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Vision and Visualisation of Geometric Representations of Prospective Mathematics Teachers

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ABSTRACT

Background: The study of spherical geometry and other non-Euclidean geometries associated with the development of geometric thinking in the initial education of mathematics teachers is promising and can contribute to the teaching of geometry. **Objectives**: Discuss elements of vision and visualisation mobilised by prospective mathematics teachers in solving a spherical geometry task. **Design**: Qualitative research of an interpretative nature. **Setting and participants**: Thirteen prospective mathematics teachers (PMTs) enrolled in the Geometry Teaching subject at a state university in Paraná participated. **Data collection and analysis**: The information obtained includes the audio recordings of pair and collective discussions, PMTs' written production during task exploration, and the field diary registers. The analysis focused on PMTs' task resolution and discussion on mobilising elements of vision and visualisation (Duval, 1999; Gutiérrez, 1996) to develop geometric thinking. **Results**: The results show the mobilisation of visual perception (vision) and elements of visualisation, such as the creation of mental images and representations of geometric objects, external representations, and visualisation processes and skills (namely, visual interpretation of information and mental images, figure-ground perception, perceptual constancy, mental rotation, and perception of spatial positions. **Conclusions**: The mobilisation of these elements reveals learning of spherical geometry concepts, allowing us to understand the differences and similarities between spherical geometry and Euclidean geometry.

Keywords: Initial education of mathematics teachers; vision; visualisation; teaching of spherical geometry; geometric thinking.

Visão e Visualização de Representações Geométricas de Futuros Professores de Matemática

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RESUMO

Contexto: O estudo da geometria esférica, assim como das outras geometrias não euclidianas, associada ao desenvolvimento do pensamento geométrico na formação inicial de professores de matemática é uma temática promissora e que pode contribuir para o ensino de geometria. **Objetivos**: Discutir elementos da visão e da visualização mobilizados por futuros professores de matemática na resolução de uma tarefa de geometria esférica. **Design**: Investigação de natureza qualitativa, de cunho interpretativo. **Ambiente e participantes**: Participaram treze futuros professores de matemática (FPMat) matriculados na disciplina "Ensino de Geometria" de uma universidade estadual paranaense. **Coleta e análise de dados**: As informações obtidas referem-se às gravações em áudio das discussões em pares e coletivas, à produção escrita dos FPMat promovida pela exploração da tarefa e aos registros em diário de campo. A análise incidiu na resolução e na discussão da tarefa pelos FPMat sobre a mobilização de elementos da visão e visualização, Duval (1999) e Gutiérrez (1996), para o desenvolvimento do pensamento geométrico. **Resultados**: Os resultados evidenciam a mobilização da percepção visual (visão) e de elementos da visualização, como a criação de imagens mentais e representações de objetos geométricos, representações externas e processos e habilidades de visualização (nomeadamente, interpretação visual de informações e de imagens mentais, percepção figura-fundo, constância perceptiva, rotação mental e percepção de posições espaciais. **Conclusões**: A mobilização destes elementos desvela aprendizagens de conceitos de geometria esférica que permitiram compreender diferenças e semelhanças entre a geometria esférica e a geometria euclidiana.

Palavras-chave: Formação inicial de professores de matemática; visão; visualização; ensino de geometria esférica; pensamento geométrico.

INTRODUCTION

Researchers in the mathematics education field have been considerably concerned with geometry teaching. Several factors contribute to this concern, among which the history of marginalisation of geometry in educational curricula, the predominance of axiomatic teaching, and teachers' insufficient exploration or, in many cases, neglect of the topic (Barros & Pavanello, 2022; Elia et al., 2018; Marmolejo & Astudillo, 2015; Pavanello, 1993; Silva & Cyrino, 2024).

Although the current documents[1] guiding the curriculum proposals for basic education foresee the study of Euclidean and non-Euclidean geometries from the early years[2], this knowledge has not been explored much in mathematics classes. Regarding non-Euclidean geometries (NEGs), they are rarely addressed (Silva & Cyrino, 2024).

Inside and outside Brazil, researchers draw attention to geometry abandonment in basic education and mathematics teachers' initial education (Blanco, 2014; Gonzato et al., 2011; Marmolejo & Astudillo, 2015; Marmolejo et al., 2017). Most investigations focus on discussing concepts of the plane and spatial Euclidean geometry to analyse basic education students' and teachers' thinking and problem-solving strategies when proving theorems and mathematical properties and using images and geometric reasoning when solving Euclidean geometry (EG) tasks (Blanco, 2014; Elia et al., 2018; Fujita et al. 2017; Marmolejo et al., 2017; Swoboda, 2008).

We identified a few studies on the exploration of NEG concepts, both with basic education students and in contexts of initial and continuing teacher education (Carvalho & Azevedo, 2020; Fonseca, 2020; Gomes, 2014; Reis, 2006; Silva, 2018), although such concepts are part of both the educational proposals of some states (such as Paraná) for basic education and the National Common Curriculum Base (Base Nacional Comum Curricular - BNCC).

The Study and Research Group on the Education of Teachers who Teach Mathematics (Grupo de Estudos e Pesquisas sobre a Formação de Professores que Ensinam Matemática - Gepefopem) has been dedicated to investigating different approaches to geometric thinking and its development in teacher education contexts (Cybulski & Cyrino, 2022; Silva & Cyrino 2024; Vieira & Cyrino, 2022). In these studies, the elements associated with vision and visualisation in the development of geometric thinking caught our attention since the authors frequently mention them but lack further exploring teacher education and spherical geometry teaching.

Studies on those elements in mathematics teachers' initial and continuing education with discussions that associate them with the teaching and

[1] In Brazil, the educational curriculum that guides the content that should be taught in basic education is the National Common Curriculum Base (Base Nacional Comum Curricular - BNCC), implemented in 2018.
[2] Paraná created its own curriculum based on the guidelines proposed by the BNCC, entitled the Paraná State Network Curriculum (Currículo Estadual da Rede Paranaense), implemented from 2019 onwards.

learning processes of geometry to mobilise and develop geometric thinking are crucial because vision and visualisation can help us access mathematical objects through verbal and graphic representations (Blanco, 2014; Elia et al., 2018; Marmolejo et al., 2017; Swoboda, 2008; Torregrosa & Quesada, 2007).

In this research, we analysed elements associated with vision and visualisation mobilised by prospective mathematics teachers (PMTs) enrolled in the subject "Geometry Teaching" in the 2nd year of the mathematics teaching degree course at the State University of Paraná – Paranavaí Campus while solving a spherical geometry task[3]. We chose to address it, recognising its relevance for understanding everyday situations, often interpreted based on common sense and disseminated on social media, for example, that the Earth is flat.

VISION AND VISUALISATION AS MOBILISING ASPECTS OF GEOMETRIC THINKING

Vision cannot be considered the only and most relevant way to guide the processes of teaching and learning mathematics (Gutiérrez, 1992, 1996). The act or effect of seeing enables a perception of the outside world through the eyes, among many other meanings attributed to it in our daily lives.

Vision, specifically when understood as the sense of sight, refers to visual perception, which is directly associated with the development of a mental image from what is observed. However, this visual perception is often limited to what is captured by the sense of sight, which implies an imperfect apprehension of the object or situation in question (Duval, 1999). This is due to our experience in a three-dimensional world, where we can only perceive part of a physical object at a time. To have a complete visualisation of the observed structure, we must perform additional exploration, either through the movements of the observer or the object.

Visual perception is relevant for many everyday tasks, including geometry learning. When using such a term, the researcher must be aware of the meaning adopted to avoid philosophical and epistemological confusion. Gutiérrez (1992) rightly states that: "The basic central element in all

[3] By tasks, we understand them to be those such as problems and exploration tasks that fit "into the levels of cognitive demand with procedures with connections to meanings and doing mathematics, categorized as a high level of cognitive demand" (Jesus et al., 2018, p. 22).

conceptions of visual perception are mental images, that is, the mental representations that people can make of physical objects, relationships, concepts, etc." [4] (p. 44).

The term visualisation in mathematics, especially in geometry, is not only related to what can be seen and touched with the hands. As Arcavi (1999) states, subjects see not only what is given to them to be seen but also what is not visible to the eyes. The ability to understand and use concepts is essential to develop mathematical and geometric knowledge, both for students and teachers.

Gutiérrez (1996) describes visualisation "as a type of reasoning activity based on the use of visual or spatial, mental, or physical elements, carried out to solve problems or prove properties" [5] (p. 9). Based on the definition of visualisation, the author describes it as

> the set of types of images, processes, and skills necessary for geometry students to produce, analyse, transform, and communicate visual information related to real objects, models, and geometric concepts. The visual information produced (images) can be either physical (figures or diagrams) or mental (mental images). Visual information analysis refers to images produced by students themselves and those received from outside (from students, teacher, text, etc.). Transformations can be made between an image and verbal information (oral or written) or from one image to another. Communication can be graphic, verbal, or mixed. [6] (Gutiérrez, 2006, p. 27)

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[4] "El elemento básico central en todas las concepciones de percepción visual son las imágenes mentales, es decir las representaciones mentales que las personas podemos hacer de objetos físicos, relaciones, conceptos, etc."

[5] "as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties."

[6] "el conjunto de tipos de imágenes, procesos y habilidades necesarios para que los estudiantes de geometría puedan producir, analizar, transformar y comunicar información visual relativa a objetos reales, modelos y conceptos geométricos. La información visual producida (imágenes) puede ser tanto física (figuras o diagramas) como mental (imágenes mentales). El análisis de información visual se refiere tanto a las imágenes producidas por el propio estudiante como a las recibidas desde el exterior (de estudiantes, profesor, texto, etc.). Las transformaciones pueden hacerse entre una imagen e información verbal (oral o escrita) o de una imagen em otra. La comunicación puede ser gráfica, verbal o mixta."

Gutiérrez (1996) argues that visualisation consists of four important elements, namely "mental images, external representations, processes of visualisation, and abilities of visualisation" (p. 9). According to the author, the mental image is the fundamental element for visualisation, being "any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements" (p. 9).

External representation is "any kind of verbal or graphical representation of concepts or properties including pictures, drawings, diagrams, etc. that helps to create or transform mental images and to do visual reasoning" (Gutiérrez, 1996, pp. 9-10).

The visualisation process, according to Gutiérrez (1996), "is a mental or physical action where mental images are involved. There are two processes performed in visualisation: 'Visual interpretation of information' to create mental images, and 'interpretation of mental images' to generate information" (p. 10). Visual interpretation of information is associated with converting abstract relationships and non-figural data into visual terms, manipulating visual images, and transforming visual images into other images. The interpretations of mental images are associated with visual conventions and special vocabulary used in geometric works, diagrams, tables, and graphs, and the interpretation and reading of visual, mental, or physical images to obtain relevant information that can contribute to solving a problem.

Gutiérrez (1996) describes visualisation skills as a set of skills that individuals must develop to carry out processes with mental images to solve a given problem. The main skills highlighted by the author are figure-ground perception, perceptual constancy, mental rotation, perception of spatial positions, perception of spatial relationships, and visual discrimination that individuals use when solving problems.

Figure-ground perception skill is associated with identifying and isolating a specific figure within a complex situation. Perceptual constancy is the ability to recognise specific properties of an object, whether physical or mental and recognise when an object is real or a representation and how it is perceived in different positions. Mental rotation is related to the ability to construct mental images and move them mentally, i.e., to create a mental movement for the image produced in the mind.

The perception of spatial positions is the individual's ability to relate an object, a representation, or a mental image to him/herself (Gutiérrez, 1996). The perception of spatial relationships concerns the ability to relate different objects (or representations or mental images) to each other or simultaneously to oneself. Finally, visual discrimination is the ability to identify similarities and differences when comparing different objects (or representations or mental images).

We will assume visualisation in geometry based on Gutiérrez's (1996) studies, as it is a process that involves the construction of mental images for the construction of convincing mathematical arguments, which provides the practice of mathematics and can contribute to the development of PMTs' geometric thinking.

CONTEXT AND METHODOLOGICAL PROCEDURES

We developed a[7] qualitative and interpretative investigation (Erickson, 1986, 2012) involving thirteen prospective mathematics teachers (henceforth PMTs) regularly enrolled[8] in the subject "Geometry Teaching" in the 2nd year of the mathematics teaching degree course at a state university in Paraná. The course was imparted in the second term of 2022, starting in September 2022 and ending in February 2023[9].

The classes, taught by the first author of the article and the class's head teacher, were held in person in the evening, lasting four contact hours (class hour) per week, totalling a workload of 72 contact hours. For this investigation, eight contact hours weekly were allocated to develop the task.

Before starting the research, we developed a task with the PMTs to discuss elements and characteristics found in the geometric solids that can be associated with EG. Then, to discuss elements of vision and visualisation mobilised by PMTs, we proposed a task called "Understanding spherical geometry," which was adapted by Gomes (2014) based on the publication by Coutinho (2001), which we readapted and organised into three items (Tables 1, 2, and 3).

[7] Approved by the Research Ethics Committee of the State University of Londrina - UEL (Opinion: 5.001.063; CAAE: 50991921.1.0000.5231)

[8] To preserve the identity of the PMTs in the analysis process, the participants are identified with fictitious names: Celeste, Aria, Luna, Órion, Zora, Aurora, and Balthazar.

[9] The calendar was changed due to restrictions caused by the COVID-19 pandemic.

Chart 1

Bear problem, item 1. (Adapted from Gomes, 2014)

1. A bear left its den searching for food and travelled 10 km south. Then it changed direction and travelled another 10 km east. Finally, it changed direction again, heading north, travelling 10 km, thus reaching its den. With this information, draw the path taken by the bear: a) On an A4 sheet of paper. b) On the Styrofoam ball. c) After constructing the drawings, discuss them with your partner, write down all the observations made, and give an explanation.

The purpose of item 1, the "*Bear Problem*," was for the PMTs to recognise that in plane geometry, they would not be able to draw the path taken by the bear so that it could return to its den. However, they could do it in spherical geometry.

Chart 2 shows the statement of item 2, entitled "*Constructing straight lines on flat and spherical surfaces."* In this item, the PMTs should observe that the line on the flat surface grows indefinitely to both sides, while the line on the surface of a sphere, the Styrofoam ball, will always return to the starting point.

Chart 2

Constructing a straight line on a flat and spherical surface, item 2. (Adapted from Gomes, 2014)

2. On a sheet of A4 paper, mark any point and, using a ruler, draw a line that passes through this point. Then, mark any point on the Styrofoam ball and, using an elastic band, trace a path that passes through the marked point. In which situation is it possible to return to the starting point? Why? Describe your answer in detail and explain it.

In Chart 3, we present the statement of item 3 entitled "*Pasting flat triangles onto a flat and spherical surface"* so that PMTs would represent two triangles on a coloured sheet, which could be triangles of different sizes. Next, they were asked to cut out and paste each representation, one on the flat surface and the other on the spherical surface, noting their observations regarding what was happening while pasting them.

Chart 3

Constructing flat triangles on the flat and spherical surface, item 3 of task I. (Adapted from Gomes, 2014)

3. You have received two sheets of A4 paper, one white and one coloured. On the coloured sheet, draw and cut out the representation of any two triangles. Then, glue one of these cut-out triangle representations onto the white sheet and the other triangle representation onto the Styrofoam ball. What did you notice in the triangle representations after the triangles were pasted? Discuss with your partner and describe in detail your conclusions.

The PMTs were instructed to work in pairs and write down all their resolution strategies, post-class reports, and possible questions related to the content and pedagogical knowledge involved in solving the task to present in the collective discussion (large group).

We allocated 20 minutes to resolve each task item and, when necessary, allowed an additional 10 minutes. During this time, the educator guided and encouraged the pairs in the resolution and discussion process. To this end, we analysed the prospective teachers' strategies and resolutions and questioned them about their intentions with the answers and the reports described. Next, we selected pairs to present and explain their solutions on the board, followed by a collective discussion to systematise learning related to geometry content, both flat and spherical.

The sequence of presentation of the pairs had as a criterion the first incorrect resolutions, followed by partially correct resolutions and correct resolutions, disregarding those considered similar. We chose to start with the incorrect resolutions so that it would be possible to draw the attention of the PMTs to some theorems and properties that were being misused.

The task items were delivered one by one separately so that the prospective teachers could write down the resolution strategies for each item, their conclusions, and class reports about the discussions in pairs and the large group before receiving the next item. Each pair received teaching materials to solve each item: white A4 sheets of paper, coloured A4 sheets of paper, Styrofoam balls, a ruler, scissors, glue, coloured pens, measuring tape, and rubber bands.

The instruments to collect data were the PMTs' written registers, the educator's field diary, and the audio recordings from class during work on the task (eight class hours total). We considered the writings during the proposition, exploration, and resolution of the task, as well as notes in the PMTs' notebooks with doubts or particular questions that had not been discussed collectively.

The audio recordings involved the discussions of each pair during the resolution of the task and the collective discussions. Episodes that were the subject of analysis in the investigation were selected and transcribed.

In this research, we assumed learning as a process of negotiating meanings that develops in the context of the everyday experience of participating in the world, one that promotes interaction between a competence regime and an experience of meaning (Schneider, 2012; Wenger, 2003, 2013). That is, competence is historically and socially developed and in interaction with each subject's experience (Wenger, 2003).

First, we identified elements of vision and visualisation mobilised by the PMTs in the set of information obtained. Second, we selected episodes in which these elements were evident. We sought to identify similarities and differences in the identified elements in the third moment. In the fourth moment, we carried out groupings in analysis units articulated with the literature adopted in the investigation, specifically studies on vision and visualisation by Gutiérrez (1996). Finally, in the fifth moment, we made inferences between the information and these theoretical contributions to meet the article's objective.

ELEMENTS OF VISION AND VISUALISATION MOBILISED IN THE TASK - UNDERSTANDING SPHERICAL GEOMETRY

Analysis of the information indicates that the resolution and discussion of the task "*Understanding spherical geometry*" allowed PMTs to mobilise aspects related to geometric thinking, particularly elements of vision and visualisation: visual perception (vision) and elements of visualisation, such as the creation of mental images and representations of geometric objects, external representations, and visualisation processes and skills. These elements are conducive to the development of geometric thinking.

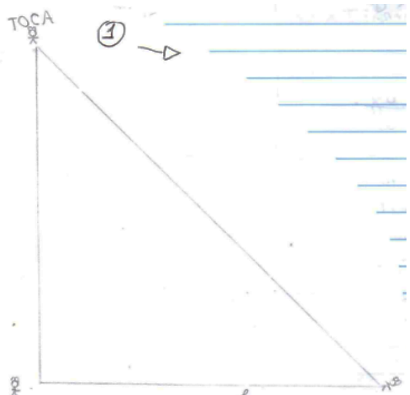
Visual Perception and Learning of the PMTs

Visual perception is one of the elements associated with geometric thinking that is most present in the strategies and resolutions presented by PMTs. This is a global perception in which the subject's attention focuses on visual and physical characteristics of the geometric figures, such as the identification of sides, angles, vertices, colour, and texture, giving more importance to physical than mathematical aspects.

From reading the statement in item 1, Celeste **drew a triangle** to represent the path the bear had taken (Figure 1). This action is presented by **direct access to the statement**, taking the expression "thus arriving at its den" as a point of departure.

Figure 1

Representation on Celeste's A4 paper. (Notebook register on 10/31/2022)



When representing the bear's path on the A4 sheet of paper, Celeste did not comply with the condition of following the north direction. That is, when tracing the bear's path, connecting the point to the east, where it meets with the point to the north, she is taking the northwest (Figure 1) instead of the north direction, as requested in the statement of item 1.

This strategy allows us to question **Celeste's geometric knowledge of NEGs** when trying to represent a triangle on the plane, which would be the bear's possible path, or think about **EG influences** on how we conceive the real world, that is, how we conduct the situations we experience (Chart 4).

Chart 4

Celeste's visual perception. (Audio recording on 10/31/2022)

Celeste*: Here, I actually thought about drawing because I went with the idea of the triangle, it was the first thing, but here it worked, the measurements in my notebook didn't work, so I said they were out of scale.* ***Teacher educator****: You thought of the triangle, why think of the triangle right away? What led you to think about the triangle straight away?* ***Celeste****: Because if it goes down, then to the right, for it to return to it place, it would have to go diagonally.* [...] ***Celeste****: And here I thought the same but now thinking about the ball, which helped me a lot.* ***Educator****: In letter b?* ***Celeste****: In letter b, I marked 10 for this point here, then I marked it here and then I marked 10 back, but I rounded a bit.*

Celeste **drew a triangle** intending to represent the path taken by the bear on the surface of the Styrofoam ball on the A4 sheet of paper (Figure 2). She believed that this representation would be correct and that all sides of the geometric representation constructed on the plane **had equal measurements**, and she argued that **the sides of the triangle were somewhat rounded** because it was on the surface of a ball.

Figure 2

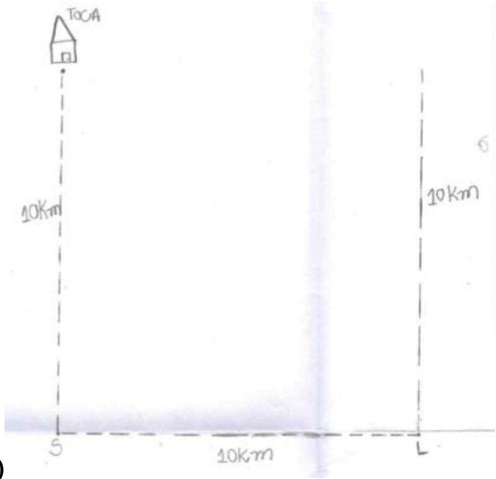
Flat representation of the resolution in the Styrofoam ball, Celeste. (Notebook register on 10/31/2022)

The visual perception Celeste mobilised was also present in item 2, being that **reading the statement may have influenced her to construct her visual image** for a possible solution. Another possibility of influence is, having handled and explored the Styrofoam ball in item 1, this favoured the development of her visual perception, contributing to the construction of a mental representation when noticing that the elastic runs across the entire spherical surface of the Styrofoam ball until returning to the starting point.

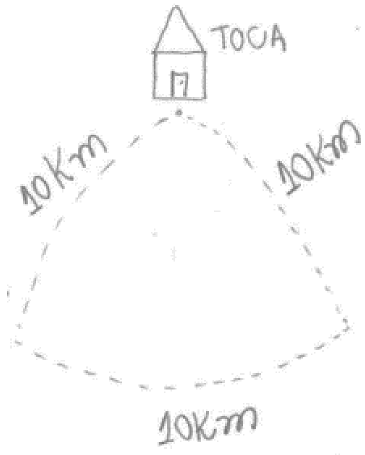
The situation reported by Celeste was repeated with Órion when he argued that **the path the bear took was a triangle** (Figure 3), based on the reading statement in item 1. The construction of a **visual image** may be **associated with their knowledge of EG**, based on the mobilisation of their visual perception.

Figure 3

Representations built by Luna and Órion. (Notebook register on 10/31/2022)



a) b)



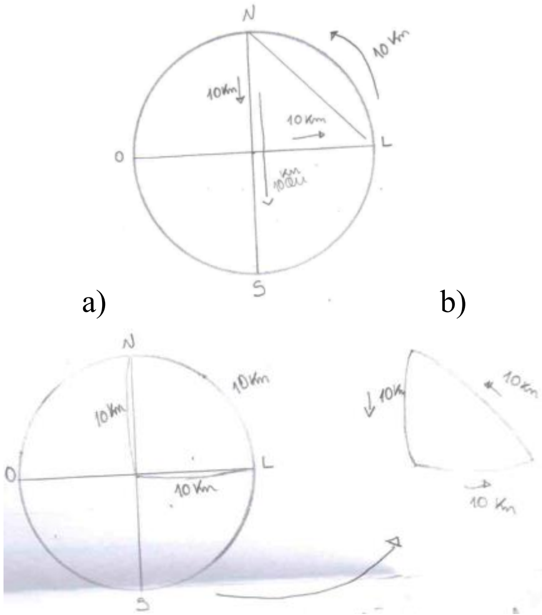
Luna was confused and asked Órion where the bear would go after travelling the 10 km to the east. Órion **explored possible situations** to explain his reasoning to her, building **mathematical arguments** to affirm that the path taken **was a triangle**.

By exploring other strategies with Luna, Órion **moved the visual image** to be able to answer her question. Órion's manipulations make it possible to develop processes and skills associated with geometric thinking, such as visual interpretation of information and interpretation of mental images, and the skills of figure-ground perception, mental rotation, and perception of spatial positions.

Aurora and Balthazar presented other possibilities to represent the path taken by the bear for the situation of the A4 sheet and the Styrofoam ball (Figure 4).

Figure 4

Aurora's resolutions. (Notebook register on 10/31/2022)



Aurora argued that the bear should reach its den, even though the instruction was that the path to follow would be north simply because, **in the statement of item 1, the expression "thus arriving at its den" should be included**.

Aurora stated that the answer to the question was the **same as Celeste's representation** (Figure 1) since the path traced led the bear to its den, **disagreeing with Luna's answer** (Figure 3a), in which the bear headed north without reaching its den.

In the collective discussion of item 1 of the task, Balthazar acknowledged that there had been a **difference between flat and spherical surfaces** and that **representing a spatial object on a plane** was not simple. Balthazar told Aurora that it would be necessary to **take a perspective** to understand the bear's path (Chart 5) and look at the whole without focusing on something specific, as she was doing.

Chart 5

Balthazar's perspective view. (Audio recording on 10/31/2022)

***Balthazar**: Now your perspective is different. Now your perspective is here. Your look in perspective! Aren't you going straight?* ***Aurora**: Oh, okay!* ***Balthazar**: Did you understand?*

Balthazar's representation and explanations during the resolution of item 1 and in the dialogues with Aurora allow us to say that there was a **mobilisation of visual perception added to the knowledge of EG**, **including evidence of knowledge about** NEG, such as the notion of depth and projection in projective geometry.

Visualisation elements mobilised by PMTs

Below, we describe elements of the visualisation mobilised by PMTs in solving and discussing the task "*Understanding spherical geometry*": creating mental images and representations of geometric objects, external representations, and visualisation processes and skills.

Creation of Mental Images and Representations of Geometric Objects

During task resolution, the PMTs presented difficulties in constituting mental images, mainly when they analysed the possible solutions for what was proposed in each item in search of **recognising and validating** some geometric definition or property.

In Chart 6, there is a representative excerpt that illustrates how PMTs seem to mobilise the element associated with visualisation, in this case, the mental image. Órion, in dialogue with Luna, stated that the path taken by the bear was a triangle. This element related to visualisation was also observed in interactions and conversations between other PMTs.

Chart 6

Luna's and Órion's mental images. (Audio recording on 10/31/2022)

Órion: *What is this here? Why?* ***Luna***: *It is to do the teacher's activity.* ***Órion***: *It's going to be a triangle!* ***Luna***: *Oh, is that true?* ***Órion***: *Just in case.*

When representing the path taken by the bear, Balthazar showed a graphic record that allows us to remember a triangle constructed on the surface of a sphere. That is, the representation of a spherical triangle drawn on the flat surface.

According to Balthazar, this representation developed after **attempts to draw** the path taken by the bear through EG content, specifically **circumference and distance** (Chart 7). Given this circumstance, the prospective teacher highlighted the need to work in a higher dimension, i.e., to build a representation on the sphere and, consequently, obtain the correct distance.

Chart 7

Record of discussions between Balthazar and the teacher educator. (Audio recording on 10/31/2022)

Balthazar: *We tried to represent it in a circle. Instead of taking their idea on the sphere, we tried to represent it first on the circumference. So it would go down, then it turns to the side, and then it should arrive. But at that size, to get there, it would no longer work, and it would also not be going directly north, so it would already be an error on two questions. For us to represent this correctly, I would have to represent it on a sphere, which would then*

allow us to take the other missing dimension to give it the correct size and distance.
***Teacher educator**: I get it!*

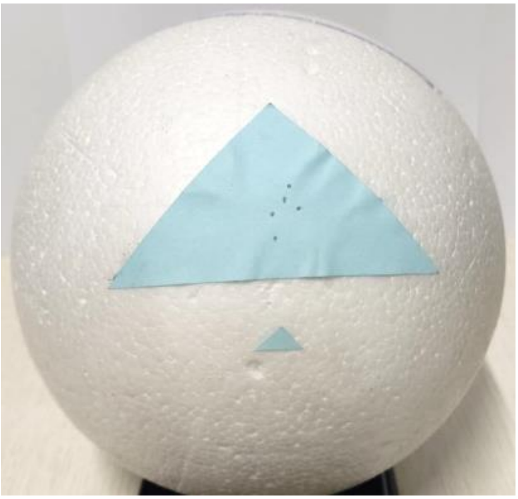
During her justification for the resolution of item 1, Aurora argued that "*the measurements drawn on the paper were not correct, only on the sphere that is not flat*" (Registered on 10/31/2022). Aurora's justification is related to the **drawing attempts** that she had made to represent the path taken by the bear, stating that it was not possible to use the measurements indicated in item 1.

For item 2 of the task, Aurora said it **would only be possible to return to the starting point on the ball**. We understand that the answer produced is part of a mental construction, that is, of a mental image that is **the cognitive representation from a visual element**, the Styrofoam ball.

In item 3, the PMTs argued that it is not possible to represent a flat triangle on the surface of a sphere because it is in a higher dimension, a three-dimensional space (Figure 5).

Figure 5

Luna's resolution of item 3. (Photographic register on 11/07/2022)



In the pair discussion, Luna and Órion argued that it was impossible to represent a flat triangle on the surface of the Styrofoam ball. Moreover, they said that the representation of the flat triangle would not touch all its points on the surface of the Styrofoam ball (Chart 8).

Chart 8

Dialogue between Luna and Órion. (Audio recording on 11/07/2022)

Órion*: Okay! I see. It is impossible to represent a triangle on a ball, on a sphere. Oh, shoot! Why is that?* ***Luna****: Because not all...* ***Órion****: Why is that?* ***Luna****: It's because the triangle doesn't touch all the sides of the sphere perfectly.* ***Órion****: And why is that?* ***Luna****: Because it is not in plane geometry, it is not in the plane. It is in a sphere.* ***Órion****: It's because we're trying to paste a representation of a triangle present in plane geometry onto a sphere, in spatial geometry. Let's put this!*

Considering such discussions, the PMTs could confirm that **it is not always possible to adequately represent in a two-dimensional space a situation that occurs in a three-dimensional space.** Therefore, it is necessary to consider the system in which the representation is produced.

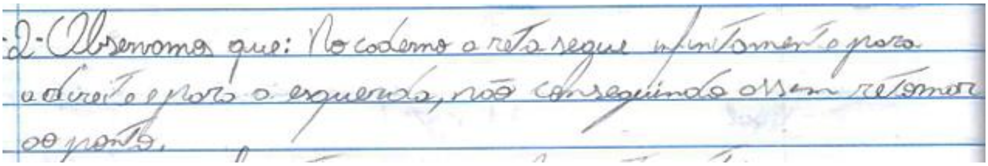
Based on the evidence presented, we can infer that **mental images and geometric representations** are essential in geometry learning, especially in constructing geometric knowledge, intending to create or transform written or graphic information into possible strategies for solving geometry problems.

EXTERNAL REPRESENTATIONS

Aurora and Balthazar used **verbal representation** to justify their answer to item 2, stating that the line grows indefinitely to the left and right, portraying one of the contents covered in the task (Figure 6).

Figure 6

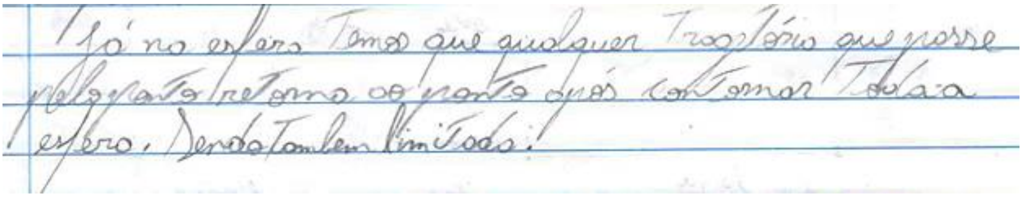
Verbal representation of Balthazar's item 2. (Notebook register on 10/31/2022)


2- Observamos que: No caderno a reta segue infinitamente para a direita e para a esquerda, não conseguindo assim retornar ao ponto.

Balthazar explained that, when passing the elastic over the surface of the Styrofoam ball, the line returns to the starting point after going around the entire Styrofoam ball, which is a **limited** straight line (Figure 7). That is, a **straight line with a limit** (a length corresponding to a turn around the surface of a sphere has a beginning and an end), but we can **go around it as many times as we want**.

Figure 7

Verbal representation of Balthazar's item 2. (Notebook register on 10/31/2022)


Já na esfera temos que qualquer trajetória que passe pelo ponto retorna ao ponto após contornar toda a esfera, sendo também limitada!

A verbal representation that resembles Balthazar's written record was made by Celeste when observing what happens on the A4 sheet of paper and the Styrofoam ball, stating that the situation in which it is possible to return to the starting point is in the Styrofoam ball.

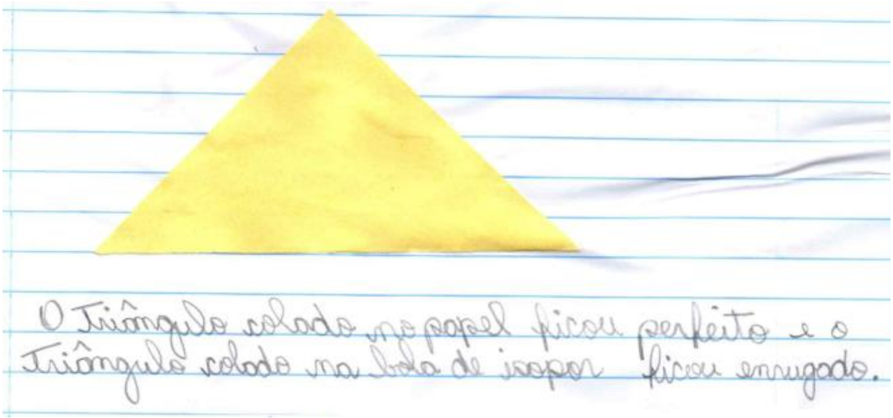
Celeste presented a justification for her resolution of item 2, which we identified as a verbal representation, written informally and without using geometric knowledge, without defining that the elastic represents a straight line on the spherical surface and that it is a limited and infinite straight line.

The resolution presented by Luna and Órion for item 2 is similar to what was offered by Balthazar and Celeste. Luna and Órion used a verbal and graphical representation. They stated that it would only be possible to return to the starting point in the Styrofoam ball and that, in plane Euclidean Geometry, it would not be possible to return to the starting point because the line grows indefinitely to both sides from any point on the plane.

For item 3, Celeste presented a graphic representation (pasting the flat triangle to the flat surface) and a verbal representation (without undertaking definitions and theorems) (Figure 8). Her arguments allow us to infer that she recognised the imperfection of the pasting of the flat triangle to the surface of the Styrofoam ball.

Figure 8

External representation for Item 3 - Celeste. (Notebook register on 11/07/2022)



In a section of the dialogue between Celeste, Zora, and Aria, in which Aria doubted pasting the flat triangles onto the flat and spherical surfaces, Celeste highlighted that **one of the triangles was on the plane and the other was on the spherical surface** (Chart 9). Aria identified the **difference between the two geometric surfaces**, asking whether the next step would be just to write the answer.

Chart 9

Dialogue between Celeste, Zora, and Aria. (Audio recording on 11/07/2022)

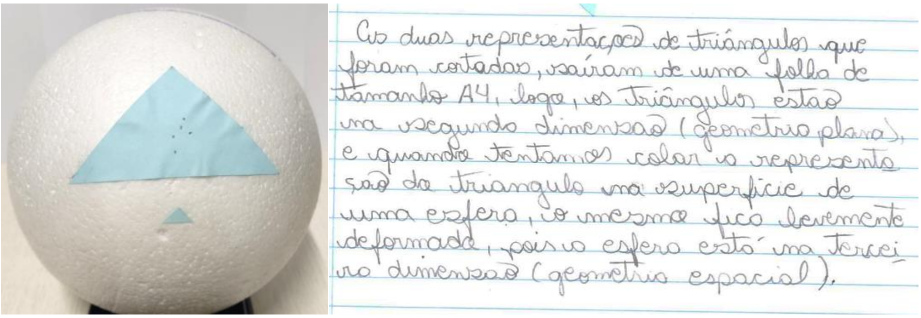
***Aria**: I don't know what happened.* ***Zora**: What?* ***Celeste**: Didn't you see that it pasted perfectly here? It doesn't paste here, oh, because it's not flat. That's the logic!* ***Aria**: Now, it is just writing.*

To motivate the PMTs, the educator asked them whether or not it was possible to stick other triangles of different sizes on the surface of the Styrofoam ball and what would happen to the flat triangle. Aria argued that if they had used a **smaller triangle** with **smaller sides**, the **plane representation of the triangle** on the surface of the Styrofoam ball **would have approximated a situation made in the plan**. Or rather, the triangle would not contain so many deformities when glued to the surface of the Styrofoam ball.

Luna and Órion reiterated Aria's statement **when they represented**, in a Styrofoam ball, a **sufficiently small flat triangle**, with **smaller measures** compared to the flat triangle initially represented, to the point where it becomes almost impossible to handle it with their hands and **found that no deformities were observed in this flat triangle** when glued to the surface of the ball (Figure 9).

Figure 9

Órion's external representations. (Notebook register on 11/07/2022)



Luna's exploration and questions allow us to say that she mobilised her mental image and her articulation with the external (verbal and graphic) representation, intending to explain whether the identified fact is valid for any flat triangle to be pasted onto the surface of a Styrofoam ball, no matter how large or small it may be.

The exploration carried out with the materials used in solving the task allowed PMTs **to work on their visual perceptions** because they have access to the task statement (an external representation) and can articulate the **written register with the representation made on paper and the ball** (verbal and graphic representation), directly associated with the mental image constructed by them.

Luna and Órion acknowledged the difficulty of representing **three-dimensional objects in two-dimensional space** and, consequently, the **difference between flat and spherical surfaces** (Chart 10).

Chart 10

Dialogue between Luna and Órion. (Audio recording on 11/07/2022)

Luna: *Why is one in plane geometry and the other???* ***Órion***: *One is in two dimensions and the other in three. Yes, that's it!* [...] ***Luna***: *Simple! One is in plane geometry and the other is in spherical geometry.*

Like Luna and Órion, Balthazar managed to articulate the **mental image constructed with external representation**, reporting his difficulty in pasting the flat triangle to the surface of the Styrofoam ball. For the **triangle to be flat,** it is "*easier*" to stick it on the flat surface than on the spherical surface.

The evidence presented highlights the importance of **external (verbal or graphic) representations in geometry teaching**, as it requires conversion and treatment processes between the information provided and/or constructed, whether verbal or graphic. It is worth highlighting the conversions between verbal and graphic information when undertaken by the PMTs, which must have intrinsic relationships with each other so that it is possible to understand the reasoning and the resolution strategy that was constructed.

VISUALISATION PROCESSES AND SKILLS

Regarding visualisation processes and skills, the PMTs mobilised the visual interpretation of information and the interpretation of mental images (visualisation process), figure-ground perception, perceptual constancy, mental rotation, and perception of spatial positions (visualisation skills).

During the task resolution, the PMTs made **changes between external representation registers**. They modified their strategies, alternating between graphic and verbal registers, sometimes looking for a justification that would suit the mental image and/or graphic register and sometimes looking to understand the verbal record from the graphic register.

The subprocess that the PMTs most mobilised in exploring the task was the "visual interpretation of information" subprocess, which is associated with interpreting abstract relationships and non-figural data in visual terms, in addition to manipulating and transforming visual images.

In Chart 11, we have an excerpt from the dialogue between the educator and the PMTs (collective discussion), in which there are signs of the presence of the subprocess of visual interpretation of information.

Chart 11

Visual interpretation mobilised in Item 2. (Audio recording on 10/31/2022)

Teacher educator*: In which of the situations is it possible to return to the starting point? Why? Describe your answer in detail and explain it.* (Reading the statement of item 2 of the task). ***Celeste****: On the ball!!!* ***Teacher educator****: On the ball, but why is it possible to return on the ball?* ***Andrieli****: It's flat around here!* ***Educator****: It's flat! And what else? Besides the plan, what happens?* ***Bruna****: In the Styrofoam ball situation!* ***Jhonatam****: They are, I don't know. By the point, they are opposite sides, like, there is no way for them to meet if it is not a sphere.*

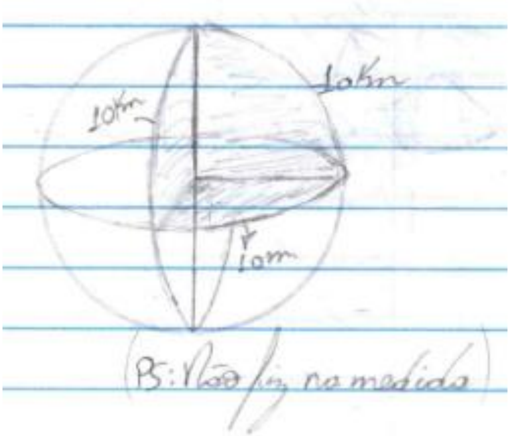
The visual interpretation of information is also witnessed in the dialogue between Luna and Órion when he said, at the end of reading the statement in item 1, that **the path taken by the bear would be a triangle**. It involves an interpretation of a written record into a mental image that will become a graphic representation.

The subprocess of “visual interpretation of images” is associated with knowledge of visual conventions, the use of spatial vocabulary used in work involving geometry concepts, in addition to work with graphs, tables, and diagrams from the most varied contexts, as well as the reading and interpretation of mental or physical images.

This subprocess could also be identified when Baltazar searched for **other strategies for solving** item 1 using **mental images and external representations** (Figure 10).

Figure 10

Balthazar's interpretation of images. (Notebook register on 10/31/2022)



Balthazar's representation takes us back to a **geometric object** specific **to spherical geometry**: the **spherical triangle tri-rectangle**. When asked about the strategies used to construct this representation, Balthazar argued that **just remembered to make the representation of a sphere on the A4 sheet of paper** and, on it, trace the path taken by the bear. After attempts to draw pictures to explain the path taken by the bear, Balthazar made a graphic representation depicting a sphere and the bear's possible path, which shows evidence of the mobilisation of visualisation processes and skills.

When Aurora and Balthazar discussed **the path in the northern direction to be taken by the bear**, they mobilised their subprocess of visual interpretation of images (Table 12).

Chart 12.

Dialogue between Aurora and Balthazar. (Audio recording on 10/31/2022)

***Balthazar**: We don't know how to draw the sphere.* ***Aurora**: So none of them go to the north if that's the case!* ***Balthazar**: It works, that one over there works! That one there is the representation of a sphere. They took a part of the sphere. As if they had only taken this here.* ***Aurora**: Independent. Still it's going West. Where is it going north? Doesn't it have to go straight up?* ***Balthazar**: Oh, it came down, after we come down...* ***Aurora**: Doesn't it have to go straight?* ***Balthazar**: It's straight!* ***Aurora**: No! Oh!* ***Balthazar**: It's just that when you look at it like that you're seeing a triangle and it's not a triangle, straightforward. It has the straight lines here!* ***Aurora**: Here, for example. If here it is standing [vertical] over here, then I come here straight.* ***Balthazar**: Now, your perspective is different. Now your perspective is here. Aren't you going straight?* ***Aurora**: Oh, okay!*

Not all PMTs could mobilise image and representation modification processes in their resolution strategies. The visualisation skills they mobilised were figure-ground perception, perceptual constancy, mental rotation, and perception of spatial positions.

The visualisation skills evidenced in the excerpts of the audio recordings and in the figures that preserve the PMTs' registers are identified as "figure-ground perception" and "perceptual constancy" when Órion **answered immediately to the educator after finishing reading** item 1.

The skill of "perceptual constancy" was observed in Balthazar's speeches and registers, and "mental rotation" was mobilised when he provided **different graphic records from his mental image**, as shown in Figure 10.

Another visualisation skill mobilised by Balthazar was the "perception of spatial positions," which occurred when he commented to Aurora about **having a perspective view to recognise that the path taken by the bear over the Styrofoam ball would be a triangle**, as per the excerpt from Chart 5.

"Perceiving spatial positions" was also observed when Aurora and Balthazar **discussed the path to the north for the bear to take**, according to the excerpt in Chart 12.

In the evidence presented, it is possible to identify that only the skills of figure-ground perception, perceptual constancy, mental rotation, and perception of spatial positions were demonstrated by Balthazar and Órion. During the resolution of the task, not all PMTs employed visualisation skills in their strategies.

VISION AND VISUALISATION AND THE MOBILISATION OF SPHERICAL GEOMETRY CONCEPTS

In the exploration of the task *Knowing Spherical Geometry,* the PMTs had the opportunity to mobilise **knowledge associated with geometries** and **geometric thinking**, like the **visual perception** related to **vision** and elements related to **visualisation**, namely the creation of **mental images and representations of geometric objects**, **external representations**, **and visualisation processes and skills**.

Regarding the **visualisation processes**, the PMTs mobilised the **visual interpretation of information** and **mental images**. As for the **visualisation skills**, the PMTs mobilised **figure-ground perception**, **perceptual constancy**, **mental rotation**, and **perception of spatial positions**.

In the interactions during the collective discussions of the task, we sought to identify **knowledge of geometries** mobilised by the PMTs, especially those who went through a **resignification process** from their engagement and participation.

In the process of negotiating meanings triggered by the resolution, discussion, and systematisation of learning, we identified **learnings** of the PMTs regarding:

- Potentialities observed in the different strategies for solving the task, some involving plane geometry and spherical geometry, which demonstrated the mobilisation by some PMTs of elements of vision and visualisation associated with geometric thinking;

- Weaknesses, when working with geometry concepts, when developing resolution strategies, indicating that some PMTs were not too familiar with the content when dealing with situations involving geometry (for example, the concept of a straight line, the notion of an angle, the concept of a sphere);

- Potential of the manipulative materials made available to PMTs as possible resources for exploring and solving the task, with the support of written and graphic registers;

- Choices of appropriate verbal and graphic representations for each task item.

From the systematisation process of item 2, the PMTs proved to be more **engaged and participatory** when working on the task. They could **re-signify concepts of plane Euclidean geometry** (for example, the notion of a straight line) and **understand concepts of spherical geometry**.

In item 2, the PMTs could represent a straight line on the flat surface of an A4 sheet of paper, and in the discussion, they stated that the straight line on the plane was infinite and unlimited. In spherical geometry, they could observe that the line on the surface of the Styrofoam ball (representation of a sphere) returned to the starting point when going around its entire surface, being a line that has a beginning and an end and also allowing it to go around as many times as we want.

This care PMTs had with the geometric space, the object of study, was observed in discussions in pairs (between pairs) during the resolution of the task items, specifically in items 1 and 3, when they explored two geometric surfaces, one flat and the other spherical. In the collective discussion, the PMTs could conclude that situations on the plane rarely occur on the surface of a sphere and vice versa.

The articulation between objects in two-dimensional geometric space with three-dimensional geometric space was challenging for PMTs (Duval, 1999; Elia et al., 2018; Swoboda, 2008).

The PMTs' **lack of knowledge of NEGs** may have some involvement in the difficulties they faced in solving the proposed task. In each item, the PMTs tried **to justify their solving strategies** through **EG knowledge** because these were the only principles they knew.

Although the EG is a subject studied in the 1st year of the undergraduate course, the PMTs revealed misunderstanding in some concepts (for example, the concept of line, half-line, and line segment, the notion of angle in plane Euclidean geometry, concepts between relative positions between lines) that were clarified during the negotiation of meanings and the systematisation of learning.

As the discussions progressed, the PMTs did not misuse the concepts, embodying the new meanings and producing convincing mathematical arguments. According to Melo (2005), "the teacher's broad and deep mastery of the teaching subject is fundamental and necessary, especially when seeking curricular innovation" (p. 39).

The PMTs **actively participated in the negotiations of meanings**, **sharing information and questions** about the geometries and **legitimising resolution strategies** produced by colleagues. The teacher educator encouraged them to share their strategies and reasoning, leading collective discussions and their participation in legitimising the knowledge produced.

While negotiating meanings, we identified evidence of PMTs' learning from their **written and graphic registers** regarding visualisation elements, namely the creation of mental images and representations of geometric objects, external representations, and visualisation processes and skills.

According to Elia et al. (2018), **visualisation** in geometry is an essential cognitive process of geometric thinking that involves recognising geometric shapes that can be identified in some figural unit beyond the possibility and manipulating and operating with these figures. Thus, visualisation makes it possible to simultaneously grasp a geometric object in its entirety, distinguishing all its figural units and their specificities.

Visualisation, like representation, is at the heart of mathematical activity carried out by students and teachers. It is not intuition because it is not mere visual perception. Visualisation in mathematics is necessary because it allows us to organise the relationships between the different forms of register for the same mathematical object (Duval, 2014; Elia et al., 2018)

The PMTs used verbal and graphic registers as they progressed through the discussions in pairs. They intended to **develop geometric strategies** (representation of flat and spatial figures) **and/or mathematics** to lead to the task solution (Gutiérrez, 1996). The PMTs established relationships between **mental images and external representations,** seeking to create and/or transform mental images into verbal or graphic registers so that they could express their ideas.

Access to external registers may have contributed to forming the **visual perception** of the PMTs, and some had more difficulties than others. Such difficulties may be related to how the PMTs interpreted the information in the item statements or even how they created the visual image of the situations.

Gutiérrez (1992) states that **visual perception is fundamental to subjects' mental representations**. That means they are the representations of physical objects or situations people make based on their observations.

Duval (1999) says that **visual perception immediately fixes the vision of some forms** (objects and situations), that this evidence makes them firm, and that geometry teaching should start from **geometric intuition** emphasised in perception.

V**isual perception** has limitations because we live in a three-dimensional world from which we can only grasp a part of the explored object. When we access any visual representation of physical objects outside of mathematics, it is only possible to focus on a specific side of this object, which can be explored by the movements of the physical object or the subject who is observing it (Duval, 1999; Elia et al., 2018).

The elements of vision and visualisation PMTs mobilised while solving the task were significant, and led them to learn spherical geometry concepts, including developing geometric thinking. PMTs could recognise the contribution of other geometries, such as NEGs, to the constitution of the field of mathematics.

FINAL CONSIDERATIONS

The research highlights the critical presence of Euclidean geometry (GE) in solving tasks and decision-making related to the construction of resolution strategies, mainly because PMTs studied geometry concepts since the beginning of their schooling, with Euclidean geometry being presented until the last century, as absolute, indisputable truth.

At the beginning of the work on the subject, some PMTs claimed to know the basics of NEGs through their basic education teachers, who had commented that other geometries existed. Other PMTs reported that they had never studied anything about NEGs and only knew about plane and spatial Euclidean geometries.

Working with the task supported by manipulative materials allowed PMTs to re-signify learning about geometries, especially EG and spherical geometry. Furthermore, it allowed the mobilisation of elements associated with vision and visualisation, intrinsically related to the development of geometric thinking.

The exploration of the tasks with the help of manipulative materials, which composed each item and the teacher educator's actions, led the PMTs on a reflective path on the concepts and processes of teaching and learning geometry. These materials worked as motivating elements, encouraging them to identify what happens in practice and to seek explanations and foundations for their actions and interactions. While exploring the proposed situations, the PMTs could develop essential elements for geometric thinking, such as vision and visualisation, reflected in the written registers in the notebooks and recorded dialogues.

One question regarding the manipulative materials PMTs use is their limitation, as they do not allow for abstraction beyond what is explored in each situation proposed in the task. Perhaps further investigations should analyse the contributions of dynamic geometry software for the development of geometric thinking and the contribution of vision and visualisation in teaching EGs and NEGs, especially in PMTs' initial and continuing education.

We emphasise the importance of working with the mathematics teachers and prospective mathematics teachers on the NGE concepts, which are so close to all of us, meeting our needs as much as EG since it can favour the teaching and learning of other mathematical concepts, including EG itself with the possibility of promoting the development of geometric thinking.

The research was conducted with a small group of PMTs with very particular characteristics. They came from a post-pandemic period, and the EG subject had been worked on completely asynchronously and axiomatically. We consider expanding it to other MTs and PMTs with different characteristics relevant, which may reveal complementary aspects.

In conclusion, research involving spherical geometry played a significant role in developing PMTs' geometric knowledge, favouring the development of geometric thinking. Furthermore, this investigation allowed the redefinition of previous knowledge of spherical geometry, allowing participants to explore situations through manipulative materials, which probably foster geometric reasoning.

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AUTHORSHIP CONTRIBUTION STATEMENT

TFL and MCCTC conceived the presented idea and developed the theory. TFL collected the data. All authors analysed the data, discussed the results, and contributed to the final version of the manuscript.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study are under the responsibility of TFL. They may be made available upon reasonable request by other interested parties for five years upon signing a liability agreement.

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