ISSN: 2178-7727
DOI: 10.17648/acta.scientiae.8220

Relationships in the Students' Process of Measuring Area

Marta Teixeira[a]
Maria de Lurdes Serrazina[b]
João Pedro da Ponte[a]

[a] Universidade de Lisboa, Instituto de Educação, Lisboa, Portugal
[b] Instituto Politécnico de Lisboa, Escola Superior de Educação, Lisboa, Portugal

ABSTRACT

Background: Several studies indicate that students have difficulties understanding the concept of measure and the measurement process. Conceptual knowledge and procedural knowledge are essential when learning to measure an area because the connection between both favours solid mathematical learning. **Objective**: Identify the relationships students establish in measuring an area by tessellating and using non-standard units of measurement (u.m.s), how these relationships develop, and what strategies they use. **Design**: We follow a qualitative-interpretative approach in the design-based research modality. **Setting and participants**: The study was conducted in a public school in Portugal in a 3rd-grade class (18 students between 8 and 9 years old). **Data collection and analysis**: Participant observation, video and photographic records, students' written productions, and field notes. Content analysis of the moment of whole-class discussion of the students' strategies and the actions of the teacher and researcher. **Results**: Three relationships established by the students were identified: between the u.m. and the corresponding value, between the u.m.s with each other, and between the values of the measure with different but related u.m.s. Three measurement strategies were also identified: full tessellation, partial tessellation, and compensation. **Conclusions**: Students developed a significant understanding of the measurement process. The whole-class discussion and the actions of the teacher and researcher made a substantial contribution. We suggest tasks that facilitate exploring these relationships, favouring the transition to standardized u.m.s and promoting more significant learning of the measurement process.

Keywords: area measure; measurement process; relationships in the measurement process; learning; mathematics.

Corresponding author: Marta Susana Meireles Teixeira. E-mail: martateixeira@edu.ulisboa.pt

Relações no Processo de Medição da Grandeza área pelos Alunos

RESUMO

Contexto: Diversos estudos indicam que os alunos apresentam dificuldades na compreensão do conceito de medida e no processo de medição. Na aprendizagem da medição da área é essencial que desenvolvam conhecimento conceptual e processual, pois a ligação entre ambos favorece uma aprendizagem matemática sólida. **Objetivo**: Identificar as relações que os alunos estabelecem no processo de medição da área, por pavimentação, usando unidades de medida (u.m.) não padronizadas, como é que essas relações se desenvolvem e que estratégias usam. **Design**: Seguimos uma abordagem qualitativa-interpretativa, na modalidade de investigação baseada em design. **Ambiente e participantes**: O estudo foi realizado numa escola pública em Portugal, numa turma de 3.º ano (18 alunos, 8 e 9 anos). **Coleta e análise de dados**: Recolha de dados por observação participante, registros vídeo e fotográficos, produções escritas dos alunos e notas de campo. Análise de conteúdo do momento de discussão coletiva das estratégias dos alunos e das ações do professor e da investigadora. **Resultados**: Identificaram-se três relações estabelecidas pelos alunos: entre a u.m. e o valor correspondente; entre as u.m.; e entre os valores da medida com u.m. diferentes, mas relacionadas. Foram também identificadas três estratégias de medição: pavimentação total, pavimentação parcial e compensação. **Conclusões**: Os alunos desenvolveram uma compreensão significativa do processo de medição. A discussão coletiva e as ações do professor e da investigadora deram um forte contributo. Sugerimos a realização de tarefas facilitadoras da exploração destas relações, favorecendo a transição para as u.m. padronizadas e promovendo uma aprendizagem mais significativa do processo de medição.

Palavras-chave: medida de área; processo de medição; relações no processo de medição; aprendizagem; matemática.

INTRODUCTION

Measure is one of the fundamental mathematics realms that allows us to link two essential themes: geometry and numbers. It not only establishes connections with other domains of knowledge but also plays an important role in the construction of mathematical concepts, such as rational numbers, and in understanding statistical principles (NCTM, 2007; Smith III & Barrett, 2017). Measurement activities develop essential skills for everyday life, reinforce fundamental mathematical concepts, and promote interdisciplinary connections. Furthermore, they encourage students' active and meaningful learning, contributing to a more structured and interconnected mathematics understanding. Although measuring is an ordinary practical activity and the ability to measure correctly is an essential skill, the teaching of measures does not always receive due attention, especially when compared with teaching numbers and operations (Smith III & Barrett, 2017), which can impair the in-depth understanding of concepts related to measure and the relationships between different magnitudes.

Several studies indicate that students have difficulties understanding the concept of measure and the measurement process, including measuring area (Clements et al., 2018; Cullen et al., 2018), a reality that several countries share (Smith III et al., 2016). Premature introduction of rules and procedures without understanding may be the origin of students' difficulties with tasks focused on measurements (Clements et al., 2018; Smith III & Barrett, 2017). In the case of the magnitude "area", learning to measure involves the development of procedural knowledge, such as identifying the space to be measured, choosing the unit of measurement (u.m.) and the appropriate measuring instruments, measuring the area by tessellating or iterating the u.m. and counting the number of u.m.s used, decomposition and recomposition of figures, use of the rectangular structure, application of specific formulas and conversion between different u.m.s. However, students must also develop their conceptual knowledge of aspects such as area, u.m., conservation, the relationship between area and perimeter, compound units, and row and column. According to Van de Walle et al. (2020), the connection between this knowledge favours solid mathematical learning, allowing students to develop effective strategies to calculate a polygon area.

This article focuses on an informal approach to measuring an area, where students resort to using and counting non-standardized u.m.s to build a progressive conceptual understanding of measure and the measurement process. In this context, our objective is to identify the relationships that students establish in the measurement process by tessellating and using a non-standardised u.m.s when solving an exploratory task. To this end, we seek to answer the following questions: (i) What relationships do students establish in the measurement process? (ii) How do these relationships develop when students solve an exploratory task, and what strategies do they use?

THEORETICAL FRAMEWORK

Concept of magnitude

A magnitude can be defined as an attribute or property of an object or phenomenon that can be compared and quantified (Silva et al., 2016). A magnitude can be seen as a set of quantities that can be organized and compared according to a criterion, making it possible to perform operations between those quantities, such as an addition, with its commutative, associative properties, and the existence of a neutral element (Ponte & Serrazina, 2000).

The area magnitude allows us to analyse different flat regions and classify and compare them. For example, if we have a set of sheets of paper of three different sizes, we can group them into small, medium, and large categories, according to the area they occupy. Each category constitutes an equivalence class (Breda et al., 2011; Passelaigue & Munier, 2015), i.e., a set of figures with the same area. Within each class, the regions are said to be equivalent. Regions that belong to different classes can be compared and ordered. For example, a medium paper sheet has a larger area than a small one. Besides classification and ordering, it is also possible to add areas. For example, if we place two A4 sheets side by side without overlapping them or leaving a gap, the total area obtained is the sum of the individual areas of each of the sheets. This principle is important for understanding the calculation of the area of composed figures. Ponte e Serrazina (2000) consider that tasks involving these comparisons and operations promote the development of the concept of magnitude. Thus, tasks encouraging students to estimate, measure, and compare areas, such as tessellating a figure with paper squares, facilitate understanding of the concept. Passelaigue e Munier (2015) add that, after this work, tasks involving measurement should be proposed.

Area, length, and volume are geometric magnitudes, as they are associated with attributes or properties of geometric figures and the relationships established between them. Understanding the magnitude area is a challenge for students, as it involves coordinating two dimensions (Outhred & Mitchelmore, 2000) and the ability to visualise subdivisions and compositions of two-dimensional figures (Smith III et al., 2016).

Area, area measurement, and measure of an area

The concept of area refers to the amount of two-dimensional region contained within a closed line (Clements & Sarama, 2009; Cullen et al., 2018; Smith & Barrett, 2017). Measuring the area consists of determining how much region is contained within that line (Lehrer et al., 2003) using suitable, standardized, or non-standardized u.m.s. The area measure corresponds to the number of u.m.s needed to cover that region, thus transforming continuous quantities into discrete quantities and dividing them into countable sets of parts of equivalent dimension (Smith III et al., 2016). Choosing the right one is essential for this process. Breda et al. (2011) consider the distinction between area and area measure fundamental for meaningful mathematical learning.

Unit of measurement and measurement process

U.m.s play a central role in the measurement process. Breda et al. (2011) define them as the quantity of a magnitude used compared to other quantities one intends to measure. In this regard, Smith III and Barrett (2017) consider the u.m. a concept instead of a physical object; that is, the u.m. is not an object but one of its attributes, although this distinction is often ignored. For the authors, the u.m. is, conceptually, only a part of the quantity to be measured, and using physical objects as u.m.s is a logical and essential practice –but one that can lead to errors, making this distinction difficult. Correa et al. (2024) highlight the importance of this conceptual understanding because it allows students to develop a deeper understanding of a measure. Using different terms as synonyms can also obliterate the distinction between an object and its attribute.

Cullen et al. (2018) indicate that students have difficulties understanding a u.m. Smith III et al. (2016) consider that this understanding develops progressively as students explore different conceptual properties. Some authors (Lehrer et al., 2003; Ponte & Serrazina, 2000; Stephan & Clements, 2003) believe that in the first experiences with the area, students must tessellate regions with an u.m. of their choice, discuss and analyse the results citing the processes used, and explain the chosen unit. Those discussions facilitate the development of mental images that help students visualise a region divided into countable sub-regions, thus overcoming the difficulties that sometimes arise in subdividing the unit. Ponte and Serrazina (2000) also highlight the importance of using different u.m.s in the tessellating process, as this leads students to discriminate the selected one, which enables the understanding that the unit can vary, while promoting the discovery of relationships between different units. Along the same lines, some authors (NCTM, 2007; Ponte & Serrazina, 2000; Smith III & Barrett, 2017) argue that students should experience several u.m.s before using standardized u.m.s and should be encouraged to select those units, taking into account the specific magnitude with which they are working and the quantity associated with that magnitude, which facilitates a gradual and more effective transition to the use of standardized u.m.s, allowing for more meaningful learning.

Smith III et al. (2016) report that students tend to select u.m.s similar to the region to be measured. Lehrer et al. (2003) highlight that the measurement process is more effective when there is a correspondence between the dimensions of the u.m. used and the dimensions of the area to be measured. In this regard, Zacharos (2006) suggests using flat figures as u.m.s to make the measurement process more suitable. Students must experiment with different u.m.s, understanding that the choice of the u.m. influences the value of the area measure. Although the space remains constant, larger u.m.s cover more space, requiring fewer iterations, while smaller u.m.s require more iterations to cover the same region, which results in a larger area measure value, as mentioned by Smith III et al. (2016). Understanding this inverse relationship between the size of the u.m. and the number of u.m.s required is essential in the measurement process, as it allows for a greater understanding of the u.m., namely how the measurement result varies depending on the required size and quantity of u.m.s, which contributes to a more appropriate choice of a u.m. for each situation. This makes the measurement process more precise and adapted to different contexts. Correa et al. (2024) add that understanding this relationship is important for comparison, establishing equivalences, and understanding scales.

According to NCTM (2007), the process of measuring any magnitude is identical and must follow a structured progression: 1) choosing the u.m.; 2) comparing the u.m. with the magnitude to be measured; and 3) determining the number of u.m.s required, obtained through iterating the u.m. and counting the number of iterations or using a measuring instrument. Through these steps, students understand how the quantity to be measured relates to the chosen u.m. and, consequently, to the number of iterations needed to obtain the measurement value. Grant and Kline (2003) attribute more emphasis on exploring the relationships between a u.m. and a measure of an area, considering the measurement process more dynamic and flexible, highlighting the following steps: 1) choosing an appropriate u.m. to measure the attribute; 2) exploring the relationship between the size of the u.m. and the number of u.m.s needed to measure, which implies understanding how the size of the u.m. influences the value of the measurement; 3) working with measures that involve whole u.m.s and parts of a u.m., allowing students to work with fractions and with higher measure precision; and 4) understanding the inverse relationship between the size of the u.m. and the number of u.m.s, helping students to visualise how changing the size u.m. influences the quantity of u.m.s needed to cover the region.

Clements and Sarama (2009), just as Stephan and Clements (2003), identify four fundamental processes in learning the area measurement process: 1) equitable partition, 2) unit iteration, 3) conservation, and 4) rectangular structuring. These processes are related to the understanding of u.m.s and the organization of space, which facilitates visualising and obtaining the measure of the area of a given region in a coherent and precise way. On their side, Smith III et al. (2016) highlight five steps related to a more formal and abstract understanding of the area measurement process: 1) conserving the area as a quantity, 2) understanding the u.m.s, 3) structuring the rectangular space into composed units, 4) understanding the area formulas, and 5) distinguishing between area and perimeter.

Exploring the magnitude of an area involves dealing with two dimensions, which makes it difficult to understand. Learning concepts requires an integrative approach, as it mobilises knowledge, ideas, and skills developed in learning the magnitude length, constituting the basis for learning the magnitude volume. Therefore, these three magnitudes must be learned progressively, interconnected and adapted to the different levels of underlying complexity, allowing students to understand their relationships and build an integrated vision of geometric quantities.

Whole-class discussion and teacher's actions

Whole-class discussion is essential to develop an understanding of concepts, and for mathematical learning to be meaningful (NCTM, 2007), it should focus on students' thinking and promote mathematical ideas (Stein et al., 2008). Thus, the whole-class discussion constitutes a fundamental moment in a mathematics class, promoting the active participation of students, encouraging them to share and justify their ideas through argumentation and to critically and constructively question the ideas of others (Canavarro, 2011). This process is essential for constructing new knowledge, as current curriculum documents recommend.

Stein et al. (2008) believe that conducting whole-class discussions is essential in teacher practice, where students' ideas are presented and complemented, transforming into more precise and robust mathematical concepts. As this is a complex practice, its preparation is essential to ensure its effectiveness and allow the teacher to be prepared to face the challenges that may arise and take advantage of unexpected opportunities that may enrich students' learning (Duarte et al., 2024).

When analysing whole-class discussions, the teacher's actions must also be considered as related interventions carried out with a specific objective (Brocardo et al., 2022). In this context, Ponte et al. (2013) propose a model that analyses these actions, organising them into two groups: actions focused on learning management and actions directly related to mathematical aspects. Among the latter, four stand out, with particular relevance for this study: 1) inviting, initiating the whole-class discussion, and encouraging students' participation in sharing their strategies and involvement in the discussion; 2) supporting/guiding, keeping students involved, leading them in the presentation of information through questions or other interventions; 3) informing/suggesting, providing information, arguments or validating answers; and 4) challenging, encouraging students to deepen their knowledge.

METHODOLOGY

This study of the interpretative paradigm (Bogdan & Biklen, 1994) follows a qualitative approach in the design-based research modality. The exploratory task analysed in this article was carried out in a public school in Portugal, in a 3rd-grade class of 18 students aged 8 and 9. This study guaranteed the anonymity and confidentiality of all participants and the public institution involved. All received fictitious names. Authorization was requested from the General Directorate of Education [Direção Geral de Educação] and from the principal of the School Cluster [Agrupamento de Escolas] to collect data in the classroom where the study was carried out. Students, their guardians, and the teacher signed a free and informed consent. Everyone's participation was voluntary. The principles and standards of the national and international codes of ethics of the community of researchers in education were also respected (AERA, 2011; IE-ULisboa, 2016; SPCE, 2014).

The task analysed is part of a sequence of tasks integrated into a study on the magnitudes: length, area, and mass. This study included carrying out a teaching experiment in 3rd and 4th-grade classes in the academic years 2021/2022 and 2022/2023 in a public school in Portugal. The study was developed in three phases: 1) experiment preparation, 2) implementation in the classroom, and 3) retrospective analysis. A conjecture was formulated, guiding the teaching experience, which considers that students develop the understanding of magnitude, and the respective measurement process, going through five levels of learning: 1) identification of the attribute to be measured, 2) informal measurement: tessellation, 3) informal measurement: iteration of the u.m., 4) measurement with standardised u.m.s, and 5) relationship between the standardised u.m.s. For each level and each magnitude, a sequence of exploratory, challenging, and articulated tasks was constructed and planned to provide a coherent learning path (Ponte, 2005).

We analysed data collected from students' interventions during the whole-class discussion (participant observation), registers of strategies (students' notes), and the researcher's field notes. The actions of the teacher and the researcher are also analysed based on Ponte et al.'s (2013) framework and Araman et al.'s (2019) indicator framework with minor adaptations. The researcher's participation was previously agreed with the teacher, given her limited experience in exploratory classes.

Our objective was to identify the relationships that students establish in the process of measuring the area of three figures by tessellation, using as u.m. the area of a triangle and the area of a square (whose area is twice that of the triangle). The task was included in a sequence of four exploratory tasks to develop level 2), informal measurement: tessellation, the second task in the sequence and the first to work on those relationships. Given its exploratory nature, implementation occurred in three distinct moments: 1) introduction, 2) autonomous work, with students working in pairs, and 3) whole-class discussion and summary of learning. As mentioned, this article focuses on this last moment.

Based on studies by Grant and Kline (2003), which highlight the importance of exploring the relationships between a u.m. and a measure of an area, the students' strategies were analysed in the following categories: 1) choosing the appropriate u.m., 2) relationship between the size of the u.m. and the number of u.m.s needed to measure, 3) using whole u.m.s and parts of the u.m., and 4) inverse relationship between the size of the u.m. and the number of u.m.s (Table 1).

Table 1

Categories and subcategories of analysis of student strategies based on Grant and Kline (2003)

Categories		Subcategories	
1.	**Choose the suitable u.m.**	1.1	Analyses the suitability of different u.m.s in measuring the same area
		1.2	Identify/choose the most suitable u.m. to measure
2.	**Understands the relationship between the size of the u.m. and the number of u.m.s needed to measure**	2.1	Recognize that when using a larger u.m., one needs fewer u.m.s to measure and when using a smaller u.m., one needs more u.m.s
		2.2	Explore the variation of the number of u.m.s by changing their size
3.	**Uses whole u.m.s and parts of the u.m.**	3.1	Uses parts of the u.m. to complete the measurement

		3.2 Composes the measure through its parts
4.	**Understands the inverse relationship between the size of the u.m. and the number of u.m.s**	4.1 Understands the relationship between the size of the u.m. and the number of u.m.s needed to measure

These categories are organized to analyse how students structure the measurement process, helping to describe and interpret the students' strategies and the relationships they establish in this process, allowing a deeper understanding of learning.

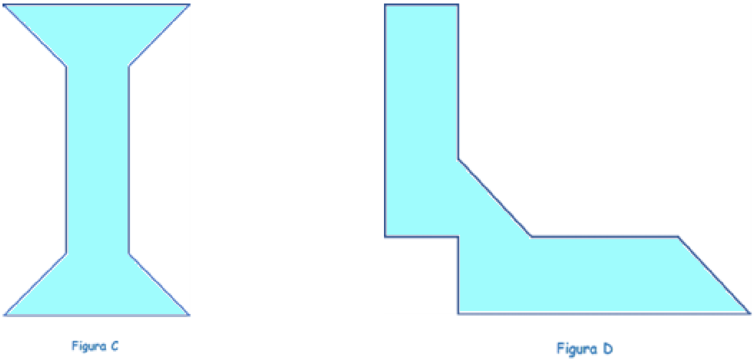
RESULTS AND ANALYSIS

According to the structure of an exploratory class, the teacher distributed to each pair of students the written statement of the task and the essential material for its completion: a set of three different figures (C, D, and E) and an envelope with squares and triangles to be used as a u.m. to measure the area of each figure, and allowed students some time to understand the task. Students should measure the areas of the figures with the u.m.s provided, concluding at the end whether they were equivalent figures or not. The teacher had already addressed equivalent figures in previous classes outside the scope of this experience. After clarifying doubts regarding understanding the statement, the teacher set 30 minutes to complete the task. During this period, the teacher and the researcher circulated among the pairs of students, monitoring their work: they ensured that the task was understood and collected information about the students' way of thinking, determining which aspects should be brought up for discussion and explored in greater depth at that time. They also photographed the resolutions that emerged, representing different strategies and, together, selected those that they considered to be positive contributions to the whole-class discussion, sequencing their presentation by the students. Learning was systematised after the whole-class discussion stimulated by the teacher with the researcher's collaboration and students' participation. This article only addresses measurement strategies.

To determine the area of the distributed figures, students used three different resolution strategies: total figure tessellation, part figure tessellation, and compensation. Below, we present each of these strategies, analysing some examples relating to just two of the figures presented, C and D (Figure 1), given that strategies and justifications were generally very similar in all the figures.

Figure 1

Figures C and D proposed to students for area measurement

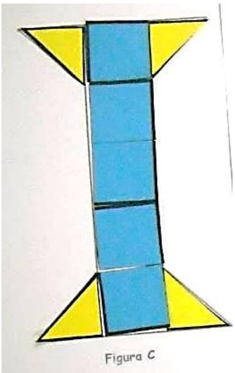


Total tessellation of the figure

Most pairs of students used this strategy. In the example in Figure 2, the students tessellated Figure C in its entirety, using the two available u.m.s

Figure 2

Total figure tessellating strategy used by Liliana and Mariana



At the teacher's invitation, Liliana presented the strategy (Figure 2) used to measure the area of Figure C.

> Liliana: *We put all the squares in a row [column], which was where they fit, and we put the triangles in the remaining gaps.*

> Teacher: *And what is the area of the figure?*
> Liliana: *5 [squares] [...] and 4 [triangles].*

The students broke down Figure C into four triangles and a rectangle, analysed the suitability of the different u.m.s for the space to be measured and identified the most suitable u.m. for measuring the area of each figure (category 1, subcategories 1.1 and 1.2). Thus, when measuring the area of Figure C, they used two different u.m.s simultaneously. The squares were used to tessellate the rectangle because “that was where they fit,” and the triangles were used to tessellate the parts of the figure where they were represented. The measure of the area involved these u.m.s, without establishing any relationship between them. Five squares and four triangles were counted, values associated with the number of u.m.s that tessellate the figure, establishing the relationship between the u.m. and the corresponding measure.

> Teacher: *Does anyone have anything to say about this group’s strategy? Catarina.*
> Catarina: *They mixed two figures [u.m.], just once. They didn’t make each figure [u.m.] once.*

The teacher challenged students to reflect on Liliana and Mariana’s work, inviting Catarina to participate in the discussion. The student mentioned the use of two different u.m.s simultaneously in this measurement. In her opinion, the students should have tessellated only with squares or only with triangles. However, her understanding of the relationship between the u.m.s is unclear, as she does not reference it. The discussion went on:

> Duarte: *In the result, there are fewer triangles than squares.*
> Teacher: *And what do you think about that?*
> Duarte: *There should be more [triangles], because there are two triangles, not just one.*
> Teacher: *Two triangles, what?*
> Duarte: *That... are the double*.
> Teacher: *Of?*
> Duarte: The *square.*

Duarte established the relationship between the two u.m.s and the inverse relationship between the size of the unit and the number of units needed to tessellate. The student recognized that when using a smaller u.m., he needs more u.m.s, exploring the variation in the number of u.m.s when changing its size (category 2, subcategories 2.1 and 2.2). In his interventions, the professor challenged Duarte to justify his statement and supported him in clarifying his speech and in his analysis of the strategy presented.

The next pair, Margarida and Luísa, presented a tessellation and strategy equal to that of the previous pair (Figure 1), but the way the students counted the u.m.s was different:

> Margarida: *We joined the two triangles… because they made squares […] seven squares [of area measurement].*

The students joined the two triangles to form a square, thus relating the two u.m.s, a justification already presented to us during the independent work (field notes). Thus, the discussion continued:

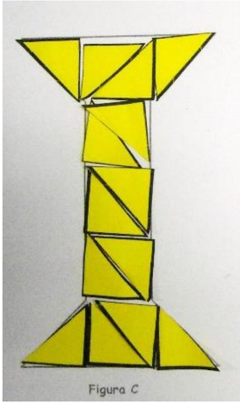
> Teacher: *If the photographs [strategies] are the same […], what is the difference? […] Rui.*
> Rui: *Because they [Margarida and Luísa] made two triangles to make a square […]. Liliana and Mariana put the triangles, but they didn't make them [count] as squares.*

The teacher challenged students to reflect on the strategies presented and invited Rui to begin this reflection. The student showed an understanding of these strategies and highlighted the relationship between the u.m.s Margarida and Luísa considered.

In measuring the area of Figure C, Rui and Simão used only triangles as a u.m. (Figure 3).

Figure 3

Total figure tessellating strategy used by Rui and Simão



The teacher invited Rui to share the strategy used (Figure 3).

> Rui: *We couldn't find a way to do it with squares and we did it with triangles.*
> Teacher: *What was the difficulty?*
> Rui: *Because… there were squares left… they didn't all fit… in some parts.*

When choosing the u.m., students analysed the suitability of the different u.m.s in measuring the area, identifying the most suitable u.m. (category 1, subcategories 1.1 and 1.2).

> Rui: *We first put the triangles here, and then we made the squares [with the triangles] in a column.*

The students also broke the figure down into four triangles and one rectangle. They started by tessellating the triangles with the u.m. because the u.m.s fit into these figures (category 1, subcategories 1.1 and 1.2) and then used the u.m. to form the squares that tessellated the rectangle, or column as Rui called it, thus relating the two u.m.s:

> Teacher: *What is the area of the figure in triangles?*
> Rui e Simão: *14.*
> Teacher: *And in squares?*
> Simão: *We made half [the number of triangles].*

To conclude, the teacher asked the students about the measurement of the area of the figure using the triangles as u.m.s. In the answer, the students established the relationship between the u.m. and the corresponding measure. Regarding the measure of the area using the squares as u.m., Simão established the relationship between the values of the measure of the area of the two u.m.s

> Teacher: *Why half?*
> Simão: *Because two triangles make a square.*
> […]
> Rui: *We saw that the triangle was half the square… and the square was twice the triangle.*

The teacher challenged both by asking why they calculated half the number of triangles to measure the area with squares. Simão and Rui highlighted the half-double/double-half relationship between the two u.m.s.

The teacher challenged the class to reflect on the justification presented by the pair of students, and Santiago intervened:

> Santiago: *Since triangles are half the size of a square, they take up less space. So we need twice as many squares.*
> Teacher: *Well done! Since the square is larger than the triangle, do we need more squares or more triangles?*
> Martinho: *More triangles.*
> Teacher: *Then…?*
> Martinho: *The measure with squares will be smaller than with triangles.*
> Teacher: *It will be smaller. How much? In this case […], these squares and these triangles.*
> Martinho: *It will be half the measure with the triangle.*

Santiago demonstrated an understanding of the inverse relationship between the size of the u.m. and the number of u.m.s needed to measure, by stating it (category 4, subcategory 4.1). In this way, he related the two u.m.s to each other, “triangles are half the square,” and established the relationship between area measurements using different units, but related to each other, “So we need twice as many squares.” In the act of informing/suggesting, the teacher validated the student’s answer and guided the students in understanding this relationship in the speech that followed. Martinho also showed understanding of this relationship (category 4, subcategory 4.1), stating it.

Luísa and Margarida, who had already measured the area of Figure C using squares and triangles simultaneously, also used triangles to repeat the measurement, despite already knowing the value of the measure they would obtain with this u.m.:

> Luísa: *We already knew how many triangles would fit, because we made twice as many.*
> Researcher: *Twice as many of what?*
> Luísa: *Of the squares.*
> Margarida: *Of seven.*
> Researcher: *So was there a need to measure the area with triangles?*
> Margarida: *No! But it was to be sure.*

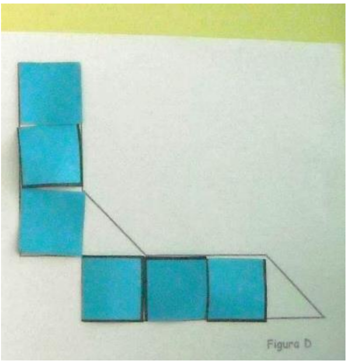
In response to Luísa’s answer, the researcher supported the students in clarifying Luísa’s statement. They showed that they were aware of the relationship between area measurements using two different but related u.m.s, which also suggests an understanding of the inverse relationship between the size of the u.m and the number of u.m.s (category 4, subcategory 4.1).

Tessellating parts of the figure

In this strategy (Figure 4), parts of the figure remained non-tessellated, and the u.m. did not entirely fit.

Figure 4

Strategy for tessellating parts of the figure used by Inês and Catarina



The teacher invited Inês and Catarina to present their strategy for measuring the area of Figure D (Figure 4).

> *Inês*: First, we put three squares in a column; then, the other three squares in a row. Then, we saw that there were two triangles left. So, we put these triangles together because they formed a square.

The students divided Figure D into two rectangles and two triangles. Although Inês did not explicitly mention this, we noticed that they analysed the suitability of the u.m.s, opting to use squares as the most appropriate u.m. (category 1, subcategories 1.1 and 1.2). They started by tessellating the rectangles, and in the remaining non-tessellated triangles, they used the relationship between them and the u.m.

During the independent work (field notes), we verified that this tessellation was not completed due to difficulties experienced by the students. They did not want to use triangles to complete it because the square was the u.m. but they also did not know how to use that u.m. when it was outside the figure. So, they chose not to tessellate it. In other words, although these students did not know how to deal with a u.m. that was larger than the space to be measured, they managed to reach the measure of the area through the relationship between the two u.m.s:

Teacher: *Using the number of squares, how did we find out, without tessellating, the number of triangles? [addressing the class]*
Duarte: *We divided the squares into two triangles.*
Teacher: *And the area would be…?*
Duarte: *Double the result with squares.*
Teacher: *Since two triangles fit in a square, there will be twice as many triangles as squares. If you counted seven squares, there will be twice as many triangles, that is, fourteen.*

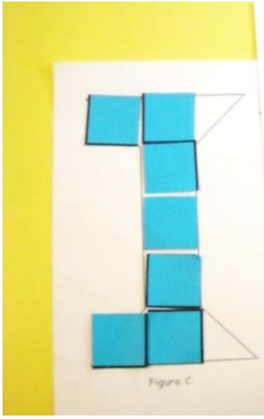
The teacher challenged the students to reflect on the resolution strategy. Duarte initially presented the relationship between the u.m.s and, with the teacher's support in clarifying and guiding the response, he related the area measures. The student showed an understanding of the inverse relationship between the size of the u.m. and the number of u.m.s (category 4, subcategory 4.1). The teacher concluded with an action of informing/suggesting.

Compensation

In this strategy (Figure 5), squares were used as a u.m. The remaining halves of the u.m. were used to complete tessellating the figure, demonstrating the capacity for abstraction.

Figure 5

Compensation strategy used by Duarte and Miguel



The teacher invited Duarte and Miguel to present the strategy used (Figure 5) in measuring the area of Figure C.

> Duarte: *We put the squares in a column, and then, we saw that there were four triangles left at the ends. Then we, with a square, saw that if we folded it in half, it would form two triangles. So, we put [imagined] a triangle on top and another on the bottom.*
> [these imaginary triangles refer to half of the square at the top of the figure and half of the other square at the bottom, both on the left side, and which fit into the figure].
> *Next... there was a triangle left [in each of the previous squares] [...] this part and this one. Then this triangle [from the lower left square], we imagined that it came here [lower right triangle], because the square was divided into two triangles. And this square was also divided into two triangles [pointing to the upper left square]. Then, this part that came out is a triangle, and we imagined that it came here [now pointing to the upper right triangle].*

The students considered the squares the most suitable for tessellating Figure C (category 1, subcategories 1.1 and 1.2). In this measurement, they used a compensation strategy in which the halves of the placed and remaining squares, the triangles, are imaginarily placed in the triangles that remain to be tessellated, demonstrating the capacity for abstraction. In this way, students worked with whole u.m.s and with parts of the u.m. (category 3, subcategory 3.1) and composed the measure of the figure through its parts (subcategory 3.2). A student asked:

> Martinho: *Why didn't they do it [*tessellate *it] with the triangles?*
> Duarte: *Because we divided the squares into triangles and realised that it was unnecessary to* tessellate *with the triangles [...] We knew there were two triangles here and here too [pointing to the left side of the figure, bottom and top, respectively]. Then, we just divide the squares in the column into triangles.*

Duarte's answer to Martinho's question shows that the students did not need to tessellate the figure with triangles, calculating its area measure using the ratio between both u.m.s. The teacher added:

> Teacher: *What is the area [of the figure] in triangles?*
> Duarte: *Twice the squares.*
> Teacher: *Why?*

Duarte: *Because if a square is two triangles, we will need twice as many triangles.*

To conclude the students' presentation, the teacher asked a verification question in the invitation action. Duarte indicated the ratio between the area measures in triangles and squares, demonstrating the understanding of the inverse relationship between the size of the u.m. and the number of u.m.s (category 4, subcategory 4.1). Then, the teacher challenged the student to give a justification. Duarte argued with the relationship between the two u.m.s and the relationship between the area measures.

Summary

This was followed by the phase of systematising learning. The researcher and the teacher began by challenging the students to reflect on the learning they had achieved to conclude and systematise the work.

Researcher: *What is the relationship between […] the units of measurement?*
Student: *Half and double.*
[…]
Researcher: *The area of the square is…?*
Santiago: *Twice the [area] of the triangle.*

The researcher guided the students, leading and focusing their thinking on the relationship between the two u.m.s used. The students' interventions show this relationship.

Researcher: *When we measure with the square […], what is this area measure compared to the area measure when we measure it with the triangle?*
Carlos: *Half.*
Researcher: *It's half! So, the square is twice the triangle and the area of the figure, when measured with the square, is…*
Santiago: *Half of what we measured with the triangle.*
Researcher: *What if we measure with the triangle?*
Santiago: *The triangle is half of the square.*
Researcher: *And what about the area of a figure when we measure it with a triangle…?*
Students: *It's the double.*
Researcher: *It is twice the area of the figure when we measure it with the square. Therefore, there is a relationship of double and half.*

In her interventions, the researcher continued the action of guiding, leading, and focusing students' attention on the relationship between the u.m.s and the relationship between the area measures, already indicated by the students. In the last intervention and in the action of informing/suggesting, the researcher reworked the students' answers to clarify the discourse and conclude. The discussion went on:

> Miguel: *When the unit is the double, the measurements will be half, and vice versa.*
> Santiago: *As is the case with the square. The square is twice [the triangle] and the measurements will be half.*
> Researcher: *Exactly! In the units of measurement, if the ratio is double... the measurements will be...*
> Students: *Half!*
> Researcher: *If the relationship between the units of measurement is half...*
> Miguel: *The measures will be double.*

Miguel, Santiago, and the other students established the relationship between the different ones and the corresponding measures. In informing/suggesting, the researcher validated the students' interventions, leading and challenging them towards generalisation. The inverse relationship between the size of the u.m. and the number of u.m.s was thus reinforced (category 4, subcategory 4.1).

DISCUSSION

The results of this study show different strategies students used to measure the area of figures by tessellating and using different non-standard u.m.s. These results suggest different levels of understanding of the concept of area, the measurement process, and the relationships established in this process. Table 2 shows students' strategies to measure the area of the proposed figures and the relationships they established in each strategy.

Table 2

Area measurement strategies and established relationships

Measurement strategies	Relationships established
Total tessellation of the figure	1) u.m. – the value of the corresponding measure

	2) u. m. – u.m. 3) area measure value estimated with squares – area measurement value estimated with triangles
Tessellating parts of the figure	1) u.m. – the value of the corresponding measure 2) u. m. – u.m. 3) area measurement value estimated with squares – area measurement value estimated with triangles
Compensation	1) u.m. – the value of the corresponding 2) u. m. – u.m. 3) area measure value estimated with squares – area measurement value estimated with triangles

These strategies show that students could perform the task in different ways, exploring three types of relationships: 1) between the u.m. and the corresponding measure value, the most expected relationship since it indicates the value of the measure, 2) between different u.m.s, and 3) between the area measure values estimated with different u.m.s, but related to each other.

Total tessellation of the figure

To measure the area of Figure C (Figure 2), Liliana and Mariana decomposed the figure. They used two different u.m.s, a strategy that agrees with studies by Smith III et al. (2016), which states that students select u.m.s similar to the space to be measured. However, by using two different u.m.s, the students did not consider the need to unify the u.m., ignoring one of the principles of the measurement process (Grant & Kline, 2003; NCTM, 2007), an error also pointed out by some authors (Smith III & Barrett, 2017), nor did they establish any relationship between the u.m.s In the value of the measure of the area of the figure, these students established only the expected relationship, the relationship between the u.m. and the corresponding measure for each of the u.m.s. Although Margarida and Luísa used the same strategy, they considered the relationship between the u.m.s when they counted them to determine the measure of the area of the figure, which suggests that these students have a more advanced understanding of the concept of area and the relationships between u.m.s.

Rui and Simão, by choosing triangles as the most appropriate u.m. (Figure 3) (Grant & Kline, 2003; NCTM, 2007), demonstrated more structured thinking but also revealed limitations in understanding the concept of measure. The choice of this u.m. was due to the difficulties in dealing with a u.m., in this case, the square, larger than the space to be measured (Lehrer et al., 2003; Ponte & Serrazina, 2000; Stephan & Clements, 2003), which suggests difficulties in using parts of the u.m. (category 3 and respective subcategories). Still, students could analyse the suitability of the different units, select the most appropriate u.m. and tessellate the figure (category 1, subcategories 1.1 and 1.2). In some parts of the figure, they used the relationship between the u.m.s. In the measure of the area of the figure, students counted the u.m.s, established the relationship between the u.m. and the corresponding measure and transformed this measure considering the other u.m. (the square), thus establishing another relationship, the relationship between the values of the two area measures. This approach, which Luísa and Margarida also used, shows a greater understanding than the other students, although limited for Rui and Simão in the sense that they could not work with parts of a number, which could also indicate difficulties in working with rational numbers.

Duarte, in addition to establishing the relationship between the u.m.s, stood out for recognizing the inverse relationship between the size of the u.m. and the number of u.m.s needed to measure (Grant & Kline, 2003), exploring the number of u.m.s by varying its size (category 2, subcategories 2.1 and 2.2), which demonstrates an understanding of the implications of this relationship. Luísa, Margarida, Santiago, and Martinho demonstrate a more advanced level of this understanding (category 4, subcategory 4.1).

Tessellating parts of the figure

When measuring the area of Figure D (Figure 4), Inês and Catarina resorted to breaking the figure down into two rectangles and two triangles and considered the square the most suitable u.m. (category 1, subcategory 1.2). They started by tessellating the rectangles and triangles that were left to be tessellated, relating them to the u.m. However, they did not complete the tessellation, as they showed difficulties in using a u.m. larger than the space to be measured, a difficulty Rui and Simão also expressed and identified as an obstacle in the process of developing the concept of area (Lehrer et al., 2003; Ponte & Serrazina, 2000; Stephan & Clements, 2003). However, this difficulty did not constitute an impediment in measuring the area of the figure, as the students used the relationship between the u.m.s. Duarte, once again, established the relationship between the u.m.s and related the measures of the area. The student showed an understanding of the inverse relationship between the size of the u.m. and the number of u.m.s (category 4, subcategory 4.1).

Compensation

When measuring the area of Figure C (Figure 5), Duarte and Miguel chose the square as the most suitable u.m. (category 1, subcategory 1.2) and used the compensation strategy. This strategy, in which the halves of the squares were mentally used to tessellate parts of the figure, involves the capacity for abstraction, which is essential in developing the concept of area. The fact that students work with whole u.m.s and parts of the u.m. (category 3, subcategories 3.1 and 3.2) shows conceptual progress in understanding measurement (Grant & Kline, 2003), understanding that the measure may involve subdivisions of the u.m. (Ponte & Serrazina, 2000). Duarte's interventions also show that the students used the relationship between the u.m.s and the relationship between the values of the area measure with different u.m.s, showing, once again, the understanding of the inverse relationship between the size of the u.m. and the number of u.m.s (category 4, subcategory 4.1). Thus, we can say that this strategy highlights the transition from a more concrete understanding to a more abstract and formal understanding of the measurement process, as discussed by Smith III et al. (2016).

In the analysis of the interventions of the teacher and the researcher, considered as a set of related interventions carried out with a specific objective, as mentioned by Brocardo et al. (2022), the four teacher's actions proposed by Ponte et al. (2013) we identified: inviting students to start and participate in the discussion, allowing them to get involved; supporting/guiding them in clarifying ideas by asking targeted questions, allowing the class to understand the explored ideas; informing/suggesting, not only validating answers but also guiding students to consolidate their understanding of the relationships established; and challenging them to justify and reflect, allowing them to build new knowledge.

CONCLUSIONS

Given the objective of this article, using different interrelated u.m.s in the measurement process allowed students to discover the relationship between the u.m.s with each other, as highlighted by Ponte and Serrazina (2000), and identify and explore other associated relationships. This study shows that students established three types of relationships in the process of measuring the area with different non-standard u.m.s: 1) the relationship between the u.m. and the value of the corresponding measure, 2) the relationship between the u.m.s, and 3) the relationship between the measurement values with different u.m.s related to each other. The categories and subcategories used, based on the studies by Grant and Kline (2003), which emphasize the importance of exploring the relationships between a u.m. and the measure of an area, allowed us to support those relationships and analyse how students structured the measurement process. Initially, students focused on the most direct and fundamental relationship of the measurement process (relationship 1), understanding how the u.m. relates to the value it represents (NCTM, 2007). As they progressed through the task, students also began to explore the relationship between different u.m.s (relationship 2), understanding how they relate to each other and that different u.m.s can be used to measure the same area, but that the choice can influence the number of required u.m.s (Grant & Kline, 2003; Smith III et al., 2016)). This understanding is also related to the principle of conservation, identified as one of the fundamental processes of learning area measurement by Clements and Sarama (2009), Stephan and Clements (2003) and Smith III et al. (2016). Thus, the students could compare the area measures of the same figure using the different u.m.s (relationship 3) and convert between the u.m.s, consolidating the idea that the area remains constant, regardless of the u.m. used. This progression in the construction of relationships demonstrates that understanding the concept of area and measurement is an evolutionary process. Such relationships help develop a greater understanding of the measurement process, encouraging students to think more critically about the u.m.s.

The evolution of these relationships was visible in the students' three strategies: 1) total tessellation of the figure, 2) tessellating parts of the figure, and 3) compensation. These strategies allowed students to deepen their understanding of how different u.m.s and measurements can relate to each other and develop the measurement process, revealing different levels of this understanding.

The difficulties observed, such as the use of larger u.m.s than the space to be measured, reveal the challenges in understanding the measurement process, namely in the subdivision of the u.m. (Lehrer et al., 2003; Ponte & Serrazina, 2000; Stephan & Clements, 2003). These difficulties highlight the need to propose diversified tasks that, in addition to promoting the visualisation and understanding of the relationships between one, encourage the subdivision of the u.m., allowing students to develop a more flexible understanding of the measurement process.

The whole-class discussion proved to be a crucial moment in the class due to the sharing and discussion of the strategies used by the students, the justification and argumentation of their ideas, the discussion and clarification of the error in the comparison of strategies, promoting the construction of new knowledge (Stein et al., 2008). The actions of the teacher and the researcher, framed within the four actions proposed by Ponte et al. (2013), were essential for this moment to become meaningful and an integral part of the whole-class construction of mathematical knowledge. As a result, students showed conceptual progress, evolving from a more concrete understanding to a more abstract understanding of the concept of area and the measurement process, which is essential for the transition to the use of standardised measures to be more effective and meaningful (NCTM, 2007; Ponte & Serrazina, 2000; Smith III & Barrett, 2017).

This study highlights the importance of the relationships identified by students in the measurement process, highlighting that the development of the concept of area should not be limited to just counting the u.m.s, but should involve the construction and exploration of these relationships, which are essential for an in-depth and meaningful understanding of the measurement process. Thus, the study emphasises the need for a balanced approach in considering and exploring these relationships. It also reinforces the importance of providing students with opportunities that encourage them to explore different measurement strategies, promoting a solid and flexible understanding of this process. To conclude, we highlight the importance of further researching other geometric magnitudes to understand what is shared and what is different in their learning.

ACKNOWLEDGEMENTS

This work is funded by national funds through FCT – Foundation for Science and Technology, through a doctoral scholarship awarded to the first author (2021.04798.BD) and within the scope of UIDEF – Research and Development Unit in Education and Formative Processes, UIDB/04107/2020, https://doi.org/10.54499 /UIDB/04107/2020.

AUTHORSHIP CONTRIBUTION STATEMENT

MT collected the data, analysed it and wrote the first version of the article. All authors participated in the general design and discussion of the article, reviewed it, and approved the final version of this work.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, MT, upon reasonable request.

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