's class do not practice sports?

of the quantity of “teacher Maria’s class”? If so, in relation to which unit of measurement?

Let us note that announcing another numerical value to the practitioners of the sport in teacher Maria’s class does not make this group smaller or larger than it is (we know that $\frac{3}{4}$ of 28 students correspond to 21 students).

> A conceptual breakthrough underlying students’ understanding of unit substitutions is their realisation that the magnitude of a quantity (its “amount”) as determined in relation to a unit does not change even with a substitution of unit. Wildi (1991) emphasised this point by making two distinctions. The first was between a quantity’s measure and its magnitude. A quantity’s magnitude (it’s “amount of stuff” or its “intensity of stuff”) is independent of the unit in which you measure it. […] A change of unit does not change the quantity’s magnitude—making the unit 1/4 as large makes the measure 4 times as large, leaving the quantity’s magnitude unchanged. (Thompson & Saldanha, 2003, pp. 16-17).

That said, we consider that $\frac{3}{4}$ is a measurement “of teacher Maria’s class,” referring to students who participate in sports activities. It is given from the unit of measurement, “teacher Maria’s class.” Note that this unit of measurement cannot be used *directly* to measure a *proportion* of any group of students –which is possible with the unit of measurement being “one student.” In the same sense, knowing that the measure of “teacher Lucas’ class” is not “*1* class of teacher Maria,” the numerical value $\frac{3}{4}$ in examples 1 and 2 does not correspond to the same quantity. For this reason, in the world of quantities and measurements, we have $\frac{3}{4}$ *bigger* or *smaller* than others, in the same way as we have 2 *bigger* or *smaller* than others (such as 2 eggs or 2 dozen eggs), depending on the unit of measurement used.

The question that arises is, for example, whether the problem situation in example 3, presented below, should be in the same category as the problem situation in example 1[5]:

5 Example 1: In Professor Lucas' class, 22 students do some sporting activity and 7 students do not do any sporting activity. How many students are there in teacher Lucas' class?

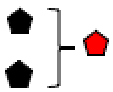
- **Example 3:** In teacher João's class, $\frac{3}{7}$ of the students only play volleyball, $\frac{2}{7}$ of the students only do judo, and the others do not practice sports activities. What fraction of students in teacher João's class practice sports, that is, playing volleyball or judo?

Based on the arguments presented above, we do not doubt that these are two problems of composition of measurements. Let us note, however, that in the case of example 1, the chosen unit of measurement does not derive from an exclusive quality of the measured object. The *whole* wanted, *T'* (union of the parts), in this case, is given by $T' = \varphi u$ ($\varphi \in \mathbb{Q}$), where *u* It is a unit of measurement that can be easily used to measure other quantities of the same nature. In the case of example 3, two species of *whole* coexist: the *whole* resulting from the union of two parts *T'* (the unknown measurement in the problem) and the *whole* used as a unit of measurement *T* (teacher João's class), where $T' = \varphi T$ ($\varphi \in \mathbb{Q}$), with u = T. That is, in the second case, the measured object is itself its unit of measurement.

In this way, we are led to consider that the change in the unit of measurement in the two cases studied not only produces a numerical change but strongly impacts knowledge and schemes[6] at stake in each situation. Thus, we decided to consider two new categories that are in some way symmetrical to those that had already been identified by Vergnaud (1990):

Figure 3

Category 3, $\{a, b, c, d, e, f\} \subset \mathbb{N}$; $b \neq 0$, $d \neq 0$, and $f \neq 0$ (authors of this research)

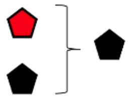


$\frac{a}{b} + \frac{c}{d} = \frac{e}{f}$, where $\frac{e}{f}$ is unknown

[6] Invariant organization of conduct by the subject (Vergnaud, 1990).

Figure 4

Category 4, $\{a, b, c, d, e, f\} \subset \mathbb{N}$; $b \neq 0$, $d \neq 0$, and $f \neq 0$ (authors of this research)



$\frac{\boldsymbol{a}}{\boldsymbol{b}} + \frac{\boldsymbol{c}}{\boldsymbol{d}} = \frac{\boldsymbol{e}}{\boldsymbol{f}}$, where $\frac{\boldsymbol{c}}{\boldsymbol{d}}$ is unknown

Example 3 is, therefore, modelled by the sagittal diagram and mathematical equation shown in Figure 3. Similarly, example 4, described below, illustrates the category presented in Figure 4.

- **Example 4:** In teacher Jorge's class, $\frac{7}{9}$ do some sporting activity. We know that $\frac{3}{9}$ of the students only play soccer and the others only play basketball. What fraction of students in teacher Jorge's class play basketball?

We want to draw attention, at this point, to another phenomenon typical of situations involving the composition of measurements with fractional numbers: the *evaporation* of the unit of measurement mentioned previously allows the emergence of problems containing only one numerical piece of data explicitly provided in the statement, as in example 5.

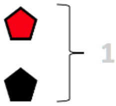
- **Example 5:** In teacher Beatriz's class, $\frac{4}{5}$ of the students do some sporting activity and the others do not do any sporting activity. What fraction of students in teacher Beatriz's class do not participate in sports activities?

Stella Baruk (2019), in the dictionary Matemática Elementar [Elementary Mathematics] she proposes, speaks of the historical difficulties in considering the "*one*" as a number, as we currently consider it. This is due to the lack of need to count when we have a single object. The different "ones" implicit in mathematical expressions (e.g.: $x = 1x^1$) may also be linked to this aspect. We therefore consider that this "*1*" that serves as a *unit of measurement* can constitute an epistemological obstacle in learning situations involving fractions. The correspondence between a "whole" and number "1," as well as the equality "$1 = \frac{b}{b}, com\ b \in \mathbb{N}^*$", are, therefore, elements that should not be disregarded by educational institutions nor by research interested in the dissemination of this knowledge.

We, therefore, consider that example 3 deserves to be included in another category apart from the two previously identified. So, we propose to introduce category 5, modelled as follows:

Figure 5

Category 5, $\{a, b, c, d, e, f\} \subset \mathbb{N}$ *;* $b \neq 0$*,* $d \neq 0$*, and* $f \neq 0$ *(research authors)*



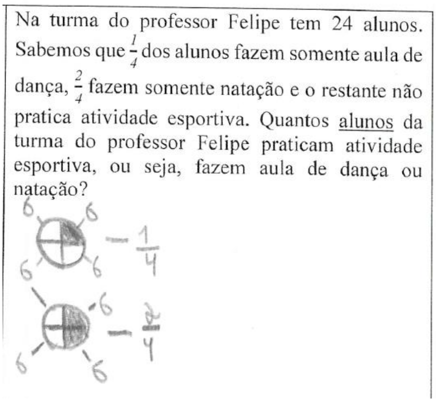
$\frac{a}{b} + \frac{c}{d} = 1$, where $\frac{c}{d}$ is unknown and "1" is implied

In some problem situations found in teaching materials or experienced in our daily lives, we also come across measurements that derive from a multitude of units of measurement. Example 6 illustrates it.

- **Example 6:** There are 24 students in teacher Felipe's class. We know that $\frac{1}{4}$ of the students only take dance classes, $\frac{2}{4}$ only swim, and the others do not practice any sports. How many students in teacher Felipe's class practice sports activities, i.e., take dance or swimming lessons?

Figure 6

Student's production regarding example 6 (research data)



In situations of this type, we observe the superposition of different units of measurement that allow us to describe the same quantity in different ways. In this case, we cite the following units of measurement (in red) and their respective measurements (in green):

$$1 \times 1 \text{ class of teacher Felipe}$$
$$24 \times 1 \text{ student}$$
$$4 \times \frac{1}{4} \text{ of teacher Felipe's class}$$
$$4 \times 6 \text{ alunos(as) of teacher Felipe's class}$$

Let us note that from the unit of measurement "$\frac{1}{4}$ of teacher Felipe's class", we have that "teacher Felipe's class" measures 4. This result is symbolised in the student's production (Figure 6), by the 4 parts of the disk that make up the whole. A mathematical interpretation –not necessarily formatted by the student– is given by Chambris (2020) on the subject:

> […] to think that $\frac{1}{3}$ is a unit, one must think that it is a number that measures numbers and its ratio of $1 : \frac{1}{3}$ is the unit with which 1 measures 3, $3\left(\frac{1}{3}\right) = 1$ […]. (p. 190)[7]

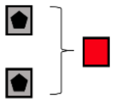
The use of unitary fractions as units of measurement is inviting and is an important resource of proportional reasoning. We can see from this that the multiplicative field is imposed in situations that combine measurements that are of the type "$\frac{\boldsymbol{a}}{\boldsymbol{b}}\mathbf{(M)}$ $a, b, \} \subset \mathbb{N}$, *with {, $b \neq 0$ and M* $\in \mathbb{R}^*$ non-fractional".

From this reflection, we consider three other classes, described below.

[7] […] pour penser que $\frac{1}{3}$ est une unité, il faut penser que c'est un nombre qui mesure des nombres et son rapport à $1 : \frac{1}{3}$ est l'unité avec laquelle 1 mesure 3, $3(\frac{1}{3}) = 1$ […] (p. 190).

Figure 7

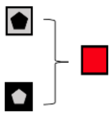
Category 6, $\{a, b, c, d\} \subset \mathbb{N}$; $b \neq 0$ and $d \neq 0$; M_α, M_β and $M_3 \in \mathbb{R}^*$ non-fractional; M_3 unknown (authors of this research)



$$\left[\frac{a}{b}(M_\alpha)\right] + \left[\frac{c}{d}(M_\beta)\right] = M_3$$

Figure 8

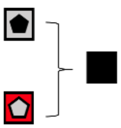
Category 7, $\{a, b\} \subset \mathbb{N}$; $b \neq 0$ and $d \neq 0$; M_2 and $M_3 \in \mathbb{R}^*$ non-fractional; with M_3 unknown (authors of this research)



$$\left[\frac{a}{b}(M_3)\right] + M_2 = M_3, \text{ with } M_2 = \frac{c}{b}(M_3)$$

Figure 9

Category 7, $\{;a, b\} \subset \mathbb{N}$; $b \neq 0$ M_2 and $M_3 \in \mathbb{R}^*$ non-fractional; withM_3 unknown (authors of this research)



$$\left[\frac{a}{b}(M_3)\right] + M_2 = M_3, \text{ with } M_2 = \frac{c}{d}(M_3)$$

Example 6 is, therefore, modelled by the sagittal diagram and mathematical equation shown in Figure 7. Similarly, examples 8 and 9, described below, illustrate the categories presented by Figures 8 and 9.

Example 7: In teacher Fernanda's class, $\frac{2}{5}$ of students practice sports activities and 15 students do not practice sports activities. How many students are there in teacher Fernanda's class?

Example 8: There are 32 students in teacher Ana's class. We know that $\frac{5}{8}$ students do some sporting activity and the others do not do any sporting activity. How many students in teacher Ana's class do not practice sports?

Vergnaud (2009) calls situations that involve more than one additive and/or multiplicative relationship complex problems. Specifically, Vergnaud (2009) calls mixed problems the complex problems involving addition (or subtraction) and multiplication (or division) operations. The problem situations mentioned above are characterised, then, as mixed problems: they go beyond the borders of the additive field, even though their primary structures are of the "composition of measurements" type.

In the following section, we propose a condensed presentation of the different classes of composition of measurements identified by our analysis.

A MODEL FOR COMPOSITION SITUATIONS INVOLVING FRACTIONAL AND NON-FRACTIONAL NUMBERS

The new categories we identified result from a long process of back-and-forth epistemological analysis. The result of the model of situations of the composition of measurements type involving fractional numbers is presented in Chart 2; for its reading, we propose the following caption:

Chart 1

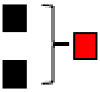
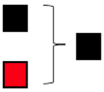
Subtitles for reading sagittal diagrams and equations (authors of this research)

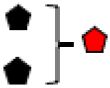
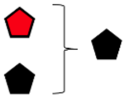
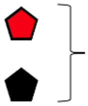
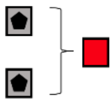
Regarding the colours present in the sagittal diagrams and equations		
Black	**Red**	**Gray**
Data provided in the problem statement	To be calculated	Information not provided by the problem statement, sometimes possible to be calculated
Regarding the shapes present in the sagittal diagrams		

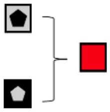
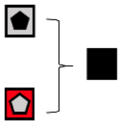
		
Non-fractional real numbers	Fractional numbers	Fraction of real numbers
Eg.: 55 km	Eg. : $\frac{2}{5}$ of the route	Eg. : $\frac{2}{5}$ of 55 km
Regarding the letters present in mathematical expressions		
M, M_α, M_β, M_1, M_2 and M_3	Measurement of a quantity: non-fractional real number	
$\frac{a}{b}, \frac{c}{d}$ and $\frac{e}{f}$	Rerepresent positive fractional numbers. $\{a, b, c, d, e, f\} \subset \mathbb{N}$; $b \neq 0$, $d \neq 0$, and $f \neq 0$	

Chart 2

Categories of composition problem situations (authors of this research)

Category	Sagittal diagram and equation	Example of a problem situation	Modelling and answer
I	 $M_1 + M_2 = M_3$	1. In teacher Lucas' class, 22 students do some sporting activity and 7 students do not do any sporting activity. How many students are there in teacher Lucas' class?	22 + 7 = ? There are 29 students in teacher Lucas' class.
II	 $M_1 + M_2 = M_3$	5. There are 28 students in teacher Maria's class. We know that 17 students do some sporting activity and the others do not do any sporting activity. How many students in teacher Maria's class do not practice sports?	17 + ? = 28 In teacher Maria's class, 11 students do not practice sports activities.

III	 $\frac{a}{b}+\frac{c}{d}=\frac{e}{f}$	2. In teacher João’s class, $\frac{3}{7}$ of the students only play volleyball, $\frac{2}{7}$ of the students only do judo, and the others do not practice sports activities. What fraction of students in teacher João’s class practice sports, that is, playing volleyball or judo?	$\frac{3}{7}+\frac{2}{7}=?$ In teacher João’s class, $\frac{5}{7}$ of students practice sports activities.
IV	 $\frac{a}{b}+\frac{c}{d}=\frac{e}{f}$	6. In teacher Jorge’s class, $\frac{7}{9}$ do some sporting activity. We know that $\frac{3}{9}$ of the students only play soccer and the others only play basketball. What fraction of students in teacher Jorge’s class play basketball?	$\frac{3}{9}+\frac{?}{?}=\frac{7}{9}$ The fraction of students who play basketball in teacher Jorge’s class is $\frac{4}{9}$.
V	 1 $\frac{a}{b}+\frac{c}{d}=1$	7. In teacher Beatriz’s class, $\frac{4}{5}$ of the students do some sporting activity and the others do not do any sporting activity. What fraction of students in teacher Beatriz’s class do not participate in sports activities?	$\frac{4}{5}+\frac{?}{?}=1$ In teacher Beatriz’s class, $\frac{1}{5}$ of students do not practice sports activities.
VI	 $[\frac{a}{b}(M_{\alpha})]$ + $[\frac{c}{d}(M_{\beta})] = M_3$	3. There are 24 students in teacher Felipe’s class. We know that $\frac{1}{4}$ of the students only take dance classes, $\frac{2}{4}$ only swim, and the others do not practice any sports. How many students in teacher Felipe’s class practice sports activities, i.e., take dance or swimming lessons?	$\frac{1}{4}24+\frac{2}{4}24=?$ In teacher Felipe’s class, 18 students practice sports activities.

VII	 $[\frac{a}{b}(\mathbf{M_3})] + \mathbf{M_2} = \mathbf{M_3}$, with $\mathbf{M_2} = \frac{c}{b}(\mathbf{M_3})$	4. In teacher Fernanda’s class, $\frac{2}{5}$ of students practice sports activities and 15 students do not practice sports activities. How many students are there in teacher Fernanda’s class?	$\frac{2}{5}? + 15 = ?$ There are 25 students in teacher Fernanda’s class.
VIII	 $[\frac{a}{b}(\mathbf{M_3})] + \mathbf{M_2} = \mathbf{M_3}$, with $\mathbf{M_2} = \frac{c}{d}(\mathbf{M_3})$	8. There are 32 students in teacher Ana’s class. We know that $\frac{5}{8}$ students do some sporting activity and the others do not do any sporting activity. How many students in teacher Ana’s class do not practice sports?	$\frac{5}{8}32 + ? = 32$ In teacher Ana’s class, 12 students do not practice sports activities.

Regarding the categories that deal only with fractions, without carrying out an exhaustive study, we observed that problem situations that result in a sum less than or equal to 1 are more common. We believe that at least two factors can explain this characteristic. The first of these is that the union of the parts of the whole that we are looking for is generally a sub-whole T’ of the whole T of the situation (T’ ⊂ T): as T assumes the function of unit of measurement, T’ has as measurement a number less than 1. A second factor may be linked to the use of fractions in everyday life, where we notice that additive decomposition into whole and fractional parts less than 1 is quite present in our oral speech. For example, we say “2 and a half bottles of soda” $(2 + \frac{1}{2})$ and very rarely, “5 half bottles of soda” $(\frac{5}{2})$. This linguistic practice certainly has consequences for the strategies used in situations of composition involving fractions. However, despite these two factors, there is no epistemological impediment to the existence of composition problem situations whose sum is greater than 1.

FROM THE EPISTEMOLOGICAL MODEL TO EMPIRICAL DATA ANALYSIS: WHAT STUDENTS ALLOW US TO UNDERSTAND

Inspired by works such as Gitirana et al. (2014) and Magina et al . (2008), we seek to study the influence of different categories on students' performance in solving problems of composition of measurements. To this end, the problem situations presented in this text were applied to 987 Brazilian students, from the 5th to the 9th grade of elementary school, distributed in 37 classes of ten public schools located in the states of Santa Catarina, Paraná, Pernambuco, and Paraíba. The distribution by school grade is shown in Chart 3:

Chart 3

School grade of research participants

School grade	5th grade	6th grade	7th grade	8th grade	9th grade
Number of students	75	195	218	271	228

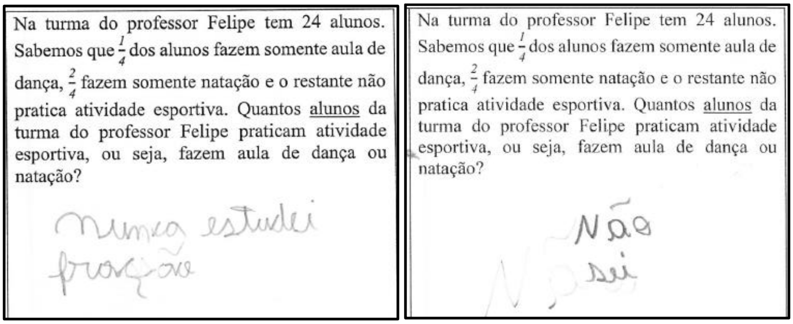
A brief a priori analysis of each problem situation is presented in Appedix I of this text, together with the protocol and information provided to the collaborating teachers who participated in the data production.

Regarding the choice of problem situations proposed to students, we sought to create a protocol in which we controlled some didactic variables to identify a possible hierarchy of complexity of the different categories in Chart 2. In this sense, we aim to reduce the possibility of errors arising from calculations, proposing, for example, fractions with equal denominators and addition and subtraction operations without reservations. Furthermore, we decided to keep the same context in all the proposed questions (students in a class who do or do not practice a sporting activity) so that the magnitude in question would not be a facilitator or cause difficulties from one question to another.

At first, three 4th-grade classes also participated in the research protocol, but these students' resolution data proved inconclusive in inferring the complexity of the proposed situations. This occurred because, when faced with questions involving fractional numbers, students were shown to lack the tools and schemes necessary to deal with the situation. This results from the fact that this is the first year in which these students are officially introduced to fractions in the Brazilian education system, therefore, there is not enough time for them to develop strategies and an invariant organisation of behaviour. This fact can be confirmed by the students' answers, as shown in the examples in Figure 10.

Figure 10

Answers submitted by 4th graders from different schools (research data)


Na turma do professor Felipe tem 24 alunos. Sabemos que $\frac{1}{4}$ dos alunos fazem somente aula de dança, $\frac{2}{4}$ fazem somente natação e o restante não pratica atividade esportiva. Quantos alunos da turma do professor Felipe praticam atividade esportiva, ou seja, fazem aula de dança ou natação?

nunca estudei fração

Na turma do professor Felipe tem 24 alunos. Sabemos que $\frac{1}{4}$ dos alunos fazem somente aula de dança, $\frac{2}{4}$ fazem somente natação e o restante não pratica atividade esportiva. Quantos alunos da turma do professor Felipe praticam atividade esportiva, ou seja, fazem aula de dança ou natação?

Não sei

ANALYSIS OF EMPIRICAL DATA: IDENTIFICATION OF THREE LEVELS OF COMPLEXITY

Once the data was collected and computed, we sought to identify the hit rate for each problem situation present in the protocol by school grade.

Chart 4

Classes of composition problem situations and hit rates (authors of this research)

Category/Hit rate	5th grade	6th grade	7th grade	8th grade	9th grade
Category I	≈76%	≈72%	≈79%	≈91%	≈91%
Category II	≈96%	≈83%	≈88%	≈96%	≈96%
Category III	≈40%	≈42%	≈49%	≈58%	≈60%
Category IV	≈33%	≈28%	≈34%	≈51%	≈49%

Category V	≈36%	≈31%	≈39%	≈48%	≈53%
Category VI	≈15%	≈14%	≈14%	≈28%	≈36%
Category VII	≈11%	≈15%	≈15%	≈38%	≈22%
Category VIII	≈8%	≈12%	≈16%	≈24%	≈25%

The data we have does not allow us to indicate a clear hierarchy between the categories, but invites us to consider the strong influence of how the measurements are described in the statement. When we regroup the data, this influence becomes easily observable:

Chart 5

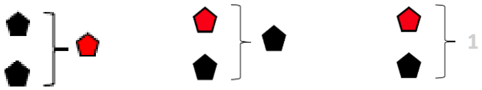
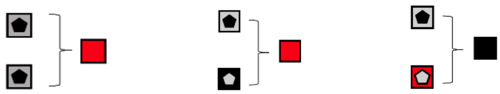
Classes of composition problem situations and success rates (authors of this research)

Category	Hit rate
Category I	≈ 84%
Category II	≈92%
Category III	≈ 52%
Category IV	≈41%
Category V	≈43%
Category VI	≈23%
Category VII	≈23%
Category VIII	≈19%

In Chart 3, we identify three levels in the hit rate: two categories with a high hit rate above 80%, three with a hit rate between 40% and 50%, and three others with a low hit rate below 30%. These categories have in common, precisely, how the measurement of a magnitude is stated, as shown in Chart 6.

Chart 6

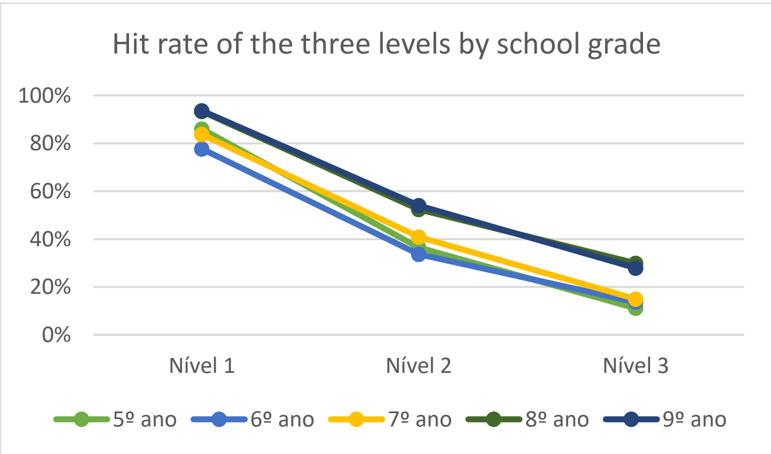
Sagittal diagrams and categories related to levels (authors of this research)

Levels	Sagittal schemes and categories
Level I	 I II
Level II	 III IV V
Level III	 VI VII VIII

Since the observed population is not uniform, we sought to analyse whether this result is confirmed by school grade. After this analysis, we found that the student performance rate decreases significantly from one grade to the next in all school grades, depending on the three levels identified:

Figure 11

Average hit rate at each level (authors of this research)

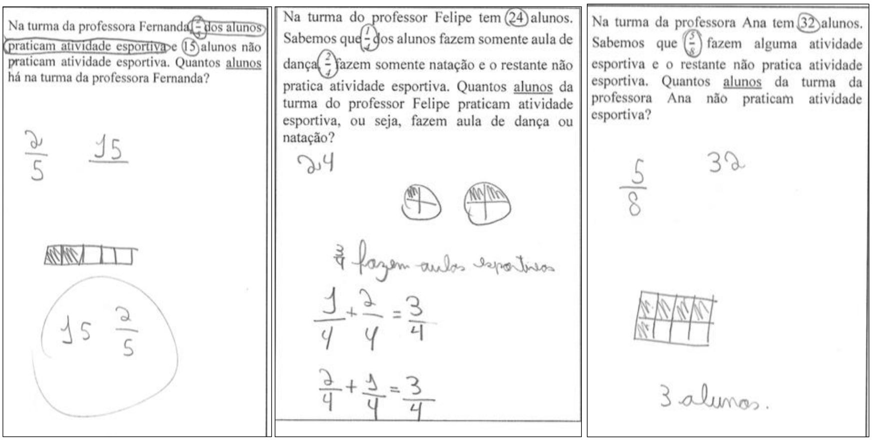


The results of the samples by school grade corroborate the existence of the three levels observed previously. Although we observed improvements in student performance from one grade to the next, we noticed that the difficulties inherent to each level of complexity remained throughout the five school grades analysed.

These results lead us to the conclusion that problem situations involving only fractions have a significantly higher hit rate than those that involve the fraction of a measurement. The difference in the number of arithmetic operations and the complexity of managing different units of measurement can explain this variation. In this sense, we observe that students who have modelling tools that allow them to represent and manipulate information as "$\frac{4}{5}$ of the class of students" are not always able to handle information like "$\frac{4}{5}$ of the class of 24 students," as illustrated in Figure 12.

Figure 12

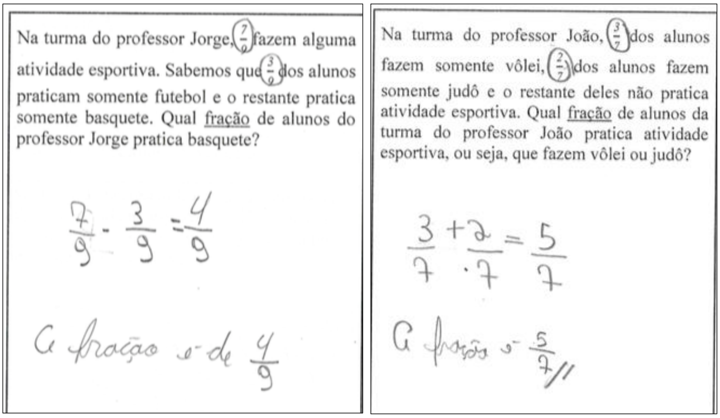
Example of a production by student A in the 9th grade (research data)



Given the three statements, the student correctly models the data from the problem situations regarding the fraction of the magnitude in play, either through figural representations (disc and rectangular representation) or mathematical models (numerical expressions). However, we note that it does not integrate the numerical information of the magnitude measurement into the models it provides, as it lacks schemes to find the value of the fraction of a measurement. The models used by the student are suitable for solving level II problem situations but do not allow for solving level III problem situations, as we can confirm in Figure 13.

Figure 13

Example of production by student A in the 9th grade (research data)



Na turma do professor Jorge, $\frac{7}{9}$ fazem alguma atividade esportiva. Sabemos que $\frac{3}{9}$ dos alunos praticam somente futebol e o restante pratica somente basquete. Qual fração de alunos do professor Jorge pratica basquete?

$$\frac{7}{9} - \frac{3}{9} = \frac{4}{9}$$

A fração é de $\frac{4}{9}$

Na turma do professor João, $\frac{3}{7}$ dos alunos fazem somente vôlei, $\frac{2}{7}$ dos alunos fazem somente judô e o restante deles não pratica atividade esportiva. Qual fração de alunos da turma do professor João pratica atividade esportiva, ou seja, que fazem vôlei ou judô?

$$\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$$

A fração é $\frac{5}{7}$//

The three identified levels of composition problem situations for fractional and non-fractional numbers reveal the importance of exploring the different types of problem situations in class. If mathematics teaching is to be anchored in the resolution of problem situations, the curriculum prescribed and practised by school institutions must consider the plurality of categories and the cognitive complexity and epistemological obstacles they involve. The didactic model and the levels of complexity presented here can, therefore, serve as an instrument to guarantee this diversity and support curriculum choices throughout schooling.

Although we used composition situations as a case study, we believe that this is a result that can be observed in other classes of the additive field.

CONCLUSIONS

> In their research, mathematicians are confronted with problems that no one can solve. A significant part of their activity consists of asking questions and solving problems. For that, they are led to create conceptual *tools* […]. For transmission to the scientific community, the concepts thus created are decontextualised and formulated in the most general way

> possible. They are then integrated into the body of knowledge already established to extend or replace some of it. They acquire the status of an *object*. It happens that researchers directly create objects to better organise a branch of mathematics, to put thoughts in order or for the needs of the exhibition. Thus, we say that a concept is a *tool* when we focus our interest on the use that one can make of it to solve a problem. The same tool can be *adapted* to several problems; several tools can be adapted to the same problem. By *object*, we mean the cultural object having its place in a larger edifice, which is scholarly knowledge at a given moment that is socially recognised. (Douady, 1986, p. 09) [8]

Inspired by the mathematical activity described by Douady (1986), we can state that scientific models of research activity also have a dual status: object and tool. Throughout this text, we sought to attribute the object status to the didactic model presented. At the genesis of this work, we have a methodological problem: the insufficiency of the models we used to describe problem situations involving fractional numbers. The need for our new object arises in the absence of a suitable tool.

The didactic model in question was developed through a dialectic between epistemological analysis and analysis of empirical data. Thus, we must emphasise that the linearity used in this article to present these two analyses should not be confused with the methodological trajectory of the work carried

[8] Dans leurs recherches, les mathématiciens sont confrontés à des problèmes que personne sait résoudre. Une part importante de leur activité consiste à poser des questions et résoudre des problèmes. Pour ce faire, ils sont amenés à créer des *outils* conceptuels […]. Pour les besoin de la transmission à la communauté scientifique, les concepts ainsi crées sont décontextualisés, formulés de la façon la plus générale possible. Ils s'intègrent dès lors au corps des connaissances déjà constituées pour les étendre ou se substituer à certaines d'entre elles. Ils acquièrent le statut d'*objet*. Il arrive que des chercheurs créent directement des objets pour mieux organiser une branche des mathématiques, pour mettre de l'ordre dans les pensées ou pour les besoins de l'exposition. Ainsi, nous disons qu'un concept est *outil* lorsque nous focalisons notre intérêt sur l'usage qui en fait pour résoudre un problème. Un même outil peut être *adapté* à plusieurs problèmes, plusieurs outils peuvent être adaptés à un même problème. Par *objet*, nous entendons l'objet culturel ayant sa place dans un édifice plus large qui est le savoir savant à un moment donné, reconnu socialement. (Douady, 1986, p. 09)

out. As is almost always the case, this linearity is a textual illusion: the researcher's work is often carried out through adjustments and misadjustments, with comings and goings to different sources of reflection, as we did here. We included categories after the pilot protocol and excluded one after data analysis from the latest protocol version, which highlights the interest in combining different methodological approaches.

Regarding the limits of this study, we emphasise that in the research protocol, we intentionally disregarded different didactic variables that strongly impact the hit rate. For example, We do not study the role of magnitudes in problem situations involving fractions nor the impact of proposing fractions whose denominators are different. These and many other didactic variables are equally important and should be considered by those interested in teaching and learning fractions.

For a better understanding of our results, the levels of complexity presented here also deserve to be correlated with the Brazilian educational system's institutional practices. In other words, the significant difference between the levels of complexity observed by this research data, especially regarding levels II and III, can also be explained by a curriculum analysis of what is prescribed and carried out in class. Recognising the institutional conditions and restrictions that impact the dissemination of mathematical knowledge invites us to rethink the current curriculum and, thus, to envision other forms of teaching that could affect students' hit rate. In this context, we can even ask ourselves whether it would be possible to have a curriculum that would make the levels of complexity identified here obsolete.

Besides contributing to the study of different research questions, using the composition situations model as a tool will allow its constant validation and eventual evolution, given the limits it may present. After all, the stability of a scientific model is a fantasy of the present.

AUTHORSHIP CONTRIBUTION STATEMENT

Both authors equally developed this research. The first author developed the teaching model that includes eight composition-type problem situations, six involving fractional numbers. This model was developed during her work as a researcher at Université Grenoble Alpes, as part of the French project "Pégase - Pôle pilote de formation des enseignants et de recherche pour l'éducation.". The second author contributed mainly to the production of data with 987 students from the southern and northeastern regions of Brazil.

DATA AVAILABILITY STATEMENT

The data supporting the empirical development of this research are stored with the researchers, respecting ethical principles.

REFERENCES

ALLARD, C. *Etude du processus d'Institutionnalisation dans les pratiques de fin d'école primaire:* le cas de l'enseignement des fractions. Thése de l'Université de Paris 7, 2015.

BARUK, S. *Dictionnaire de mathématiques élémentaires*. Editions du Seuil. IBN 978-2-02-143655-6. 2019.

BEHR, M. L., R., POS,T., SILVER, E. Rational Number Concepts. In R.LESH & M.Landau (Eds) *Acquisition of Mathematics Concepts and Processes,* pp. 91-125. New York: Academic Press, 1983

BRASIL, Ministério da Educação. *Base Nacional Comum Curricular:* Educação é a base. Brasília, DF: Ministério da Educação, Secretaria de Educação Básica, 2018.

BRASIL. Ministério da Educação. Secretaria de Educação. *Parâmetros Curriculares Nacionais:* Matemática. Brasília, 1997.

CHAMBRIS, C. Petite histoire des rapports entre grandeurs et numérique dans les programmes de l'école primaire. *Repères-IREM*, v. 69, p. 5-31, 2007.

CHAMBRIS, C. Raisons d'être des grandeurs : le cas de l'arithmétique à l'école élémentaire. In: CHAACHOUA, H.; et al. (Eds.). *Actes de la XXe École d'été de didactique des mathématiques*. Grenoble: La Pensée Sauvage, 2020.

DOUADY, R. . Jeux de cadres et dialectique outil-objet. *Recherches en Didactique des Mathématiques,* 7(2), 5–31, 1986.

KASPARY, D. *La noosphère, un lieu de tension pour le curriculum.* Étude didactique de la mise en place d'um système d'évaluation de manuels scolaires sur l'étude du champ additif à l'école primaire. Thèse des universités UFMS (Brésil) et UGA (France), 2020.

KIEREN, T. E. (1976) On the mathematical, cognitive and instructional foundations of rational numbers.' In R. LESH (ed.), *Number and Measurement* (pp 101-144). Columbus, OH: ERIC/SMEAC.

MAGINA, S. *et al*. *Repensando Adição e Subtração:* Contribuições da Teoria dos Campos Conceituais. São Paulo: PROEM, 2001.

NEYRET, Robert. *Contraintes et déterminations des processus de formation des enseignants : nombres décimaux, rationnels et réels dans les Instituts Universitaires de Formation des Maîtres*. 1995. 2 v. Thèse (Doctorat en didactique des mathématiques) — Université de Grenoble, Grenoble, 1995.

NUNES, T., BRYANT, P., PRETZLIK, U., BELL, D., EVANS, D., e WADE, J. *La compréhension des fractions chez les enfants.* Activité humaine et conceptualisation. pp. 225-262, 2007.

VERGNAUD, G.. *A Criança, a Matemática e a Realidade.* Trad. Maria Lucia Faria Moro. Curitiba: Editora UFPR, 2009.

VERGNAUD, G. La théorie des champs conceptuels. *Recherche en Didactique des Mathématiques.* Grenoble: La Pensée Sauvage, vol. 10, n. 2.3, pp. 133 a 170, 1990.

VERGNAUD, G. . Multiplicative Structures. In Lesh R., Landau M. (Ed.). *Acquisition of mathematics concepts and processes,* New York: Academic Press., pp. 127-174, 1983.

THOMPSON, P. W.; SALDANHA, L. Fractions and multiplicative reasoning. In: *Research Companion to the NCTM Standards*. [S.l.]: [s.n.], 2003.

PERRIN-GLORIAN, Marie-Jeanne. Problèmes didactiques liés à l'enseignement des grandeurs. Le cas des aires. In : DORIER, J-L. et al. (eds.). Actes de la XIe Ecole d'été de didactique des mathématiques, Corps, 21-30 Août 2001, Grenoble: La Pensée Sauvage, 2002.

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Appendix I

Message to the collaborating teachers:
Dear Teachers,

Your contribution to this research is very valuable to us. We are immensely grateful that you found a moment to complete this question sheet in your class and took the time to send us your results.

Students may have difficulty answering some questions. However, this will not harm the research, the student, or the school. We are interested in studying the errors and the different strategies they put into practice to solve arithmetic problems. For this reason, it is important that students solve each question individually. We also ask that you please not help them solve them. If they have any difficulty, reassure them by saying that it is okay if the answer is wrong or if they do not know how to answer it. He/she can, if he/she wishes, indicate in writing, "I don't know how to solve it." If you wish, you can correct it with the students after collecting the question sheets.

This research will be conducted from the 4th to the 9th grade of elementary school with students from different Brazilian cities and regions. The data shall be used exclusively for scientific purposes. No data from the school, students, or teachers will be used. To maintain the complete anonymity of participants, we ask that students provide a fictitious name of their choice. This allows you to show the student that he/she will not be assessed and that his/her anonymity is indeed protected. Before starting the experiment, you can explain that they are participating in scientific research.

To send students' productions, please send photos (or scans) of the productions to the following WhatsApp numbers: XXXX. If preferred, copies may also be mailed to the Brazilian researcher, with expenses covered by the researchers.

These numbers can be used if you have any questions.

We appreciate the contribution to Brazilian research in mathematics education.
Prof. Dr. XXXX, researcher XXXX (France) and Prof. Dr. XXX, researcher XXXX (Brazil)

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Fictitious name of the student: School grade: Student age: Date:

At one school, a survey was carried out on students' sports practices. Below, you will find information about students in nine different classes. Answer the following questions about each of these nine classes. You can solve them in whatever way you consider most convenient. We kindly ask that you register your entire decision.

In teacher Lucas' class, 22 students do some sporting activity and 7 students do not do any sporting activity. How many <u>students</u> are there in teacher Lucas' class?	There are 28 students in teacher Maria's class. We know that 17 students do some sporting activity and the others do not do any sporting activity. How many <u>students</u> in teacher Maria's class do not practice sports?	There are 24 students in teacher Felipe's class. We know that $\frac{1}{4}$ of the students only take dance classes, $\frac{2}{4}$ only swim, and the others do not practice any sports. How many <u>students</u> in teacher Felipe's class practice sports activities, i.e., take dance or swimming lessons?
Category I	Category II	Category VI
<u>Underlying mathematical model:</u> $M_1 + M_2 = M_3$ <u>Additive structure:</u> • Known data: two measurements M_1 and M_2, such that $M_1 \cap M_2 = \emptyset$ (*parts of a whole).* • Given a wanted M_3: union set of M_1 and M_2 (or *whole*).	<u>Underlying mathematical model:</u> $M_1 + M_2 = M_3$ <u>Additive structure:</u> • Known data: two measurements M_1 and M_3, such that $M_1 \subset M_3$ (one *part of* the *whole* and the *whole*). • Data sought: M_2, the complement of M_1 in M_3 (one *part of the whole*).	<u>Underlying mathematical model:</u> $[\frac{a}{b}(M_\alpha)] + [\frac{c}{d}(M_\beta)] = M_3$ <u>Additive structure:</u> • Known data: two fractional values of a quantity whose measurement is known. • Data wanted: the union of two measurements (the *whole*).

Possible strategies and difficulties: • Addition operation: ○ armed algorithm; overcounting with pictorial, gestural, oral, or mental resources; mental calculation. ➢ an incorrect answer results from the calculation performed. • Operation other than addition: ➢ misunderstanding of the structure of the problem; ➢ marked difficulty in other situations.	Possible strategies and difficulties: • Addition operation: ○ overcounting with pictorial, gestural, oral, or mental resources; mental calculation. ➢ an incorrect answer results from the calculation performed. • Subtraction operation: ○ armed algorithm; mental calculation; ➢ an incorrect answer results from the calculation performed. • Another arithmetic operation: ➢ misunderstanding of the structure of the problem; ➢ marked difficulty in other situations.	Possible strategies and difficulties: • Addition operation and multiplicative field ○ calculation of $\frac{a}{b} + \frac{c}{d} = \frac{e}{f}$ and calculation of $\frac{e}{f}$(M) = M_3. ○ calculation of $\frac{a}{b}$(M) = M_1 and $\frac{c}{d}$(M) = M_2 and calculation of $M_1 + M_2 = M_3$. ➢ error when operating with fractional numbers, for example, $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$. ➢ difficulty in calculating the fraction of a measurement. • Another arithmetic operation: ➢ misunderstanding of the structure of the problem.

In teacher João's class, $\frac{3}{7}$ of the students only play volleyball, $\frac{2}{7}$ of the students only do judo, and the others do not practice sports activities. What	There are 30 students in teacher Bianca's class. We know that 23 students practice some sporting activity. What <u>fraction</u> of students in	In teacher Beatriz's class, $\frac{4}{5}$ of the students do some sporting activity and the others do not do any sporting activity. What <u>fraction</u> of students in

fraction of students in teacher João's class practice sports, that is, playing volleyball or judo?	teacher Beatriz's class do not participate in sports activities?	teacher Beatriz's class do not participate in sports activities?
Category III	Uncategorised	Category V
Underlying mathematical model: $\frac{a}{b}(M) + \frac{c}{d}(M) = \frac{e}{f}(M)$ Additive structure: • Known data: two fractional values of a quantity whose measurement is unknown (*parts of the whole).* • Data wanted: fractional value union of a quantity whose measurement is unknown (the *whole*) Possible strategies and difficulties: • Addition operation: ○ with pictorial resources; mental calculation; overcounting of numerators; mathematical model.	Underlying mathematical model: $\frac{a}{b}(M_1) + \frac{c}{b}(M_2) = \frac{d}{b}(M_3)$ Additive structure: • Known data: two measurements M_1 and M_3, such that $M_1 \subset M_3$ (one *part of* the *whole* and the *whole*). • Data wanted: fractional values of a quantity whose measurement is known (a fraction of the *part of the whole*) Despite being part of the protocol, problem situations of this type do not constitute a class in the model proposed in this article. Let us note that the problem can be solved in two ways:	Underlying mathematical model: $\frac{a}{b}(M_3) + \frac{c}{d}(M_3) = 1$ ou $\frac{a}{b}(M) + \frac{c}{d}(M) = \frac{e}{e}(M)$ Additive structure: • Known data: a fractional value of a quantity whose measurement is unknown (a *part of the whole*) • Data wanted: a fractional value of a quantity whose measurement is unknown (a *part of the whole*) • Implicit data: the *whole*, the unit. Possible strategies and difficulties: ➢ absence of models to represent the *whole*. • Addition operation:

➢ error when operating with fractional numbers, for example, $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$. • Assigning a fictitious measurement to M: ➢ difficulty in assigning a strategic value, multiple of the denominators; ➢ passage through the multiplicative field. • Another arithmetic operation: ➢ misunderstanding of the structure of the problem; marked difficulty in other situations.	- 30 – 23 = 7, therefore $\frac{7}{30}$ - $\frac{30}{30}$ - $\frac{23}{30}$= $\frac{7}{30}$ We can observe that this is a mixed problem and that in each strategy the composition structure in play is already described by the other classes of the model: classes I and VII, respectively.	○ mental calculation; overcounting of numerators. ➢ error when operating with fractional numbers. • Subtraction operation: ○ with pictorial resource; mental calculation; mathematical model. ➢ error when operating with fractional numbers. Assigning a fictitious measurement to M: ➢ difficulty in assigning a strategic value, multiple of the denominators; ➢ passage through the multiplicative field.
In teacher Jorge's class, $\frac{7}{9}$ do some sporting activity. We know that $\frac{3}{9}$ of the students only play soccer and the others only play basketball. What fraction of students in teacher Jorge's class play basketball?	There are 32 students in teacher Ana's class. We know that $\frac{5}{8}$ students do some sporting activity and the others do not do any sporting activity. How many students in teacher Ana's class do not practice sports?	In teacher Fernanda's class, $\frac{2}{5}$ of students practice sports activities and 15 students do not practice sports activities. How many students are there in teacher Fernanda's class?
Category IV	Category VIII	Class VII

Underlying mathematical model: $\frac{a}{b}(M) + \frac{c}{d}(M) = \frac{e}{f}(M)$ Additive structure: • Known data: two fractional values of a quantity whose measurement is unknown (a *part of the whole* and the *whole*) • Data wanted: a fractional value of a quantity whose measurement is unknown (a *part of the whole*). Possible strategies and difficulties: • Addition operation: ○ mental calculation; overcounting of numerators. • Subtraction operation: ○ with pictorial resource; mental calculation; mathematical model. ➢ error when operating with fractional numbers. Assigning a fictitious measurement to M: ➢ difficulty in assigning a strategic value, multiple of the denominators; ➢ passage through the multiplicative field.	Underlying mathematical model: $[\frac{a}{b}(M_3)] + M_2 = M_3$, with $M_2 = \frac{c}{b}(M_3)$ Additive structure: • Known data: a fractional value of a quantity whose measurement is known and the value of a measurement (a *part of the whole* and the *whole*) • Data wanted: value of a measurement (a *part of the whole*). Possible strategies and difficulties: • Operation with non-fractional numbers: ○ calculation of $\frac{a}{b}$ (M_3) and calculation of the difference between M_3 in$_2$. • Operation with fractional numbers: ○ calculation of $\frac{b-a}{b} = \frac{c}{b}$ and calculation of $\frac{c}{b}(M_3$. ➢ error when operating with fractional numbers. ➢ absence of models to represent the *whole*.	Underlying mathematical model: $[\frac{a}{b}(M_3)] + M_2 = M_3$, com $M_2 = \frac{c}{b}(M_3)$ Additive structure: • Known data: a fractional value of a quantity whose measurement is unknown and the value of a measurement (*parts of the whole*) • Data wanted: the union of two measurements (the *whole*) Possible strategies and difficulties: ➢ absence of models to represent the *whole*. • find fraction $\frac{b-a}{b}$ and perform the proportional calculation passing through the unit fraction $\frac{1}{b}$ and then calculate the sum of two measurements relative to $\frac{a}{b}(M_3)$ and $\frac{c}{b}(M_3)$. • Arithmetic operations with the two data of the problem. ➢ misunderstanding of the structure of the problem.

• Another arithmetic operation: ➢ misunderstanding of the structure of the problem; marked difficulty in other situations.	• Another arithmetic operation with the two data from the problem: ➢ misunderstanding of the structure of the problem;	

Chart 3 – Protocol and a priori analysis of the problem situations proposed to the students
Source: Prepared by the authors.