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Argumentation and proof in the study of Conic Sections: the construction of the concept of Parabola through Problem Solving

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ABSTRACT

Background: Curriculum documents and research in Mathematics Education highlight the importance of engaging students in problem solving activities, integrated with mathematical argumentation, proof, and demonstration. **Objective:** To analyse the contributions of the Teaching-Learning-Assessment Methodology of Mathematics through Problem Solving to the construction of mathematical knowledge regarding the concept of the parabola and the deduction of its equation. **Design:** This is an empirical study with a qualitative approach. **Setting and participants:** The research was conducted in two classes of the third year of a Vocational Upper Secondary School, involving students aged between 17 and 19. **Data collection and analysis:** Data were obtained through participant observation and documentary analysis, recorded via audio, video, and field diary, and analysed using Discursive Textual Analysis. **Results:** The findings indicate that the Teaching-Learning-Assessment Methodology of Mathematics through Problem Solving supported the construction and understanding of the concept of the parabola, the relation used to generate the conic section, and the deduction of its equation, through the consistent and coherent use of both inductive and deductive reasoning. The use of GeoGebra enhanced learning in a reflective and critical manner by coordinating different registers of representation and supporting problem solving. **Conclusions:** The methodology proved effective in fostering conceptual understanding and mathematical demonstration. The assessment process was integrated throughout the activity, allowing for student guidance during the problem-solving process and promoting mathematical learning.

Keywords: Problem-Solving; Argumentation and Proof; Parabola; Technical High School; GeoGebra.

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Argumentação e prova no estudo das secções cônicas: a construção do conceito de parábola através da Resolução de Problemas

RESUMO

Contexto: Documentos curriculares e as pesquisas em Educação Matemática valorizam o envolvimento dos alunos em atividades de resolução de problemas, articuladas à argumentação, prova e demonstração matemática. **Objetivo**: Analisar as contribuições da Metodologia de Ensino-Aprendizagem-Avaliação de Matemática através da Resolução de Problemas na construção do conhecimento matemático sobre o conceito de parábola e na dedução de sua equação. **Design**: Trata-se de uma pesquisa de natureza empírica, com abordagem qualitativa. **Ambiente e participantes**: A investigação foi realizada em duas turmas da 3ª série do Ensino Médio Profissionalizante, com alunos entre 17 e 19 anos. **Coleta e análise de dados**: Os dados foram obtidos por observação participante e análise documental, registrados em gravações, filmagens e diário de campo, e analisados por meio da Análise Textual Discursiva. **Resultados**: Os resultados indicam que a Metodologia de Ensino-Aprendizagem-Avaliação de Matemática através da Resolução de Problemas favoreceu a construção e a compreensão do conceito de parábola, e da relação para gerar a cônica e a dedução de sua equação, empregando elementos dos raciocínios indutivo e dedutivo de forma consistente e coerente. O uso do GeoGebra potencializou a aprendizagem de forma reflexiva e crítica, mediante a coordenação de diferentes registros de representação e auxiliando na resolução do problema. **Conclusões**: A metodologia mostrou-se eficaz para promover a compreensão conceitual e a demonstração matemática. O processo avaliativo se constituiu em toda a atividade, permitindo orientar os alunos no decurso da resolução do problema, promovendo aprendizagem matemática.

Palavras-chave: Resolução de Problemas; Argumentação e Prova; Parábola; Ensino Médio Profissionalizante; GeoGebra.

INTRODUCTION

The growing complexity of the present world requires the individuals to have more and more abilities of critical thinking and problem solving, and the capacity of adapting themselves to new situations. In this context, Mathematics teaching plays a crucial role in educating citizens to a high level of knowledge that might make them able to contribute both to their own development and to the society.

Official and curriculum guidance documents from several countries, such as Brazilian National Curriculum Base (Brazil, 2018), Principles and Standards for School Mathematics (NCTM, 2000), Mathematics Program (Portugal, 2013) and *Mathematics Syllabuses Secondary One to Four* (Singapore, 2020) corroborate such importance by highlighting the development of mathematical reasoning as one of the main goals of the discipline. Furthermore, researches in Mathematics Education have emphasized that learning through memorization and repetition hardly contributes to the conceptual understanding of mathematical contents and requires low level of mathematical reasoning.

Portuguese researchers João Pedro da Ponte, Joana Mata-Pereira e Ana Henrique (2012, p. 356) point out that in order to develop students' capacity of mathematical reasoning, it is essential to "[...] work in activities that, on one hand, require reasoning and, on the other hand, stimulate reasoning. This is the only way we can expect students to effectively understand the mathematical concepts and procedures".

Upon these aspects, we consider the activities based on problem solving might be an efficient teaching strategy to promote understanding of concepts and contents worked in Mathematics discipline as well as to develop students' several cognitive abilities, which are relevant to promote mathematical reasoning for new learnings and for life.

According to Brazilian researchers Mário Barbosa da Silva, Ilda Pavret Silva, Norma Suely Gomes Allevato and Janaína Poffo Possamai (2023, p. 2), the Problem Solving shows "[...] remarkable characteristics in classroom in the sense of developing learning, arousing the students' interest in a contextualized and dynamic way, and it is appropriate to the complex scenario of schools today".

The Portuguese researcher Isabel Vale (2017, p. 131) also points out that problem solving is becoming increasingly necessary in teachers' practice due to its benefits in the teaching and learning process. This is due to "[...] the need of classroom practices where students' creative capacity can be developed, allowing everyone to actively participate in their learning, do their researches and share their findings".

Therefore, in the present work, we take on Problem Solving as a teaching methodology that may contribute to the construction and development of students' mathematical reasoning. The present study aims to analyze the contributions of the Methodology of Mathematics Teaching-Learning-Assessment through Problem Solving to build mathematical knowledge related to the concept of parabola and the deduction of its equation, according to the conceptions of Allevato and Onuchic (2021). In order to reach our goal, we intend to answer the following question: How can we build knowledge on parabola and the deduction of its formula through problem solving? This is one of the questions that was answered from practices that were implemented in a larger research developed by the authors of the present work.

This Introduction will be followed by a brief theoretical discussion on Problem Solving; next, we will discuss Mathematical Argumentation and Proof, based on Mathematics Education literature and its connection with Problem Solving. Then, we will present the adopted methodological procedures and the data analyses of the performed activity. Thus, the Final Considerations and the References.

THEORETICAL FOUNDATION

Problem Solving

Nowadays, there is a significant agreement among Mathematics teachers regarding the benefits of implementing Problem Solving as a teaching methodology in schools. The American researcher John A. Van de Walle (2009, p. 9) points out that in a problem solving context, the activities focus the students, promote the development of cognitive abilities and full understanding of mathematical concepts and contents, since "[...] the understanding and abilitites are better developed when the students are allowed to investigate new ideas, create and deffend solutions for the problems and paticipate in a mathematical learning community".

The notoriety of the benefits of working on problem solving in school context is also emphasized by Spanish researchers Antoni Vila and Maria Luiz Callejo (2006), mainly concerning the understanding of mathematical contents and the development of students' autonomy. Those researchers highlight:

> [...] a problem is not just a math assignment, but rather a tool to think mathematically, a means for creating a learning environment that forms autonomous, critical and propositional individuals, able to question themselves about facts, interpretations and explanations, to have their own criteria and to be, at the same time, open to other people's (Villa & Callejo, 2006, p. 10).

Similarly, the American and Chinese researchers Frank Lester and Jinfa Cai (2016), respectively, emphasize that activities focused on problem solving aim to propose intellectual challenges to students, in order to promote the understanding of the approached mathematical concepts and contents. Besides, when trying to solve the problem, students can use their former knowledge to help them solve the problem and build knowledge regarding the new mathematical content. Those researchers believe that the problem solving context provides a dynamic process in which students may reshape their ideas, conjectures and learning in each developed resolution, because "The power of problem solving lies in the fact that, **to obtain a successful resolution, students need to improve, combine and change the knowledge they had already acquired"** (Lester & Cai, 2016, p. 120, our emphasis).

The researches developed by Brazilian Mathematics professors Lourdes de la Rosa Onuchic and Norma Suely Gomes Allevato (2011) converge with the aspects of the above mentioned researches. They point out that implementing Problem Solving as a teaching methodology, which they call Methodology of Mathematics Teaching-Learning-Assessment through Problem Solving, can contribute significantly for understanding and learning new mathematical concepts and contents, since:

> [...] the problems are proposed to students before they are formally shown the necessary mathematical content or the most suitable resolution intended by the teacher, according to the syllabus demanded for that grade. So, the teaching-learning process of a mathematical topic begins with a problem [generator problem] that shows key aspects of such topic, and mathematical techniques must be developed in search of reasonable answers to the given problem. The evaluation of students' progress is continuously done during the resolution of the problem (Onuchic & Allevato, 2011, p. 85).

In order to promote learning through Problem Solving, the researchers suggest ten stages for the development of that Methodology in class:

> (1) problem proposition; (2) individual reading; (3) group reading; (4) problem resolution; (5) observing and stimulating; (6) registration of resolutions on the board; (7) plenary; (8) search for consensus; (9) formalization and (10) proposition and resolution of new problems (Allevato & Onuchic, 2021, p. 52).

Those stages aim to guide teachers to promote Mathematics understanding and learning in class, according to Allevato and Onuchic (2021). The work through Problem Solving begins with the proposition of a generator problem, which might have been elaborated, adapted or selected by the teacher based on the teaching goals; the problem might even be suggested by the students in order to promote learning of mathematical concepts and contents that haven't been learned yet.

On the second stage, students read the problem individually and, based on their interpretation and understanding, they go for the resolution. On the third stage, they work in small groups to share their knowledge, improve their ideas, interpretation and mathematical language.

The fourth stage is the moment students solve the problem. At this time, it is important for the teacher to monitor the groups attentively, mediating conflicts, clarifying questions and doubts that emerge, encouraging students to use their knowledge and questioning them about the coherence of their adopted procedures (fifth stage). On the sixth stage, there is the socialization of the resolutions, when each group presents their answers, correct or not, to all students, by writing them on the board.

On the seventh stage, in the plenary, the teacher stimulates a great debate in order to make all students strive to come to an consensus 0about the correct result (eighth stage), to build learnings about operatory techniques and mathematical notation, and to deepen their understanding of the generator problem contents. The penultimate stage concerns the formalization, by the teacher, of the problem content, and finally, new problems are proposed aiming to consolidate the learning that was built in the former stages.

Those aspects are corroborated by the normative Brazilian document BNCC (Brazil, 2018) that emphasizes:

> The mathematical processes of **problem solving**, investigation, project development and modeling may be considered **privileged forms of mathematical activity,** for which reason they are, at the same time, object and **learning strategy** over the entire Elementary school. Those **learning processes** are potentially rich for the development of fundamental competences for the mathematical literacy (reasoning, representation, communication and argumentation) and for the development of computational thinking (Brazil, 2018, p. 266, our emphasis).

At the international level, Problem Solving has been strongly recommended to be introduced as a teaching-learning methodology or strategy in the United States, by the *Principles and Standards for School Mathematics,* developed by *National Council of Teachers of Mathematics* (NCTM, 2000). The American curriculum guidance has a relevant view on problem solving, emphasizing that it:

> [...] implies the involvement in an assignment whose resolution method is not known beforehand. In order to find the solution, students have to explore their knowledge and, through this process, they frequently develop new Mathematical knowledge. **Problem Solving is not just a Mathematics learning goal, but also an important means by which students learn** Mathematics (NCTM, 2000, p. 57, our emphasis).

Similarly, a Singapore curriculum document (2020) also advocates that Problem Solving is not only the final objective, but also a teaching process that favours the use of general strategies and heuristics to address problems in a systematic and coherent way. According to that document, Secondary School students need to:

> • Acquire mathematical concepts and abilities for advanced studies in such area and to support learning in other disciplines, with emphasis on Sciences, but not limited to them;
>
> • **Develop thinking, reasoning, communication, application and metacognition abilities through a mathematical approach, such as problem solving;**
> • Connect ideas in Mathematics and between Mathematics and Sciences through their applications; and
> • Appreciate the abstract nature and the power of Mathematics (Singapore, 2020, p. 14, our emphasis).

In a learning context through Problem Solving, the understanding of concepts and contents will emerge; there will be search for justifications based on concepts, to agree or to refute the presented resolution. Besides, it will be possible to learn several forms of resolution, develop mathematical reasoning and build evaluative data in process, i.e., in the course of learning. After having clarified those ideas, we will present some theoretical aspects of mathematical argumentation and proof.

Mathematical Argumentation and Proof

Mathematics Education literature (Balacheff, 2019; Boavida, Gomes & Machado, 2002; Costa, 2023; De Villiers, 2010; Krakecker, 2022; Silva, 2016), as well as official documents (Brazil, 2018; France, 2015; England, 2014; NCTM, 2000; Singapore, 2020) strongly emphasize the need for developing mathematical argumentation and proof activities in all grade levels, from Basic Education on. This emphasis regards the development of fundamental cognitive abilities to make a reflexive, critical, creative and intellectually autonomous citizen, as advocated by contemporaneous society.

The French researcher Nicolas Balacheff (2019, p. 425, our emphasis) points out that the actions of "solving, **arguing, proving**, demonstrating, communicating and convincing, orally or in writing, are all dimensions of the "reasoning" competence the programs [1] require it should be acquired". However, the researcher recommends that, although there is a relation between explaining, arguing, proving and demonstrating in school context, it is necessary to make a clear distinction between those terms in order to achieve coherent understanding.

According to Balacheff (2019), the explanation intends to clarify and validate an argument from students' understanding based on their former knowledge, with no rules, as well as to favour discussion, rejection or acceptance; the proof is an explanation that is offered through convincing arguments accepted by a community; finally, the demonstration (mathematical) expresses a well defined set of rules and meets the present standards established by the community of mathematicians.

Similarly, the Italian researcher Bettina Pedemonte (2007) considers the existence of a cognitive unit to describe the structures of the student's mathematical reasoning, which is used to understand, elaborate, communicate and validate mathematical arguments. According to Pedemonte (2007), during a mathematical problem solving activity, students develop arguments to justify their answers and to come to a conjecture. In that context, "[...] the cognitive unit proposes that, in some cases, such argumentation may be used by students when building the proof, through the organization, in a logical chain, of some previously produced arguments" (Pedemonte, 2007, p. 25, our translation).

For Canadian educator and philosopher Gila Hanna (1990), three aspects of proof in Mathematics Education must be considered: (1) formal proof; (2) acceptable proof; (3) proof teaching. The formal proof was developed to avoid major mistakes and the need to reach for intuitive evidences and human judgement. The acceptable proof aims to present relevant implications for Mathematics field and to make connections with other areas of knowledge.

[1] It refers to French curriculum documents *Ministère de L'Éducation Nationale de L'Enseignement Supérieur et de la Recherche* (MENESR).

Finally, the proof teaching aims to use mathematical ideas to show the importance of the applied properties in that process in order to reach a result.

According to Hanna (1990), the proof teaching may show two characteristics: the proofs that prove and the proofs that explain. They are not different regarding the level of accuracy, and both are accepted by the mathematical community. The proofs that prove aim to highlight that a certain theorem or mathematical result is true, whereas the proofs that explain, besides that, intend to highlight the mathematical properties, ideas and concepts that were used to prove the veracity of such theorem.

British Mathematics teacher Andrea Stylianides (2007) also relates the concept of mathematical proof in school context to:

> [...] a mathematical argument, a connected sequence of statements for or against a mathematical request, with the following characteristics: 1. It uses accepted statements by classroom community (set of accepted statements) which are true and available without the need for additional justification; 2. It uses forms of reasoning (mode of argumentation) which are valid and known by the classroom community or within conceptual reach; and 3. It is communicated by modes of expression (ways of representing arguments) which are appropriate and known by the classroom community or within conceptual reach (Stylianides, 2007, p. 291).

In short, those researchers' perspectives highlight the complexity and the importance of developing activities that instigate mathematical argumentation, proof and demonstration, aiming to achieve the capacity of analysing, criticizing, justifying, conjecturing, arguing, proving and demonstrating, which are essential for the construction of students' mathematical reasoning, as Balacheff (2019) stated. Furthermore, the activities that are grounded on the work through problem solving are privileged means, because, besides they develop the abilities specified by Balacheff (2019), they also develop creativity, autonomy, and the explaining ability; and the "[...] sophisticated processes of mathematical thinking and the Mathematics teaching work take place in an investigation environment [...]" (Allevato & Onuchic, 2021, p. 53).

Considering those facts, we support the existence of a relation between problem solving and proposition and mathematical argumentation, proof and demonstration, because, besides both have aspects that are connected to mathematical knowledge and reasoning construction, they require a high cognitive level from students in those processes. Based on those considerations, we understand there is a relation between them because they present common aspects, such as:

Figure 1

Common Aspects between Problem Resolution and Proposition and Mathematical Argumentation, Proof and Demonstration in School Context (Silva, 2025, p. 98)

Aspects	Creativity
	Protagonism
	Elaboration of: resolution strategy; argumentation
	Investigation
	Connections of mathematical concepts and contents with other areas of knowledge
	Understanding of concepts and contents
	Critical thinking
	Resolution explanation
	Elaboration of conjectures
	Development of different ways of thinking: mathematical, metacognitive and of high order
	Development of inductive and deductive reasoning
	Social dimension
	Autonomy
	Multiple representations

According to Silva (2025, p. 98):

> When Problem Solving and Proposition are applied as a teaching methodology in a school context, it may promote construction and understanding of mathematical concepts and contents according to the recommendations of Allevato and Onuchic (2021) and Van de Walle (2009). Besides, such approach favours the development of mathematical argumentation, proof and demonstration, as it was highlighted by Balacheff (2019), De Villiers (2010), Hanna (1990), Pedemonte (2007) and Polya (1990).

For that reason, in the present work, we consider mathematical proof in school context as a social process in which interlocutors use their former knowledge to elaborate connected arguments; in addition, they apply concepts and mathematical writing to validate their answers to the generator problem. The generator problem is proposed to instigate and guide the construction and understanding of new mathematical contents and concepts related to conic sections, as well as to give the opportunity for argumentation e proof, and the mathematical reasoning in the deduction of its equations. Considering those aspects, in the next section, we will present the research methodology, which will be followed by the report and the description of the developed practice in school context.

METHODOLOGY

The investigation presented in this work is part of a doctoral research, in which we developed a didactic sequence founded on problem solving; it aimed to promote construction and understanding of new concepts and contents about conics, as well as the demonstration of the equations of each one. The practice reported here, with specific focus on the parabola, was developed in six meetings[2] with each participant class, in September and October, 2023. Two classes of third year of Technical High School participated in the activity, totalling 74 students. The school is located in a town in the São Paulo Metropolitan Region, Brazil[3].

It is a qualitative nature study that, as Borba, Almeida and Gracias (2018) suggest, prioritizes understanding the classroom dynamics, discussions and participant students' productions.

In order to meet the proposed objective, which was analysing the contributions of the Methodology of Mathematics Teaching-Learning-Assessment through Problem Solving to the construction of mathematical knowledge on the concept of parabola and the deduction of its equation, we agree with Lester and Cai (2016, p. 122, our translation) when they point out that "[...] mathematical problems that are truly problematic and involve meaningful Mathematics have the potential to provide the intellectual contexts for students' mathematical development". In this way, we have reformulated a textbook question related to the parabola concept, shown in figure 2.

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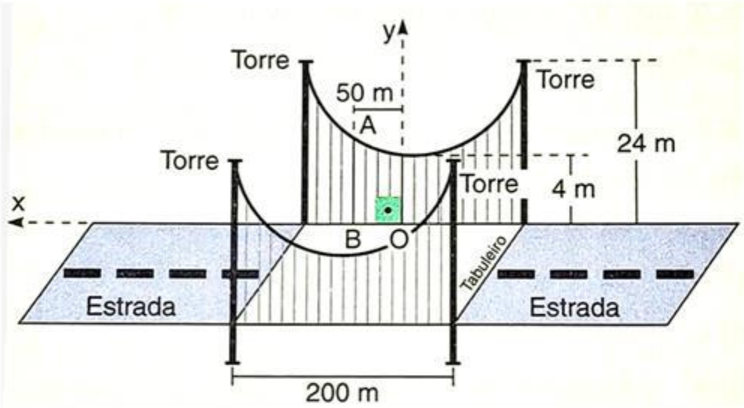
[2] Each meeting lasted 90 minutes, two 45-minute classes each.

[3] Approved by the Ethics Committee of Cruzeiro do Sul University on September 20, 2021, under number 4.985.806, complementary document attached.

Figure 2

Problem on the concept and content of parabola (Giovanni & Bonjorno, 2005, p. 131)

49 (UFMA) Os cabos da ponte pênsil, indicada na figura abaixo, tomam a forma de arcos de parábola do segundo grau. As torres de suporte têm 24 m de altura e há um intervalo entre elas de 200 m. O ponto mais baixo de cada cabo fica a 4 m do leito da estrada. Considerando o plano horizontal do tabuleiro da ponte contendo o eixo dos x e o eixo de simetria da parábola como sendo o eixo dos y, perpendicular a x, determine o comprimento do elemento de sustentação $\overline{BA}$, que liga verticalmente o cabo parabólico ao tabuleiro da ponte, situado a 50 m do eixo y.



Translate the problem: *The suspension bridge cables, shown in the figure below, have the shape of quadratic parabola arches. The support towers are 24 m high and there is a 200-meter space between them. The lowest point of each cable is 4 m from the roadbed. Considering the horizontal plane of the deck bridge containing the x-axis and the axis of the parabola symmetry as the y-axis, perpendicular to x, determine the length of the support element BA, which vertically connects the parabolic cable to the deck bridge, located 50 m from y-axis.*

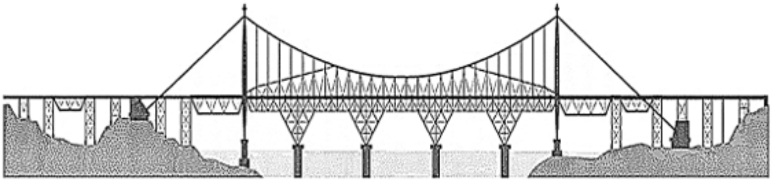
Next, in figure 3, we will present that reformulated problem according to recommendations of Lester and Cai (2016), aiming to implement it as a generator problem, i.e., to provoke students to initiate all the teaching-learning process about parabola which was intended.

Figure 3

Problem of Civil Engineering (Adapted from Giovanni & Bonjorno, 2005, p. 131)

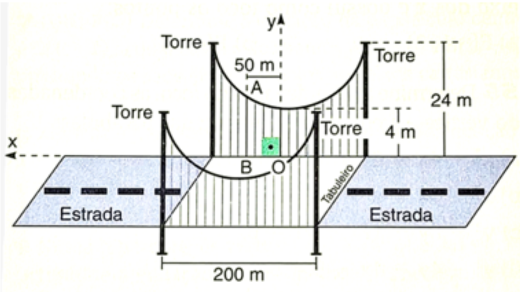
Atualmente, o desenvolvimento na área da construção civil possibilitou o acesso a diversas regiões, tanto no Brasil como em outros países, que, em época passada, só era possível por navegação. A engenharia desenvolveu um tipo de ponte suspensa, intitulada de 'Ponte Pênsil', que é sustentada por um sistema de cabos e mastros, visando não interferir no tráfego marítimo e transpor grandes distâncias. Os cabos de suspensão devem ser ancorados em cada extremidade da ponte, e qualquer carga aplicada à ponte é transformada em tensão nesses cabos principais. Os cabos principais continuam além das torres de suporte até os suportes no nível do convés e continuam ainda as conexões com as âncoras no solo, conforme a Figura 1:

Figura 1. Ponte Hercílio Luz, em Florianópolis, com 821 metros de extensão e 15,92 metros de largura de vão central.



Fonte: Conheça a história da Ponte Hercílio Luz, em Florianópolis | Santa Catarina | G1 (globo.com)

O engenheiro civil Leonardo Ibn Izzid Hassan Said, da empresa Plano&Plano, na construção da ponte pênsil precisou interromper os trabalhos devido a um problema no projeto. Ele iniciou um estudo detalhado do projeto e, para que a construção da ponte apresentasse exatidão no seu formato, os seus cálculos possibilitaram confirmar que as torres de suporte deveriam ter 24 m de altura e um intervalo entre elas de 200 m. O ponto mais baixo de cada cabo fica a 4 m do leito da estrada conforme a imagem da Figura 2:



Considere o plano horizontal do tabuleiro da ponte contendo o eixo dos x e o eixo entre as duas torres da figura como sendo o eixo dos y, perpendicular a x e passando pelo ponto mais baixo da curva. O engenheiro Leonardo Ibn Izzid Hassan Said, querendo contribuir na formação profissional do estagiário Lucas Habdala Lamek, propôs o desafio de encontrar as respostas para os seguintes itens:

Translate of the reformulated problem: *Nowadays, the development in civil construction area has made it possible to reach several regions, in Brazil as well as in other countries, which, in the past, was only possible by navigation. Engineers developed a kind of "suspension bridge", which is supported by a system of cables and poles, aiming not to interfere in the* *maritime traffic and to bridge long distances. The suspension cables must be anchored in each end of the bridge and any applied load to the bridge is transformed into tension on those main cables. The main cables go beyond the support towers up to the supports in the deck level and still keep the connections with the anchors on the soil, as shown in figure 1.*

Figure 1. Hercílio Luz Bridge, in Florianópolis, 821 meters long and 25,92 meters wide by the central span

Source: Get to know the history of Hercílio Luz Bridge in Florianópolis, Santa Catarina G1 (globo.com)- [Conheça a história da Ponte Hercílio Luz, em Florianópolis | Santa Catarina | G1 (globo.com)]

The civil engineer Leonardo Ibn Izzid Hassan Said, from Plano&Plano company, during the construction of the suspension bridge, had to interrupt the work due to a project probem. He started a detailed study of the project and, in order that the bridge had accuracy in its format, his calculations confirmed that the support towers should be 24 m high and there should be a 200 m space between them. The lowest point of each cable is 4 m from roadbed, as shown in figure 2.

Consider the horizontal plane of the deck bridge containing the x-axis and the axis between the two towers in the figure is the y-axis, perpendicular to x and passing through the lowest point of the curve. The engineer Leonardo Ibn Izzid Hassan Said, meaning to contribute to the professional training of the intern Lucas Habdala Lamek, proposed the challenge of finding the answers to the following questions:

1) Qual o possível nome para a curva formada pelos cabos entre duas torres consecutivas? É possível obter o nome da curva usando o GeoGebra? Como?
2) Como podemos representar a curva entre as torres com o software de geometria dinâmica GeoGebra?
3) Quais os elementos que compõem a curva?
4) Determine o comprimento do elemento de sustentação $\overline{BA}$, que liga verticalmente o cabo curvo ao tabuleiro da ponte, situado a 50 m do eixo y.
5) Sabe-se que essa curva é simétrica em relação ao eixo y, neste caso é possível afirmar que o comprimento do elemento de sustentação vertical, que liga o cabo curvo ao tabuleiro da ponte, agora do lado direito que está situado a 50 m do eixo y, tem o mesmo comprimento da haste $\overline{BA}$? Justifique sua resposta?
6) Determine a expressão que generalize a situação do problema? Justifique sua resposta?
7) Elabore um problema novo sobre o conteúdo que foi evidenciado nos itens anteriores para propor aos seus colegas.
8) Como você avalia a utilização do aplicativo GeoGebra nesse contexto? Quais os pontos positivos? Quais os negativos?

Translate the questions:

1) *What is the possible name for the curve formed by the cables between two consecutive towers? Is it possible to obtain the name of the curve by using GeoGebra? How?*
2) *How can we represent the curve between the towers through the dynamic geometry software GeoGebra?*
3) *What are the composing elements of the curve?*
4) *Determine the length of the support element BA, which vertically connects the curved cable to the deck bridge, located 50 m from y-axis.*
5) *It is known that this curve is symmetric to y-axis; in this case, is it possible to say that the length of the vertical support element, which connects the curved cable to the deck bridge, now on the right side, located 50 m from y-axis, has the same length of BA rod? Justify your answer.*
6) *Determine the expression that generalizes the problem situation. Justify your answer.*
7) *Elaborate a new problem for your classmates about the content that was emphasized in the previous questions.*
8) *How do you evaluate the use of GeoGebra application in this context? What are the positive aspects? What are the negative ones?*

The main objective of that generator problem, which contained this wording plus eight questions, was to start the study of a new mathematical content, according to recommendations of Allevato and Onuchic (2021), specifically regarding parabola in Analytical Geometry. Besides, the formal deduction of the equation of that conic was proposed, mobilized by the use of GeoGebra, associated with the necessary mathematical exploration to the problem resolution. Such proposition aimed to value the relations between problem solving and mathematical proof, as advocated by Balacheff (2019), Pedemonte (2007) and Silva (2025).

Next, we will present the study participant students' answers to the generator problem questions, as well as their data analyses and interpretations.

ANALYSIS AND RESULTS

The Discursive Textual Analysis (ATD) was used to analyze the research data, as it was proposed by Brazilian researchers Roque Moraes and Maria do Carmo Galiazzi (2016, p. 13), who define it as "[...] a methodology of a qualitative nature analysis of information with the objective of producing new understandings about phenomena and discourses". It is a cyclical process in which texts are deconstructed and reinterpreted, allowing the emergence of new understandings. The analysis has three stages that are developed according to the order of precedence: unitarization, categorization and the metatext.

The first stage, named unitarization, involves text fragmentation in order to identify relevant units of meaning for the research objectives. In the context of this study, students' protocols (written resolutions for the proposed problems), interview transcripts, audiovisual recordings, photos and field diary notes were analysed. After this fragmentation, the researcher starts the categorization by gathering similar units based on regularities and patterns, and progressively naming those categories more precisely. The last stage is the elaboration of the metatext, in which the researcher presents the interpretative results that were achieved along the former stages, describes and theorizes the investigated phenomena (Moraes; Galiazzi, 2016; Moraes, 2003).

> [...] Metatexts consist of description and interpretation, representing a way to understand and theorize the investigated phenomena. The quality of the texts that result from the analyses does not depend on their validity and reliability, but it is the consequence of the researcher claiming to be the author of their arguments (Moraes, 2003, p. 202).

After the guidelines of the activity were provided, each student received a copy of the generator problem for individual reading. Next, in groups, they read the problem and discussed it with their classmates in order to develop the resolution strategies. We expected the students, after understanding the problem, to elaborate a situation representation by using the resources of GeoGebra, or paper and pencil, and then they would use their previous knowledge for a possible resolution. Later, each group [4] presented their resolution on the board and explained what they had thought about. The presentation did not follow a pre-set order and each group was free to present their achieved results to their classmates.

In general, the eighteen groups were able to answer question (1) correctly when they specified that the curve between the towers form a parabola, as shown in the G3-3rd A students' answers[5] (figure 4).

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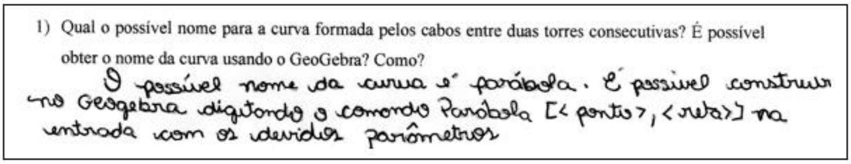
[4] Nine groups were formed in each class, four students in each group in 3rd A room and seven groups with four students and two groups with five students in 3rd B room.
[5] The groups were denoted by a number and by their class, for example: G3-3rd A corresponds to group 3 of the 3rd year A room.

Figure 4

Answer to question 1 of G3-3rd A (Data from Engineering problem)

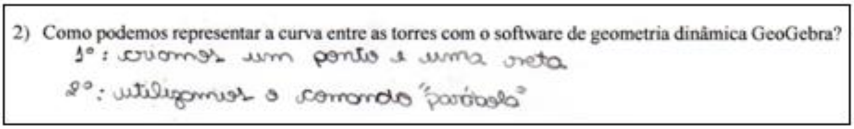

1) Qual o possível nome para a curva formada pelos cabos entre duas torres consecutivas? É possível obter o nome da curva usando o GeoGebra? Como?

O possível nome da curva é parábola. É possível construir no Geogebra digitando o comando Parábola [<ponto>, <reta>] na entrada com os devidos parâmetros

When questioned by the researcher, the group said that after reading the problem, they realized that the shape of the bridge curved cables could only be a parabola. In that moment, the group was instructed by the researcher about the importance of justifying their answers: ‘How is it possible to justify that the shape of the bridge curved cables is a parabola?’ Besides, it is possible to point out that the group used a GeoGebra command to build the parabola, i.e., the group used their previously learned knowledge in the activity about circumference, which was prior to this one, and answered that question correctly. Those facts contributed for the group to answer question (2) correctly as well.

Figure 5

Answer to question 2 of G3-3rd A (Data from Engineering problem)


2) Como podemos representar a curva entre as torres com o software de geometria dinâmica GeoGebra?

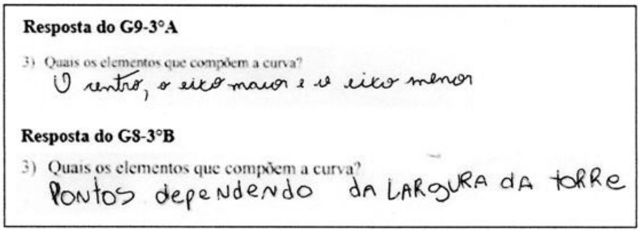
1°: criamos um ponto e uma reta

2°: utilizamos o comando "parábola"

In question (3), despite students were able to build the conic by using GeoGebra, we emphasize the lack of understanding about the elements that a parabola consists of. Those facts are shown in figure 6, with the resolutions of groups G9-3rd A and G8-3rd B.

Figure 6

Answer to question 3 of groups G9-3rd A and G8-3rd B (Data from Engineering problem)


Resposta do G9-3°A

3) Quais os elementos que compõem a curva?

O centro, o eixo maior e o eixo menor

Resposta do G8-3°B

3) Quais os elementos que compõem a curva?

Pontos dependendo da largura da torre

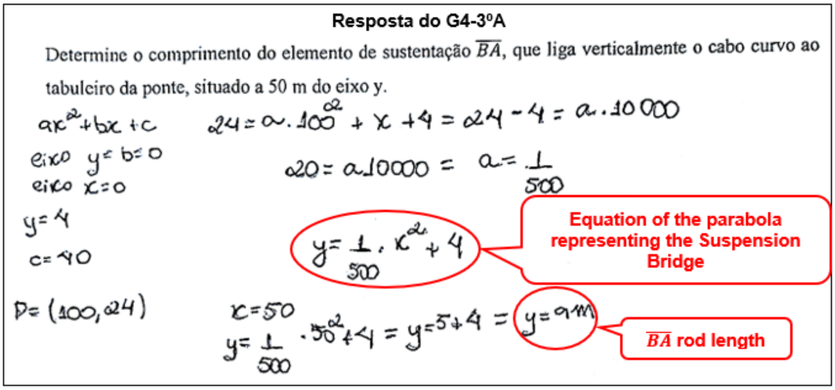
Only three groups of each class answered that question correctly, explaining that the elements are the focus, the directrix, the vertex and the symmetry axis. When questioned by the researcher, during plenary session, about how they found the elements of the parabola, the groups that answered correctly said they checked their textbook or searched on internet, while the other groups based their answers on intuition. We notice that investigative nature aspects are developed during problem solving activities, according to the researchers (Allevato & Onuchic, 2021; Van de Walle, 2009).

The first three questions were crucial to instigate questioning and promote the understanding of parabola concept and the deduction of its equation. Students related the curved shape of the bridge cables to the shape of the quadratic function graphic; they also identified the parabola elements and its algebraic expression. These aspects are highlighted in the answers to the fourth question, which asked for the calculation of *(BA)* rod length. Our objective was to emphasize how students would develop an algebraic expression for the parabola by using the values provided in figure 2 of the generator problem. That expression represents a particular case of the general equation requested in question 6, which would be the base for its deduction.

The resolution of G4-3rd A shows the group used the quadratic function concept. They deducted the parabola equation, which represents the bridge curved cable, calculated the *(BA)* rod length and answered question 5 correctly by using the concept of symmetry. The argumentation elaborated by the group for the parabola expression of the generator problem may be considered a mathematical proof, since it presents a coherent and organized logic, as Pedemonte (2007) stressed, and it also proves the veracity of the resolution, according to Balacheff (2019). The following figure shows such situation.

Figure 7

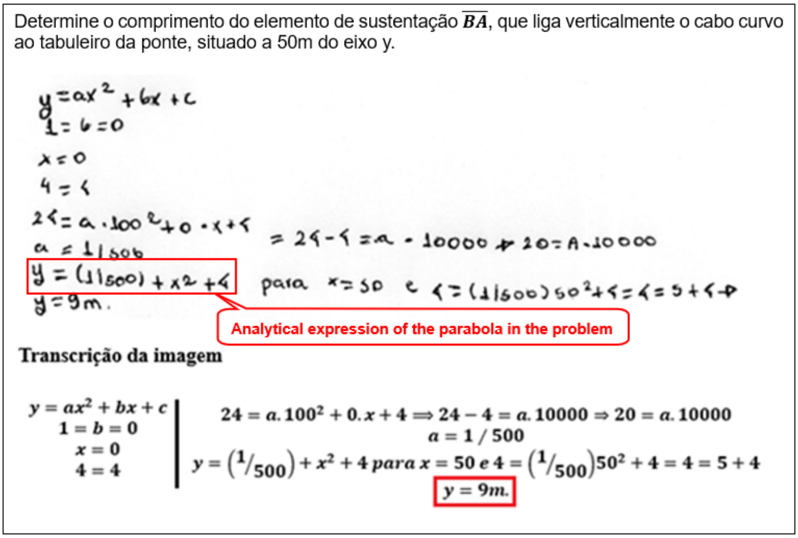
G3-3rd A resolution of question 4 (Data from Engineering problem)



The group G7-3rd B and the other groups of both classes also used the quadratic function concept to calculate the rod length. However, we found some divergence in the resolution developed by those students, as shown in figure 8, which were clarified in the plenary session.

Figure 8

G7-3rd B resolution of question 4 (Data from Engineering problem)



When the group was questioned by their classmates about the meaning of 1= b = 0 right after the expression $y = ax^2 + bx + c,$ the students explained that it referred to the order in the resolution: first, they determined the value of the numerical coefficient b of variable x and the independent term c, when they replaced the values of the ordered pair (0, 4) for x and y of the equation. At that moment, the researcher asked the group about the name of that point they used. A member of the group mentioned it was the lowest point of the curve and, after a few minutes, a classmate of another group specified it was the corresponding point to the parabola vertex. Next, the students calculated the value of the numerical coefficient a of variable x^2 by using the ordered pair (100, 24) to locate one of the bridge towers, and replaced it in the analytical expression $y = ax^2 + bx + c,$ where y is worth 24, x = 100, b = 0 and c = 4. Although the students have used the plus sign in the penultimate line, between the terms $\left(\frac{1}{500}\right)$ *and* x^2, after the replacement of the values, they operated the multiplication correctly. At that moment, students from other groups raised doubts, because they did not understand the calculation, since the presenting group had not clearly identified which incognita was being calculated, transforming the equation into a numerical expression. A student from group 7 clarified that the rod measure corresponded to y value, thus completing the answer with $y = 9m.$

Those episodes legitimate the relevance of that sharing, collaboration and discussion moment for Mathematics learning. According to Allevato and Onuchic (2021, p. 50), the objective of the plenary session in a problem solving context is to make "[...] the teacher stimulate students to share and justify their ideas, deffend their points of view, compare and discuss different solutions, i.e., evaluate their own resolutions, so that they can improve their resolution (written) presentation". By the students' answers, we consider they have developed generalization when they obtained the problem expression through logical, explanatory and coherent reasoning, and when they used mathematical writing in the deduction and to calculate the rod length. According to the guidelines of official documents, high school students need to:

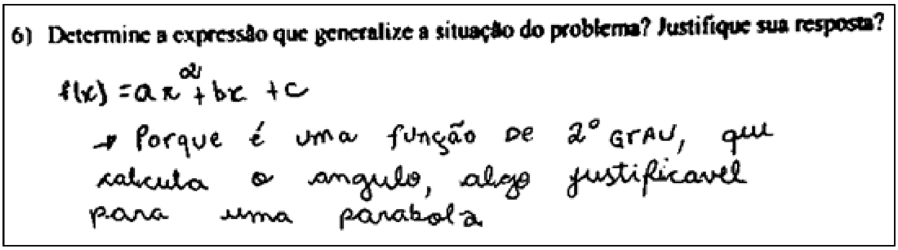
> Investigate and establish conjectures about different mathematical concepts and properties by using strategies and resources, such as pattern observation, experimentation and different technologies, identifying whether it is necessary to show a more and more formal demonstration when validating such conjectures (Brazil, 2018, p. 531).

Those facts are also recommended by the French document highlighting that “[...] it is important to allow progression in demonstration learning [...]” (France, 2015, p. 367). These aspects are also advocated by the British document, emphasizing that high school students need to “[...] think mathematically by following an investigation line, establishing relations, developing conjectures, justifications and generalizations or a proof by using mathematical language”. (England, 2015, p. 40, our translation), and by the researches of Allevato and Onuchic (2021), Balacheff (2019), Pedemonte (2007) e Stylianides (2007). It is important to point out that GeoGebra resources were essential to help students visualize the parabola and develop and test their conjectures to find the answer to the question.

The group G4-3rd A couldn’t relate the parabola expression determined in question 4 resolution to question 6. Therefore, the group was not able to elaborate the justification to their answer and only informed it was a second degree equation.

Figure 9

G4-3rd A answer to question 6 (Data from Engineering problem)


6) Determine a expressão que generalize a situação do problema? Justifique sua resposta?

$f(x) = ax^2 + bx + c$

→ Porque é uma função de 2° GRAU, que calcula o angulo, algo justificavel para uma parabola

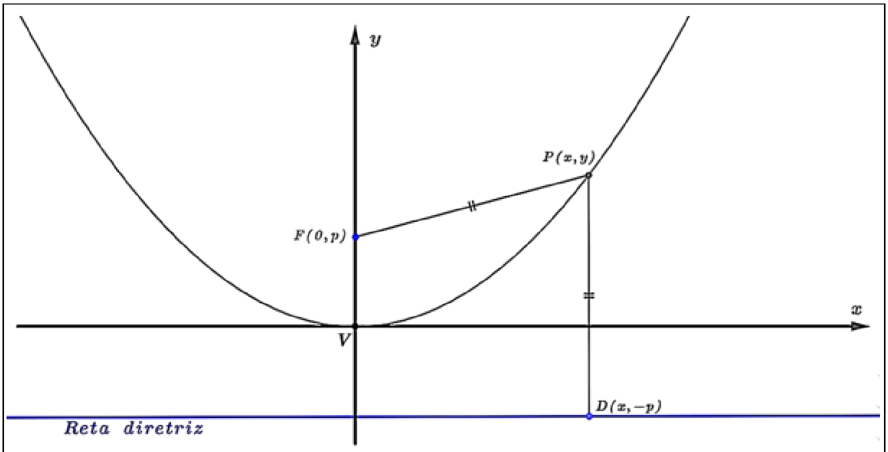
Although some groups from both classes felt difficulties in solving the problem, we emphasize these aspects are related to the recommendations of Allevato and Onuchic (2021) and Vale (2017) when they mention problem solving may contribute to teaching and learning process during students’ development of the resolutions, to promote active participation throughout all resolution process and presentation of their answers, to stimulate findings and learning sharing and to develop creative and mathematical reasoning capacity, as highlighted by official documents (Brazil, 2018; Portugal, 2013; NCTM, 2000) and by Ponte et al. (2012). Furthermore, it was possible to notice the elaborated arguments by students contributed to promote a connected and logical sequence of statements (Balacheff, 2019; Pedemonte, 2007; Stylianides,

2007) and to explain the concepts, properties and ideas they used for the resolution, as stated by Hanna (1990) about proofs that explain.

Considering the groups' answers and some researcher's interventions during resolution and plenary moments, another activity was elaborated for the groups to understand and develop the relation to deduct the equation that generalizes the representation of parabola families.

Figure 10

Activity to deduct the parabola equation with the vertex in the origin of orthogonal axis system (Data from Engineering problem)



1) Analisando a Figura 3, qual é a relação para deduzir a equação da Parábola no sistema de eixos ortogonais?
2) Qual a relação que podemos fazer do ponto V (vértice da parábola) com o ponto F (foco) e a reta diretriz?
3) Utilizando a nomenclatura da Figura 1, demonstre a expressão que generalize a equação da Parábola.

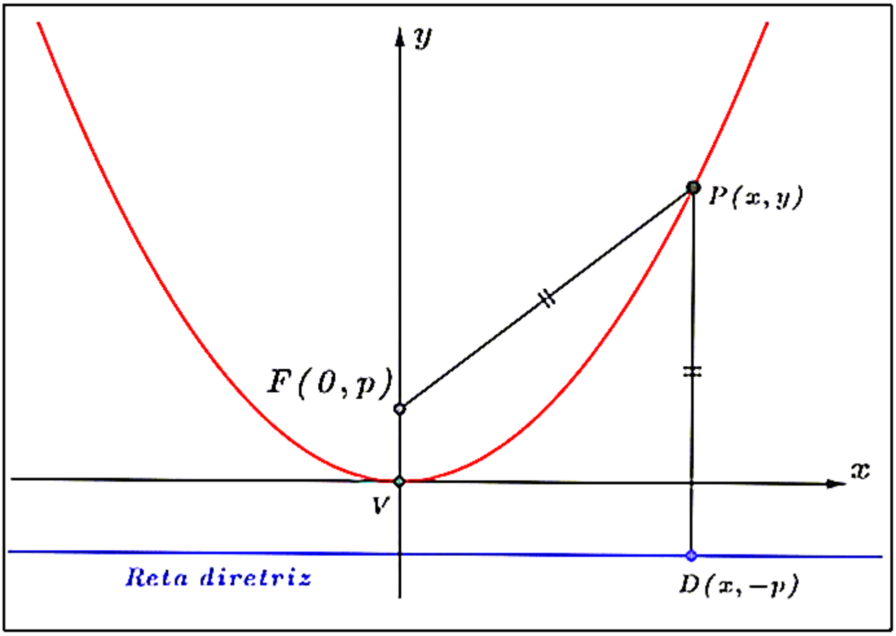
Translate the questions:

1) *By analysing figure 3, which is the relation to deduct the Parabola equation in the orthogonal axis system?*
2) *What is the relation we can make between V point (parabola vertex) and F point (focus) and directrix straight line?*
3) *By using the figure 1 nomenclature, demonstrate the expression that generalizes the Parabola equation.*

We showed them the next graphic whose link was previously provided and accessed by them:

Figure 11

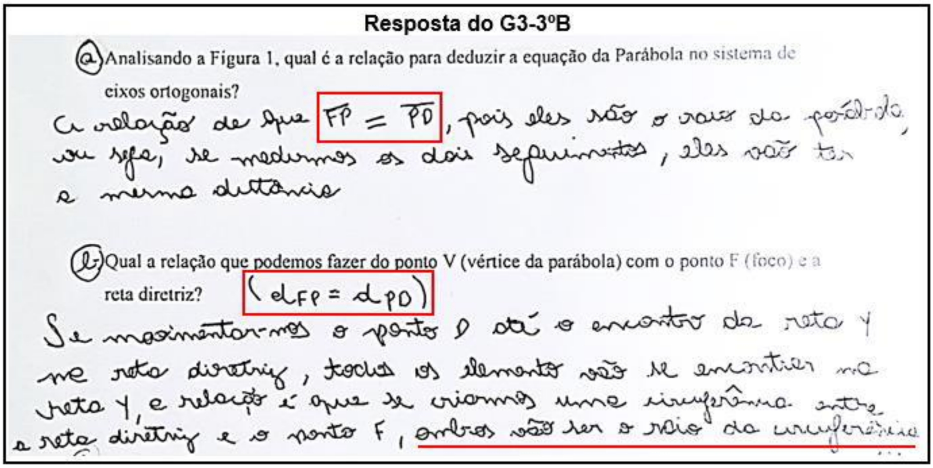
Graphic provided to students (Data from Engineering problem)



We asked students to analyse the figure and, by manipulating the parabola construction on GeoGebra, try to determine the necessary relation do deduct the parabola general equation, in the orthogonal axis system. The researcher reminded students about the relation that had been used to deduct the circumference equation and, similarly, asked which relation would lead to deduct the parabola equation. The groups started to discuss and manipulate D point in the 2D viewing window and began their resolutions. The groups were able to see that the distance between the corresponding point of F focus up to P point is the same as from P point up to D point, and so, they answered question items “a” and “b” correctly, as it is shown in the following figure.

Figure 12

G3-3rd B answer to a and b items of question 8 (Data from Engineering problem)


Resposta do G3-3ºB

(a) Analisando a Figura 1, qual é a relação para deduzir a equação da Parábola no sistema de eixos ortogonais?

A relação de que FP = PD, pois eles são o raio da parábola, ou seja, se medirmos os dois segmentos, eles vão ter a mesma distância

(b) Qual a relação que podemos fazer do ponto V (vértice da parábola) com o ponto F (foco) e a reta diretriz?

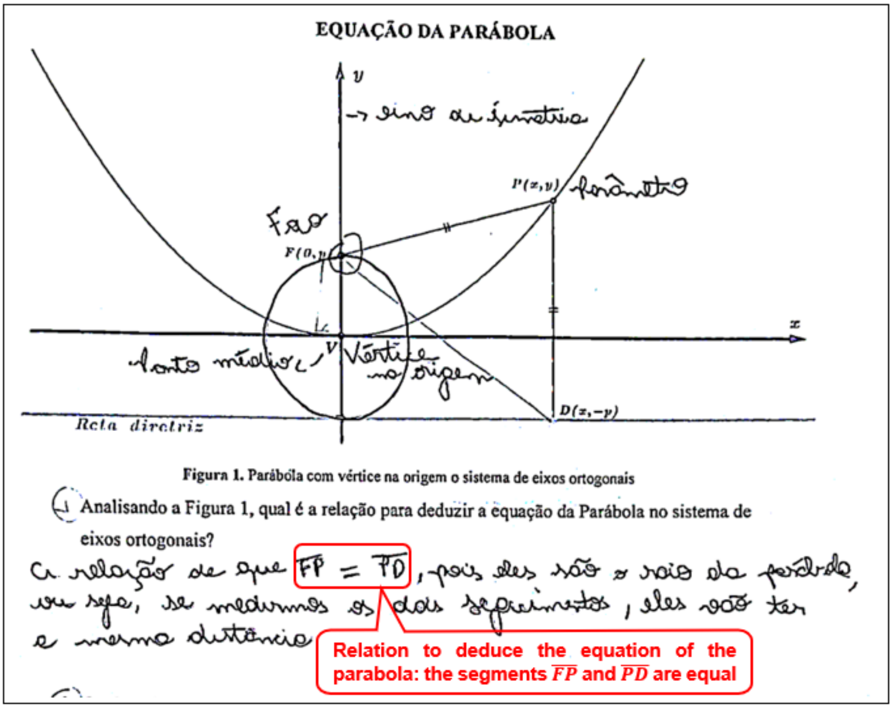
($d_{FP} = d_{PD}$)

Se movimentarmos o ponto P até o encontro da reta Y na reta diretriz, todos os elementos vão se encontrar na reta Y, a relação é que se criarmos uma circunferência entre a reta diretriz e o ponto F, ambos vão ser o raio da circunferência ...

G3-3rd B group specified those segments are circumference radii of V center, the parabola vertex. That group established a connection between the circumference concept and the relation between the distances of V vertex of the parabola to F focus and to directrix. They made a connection between two mathematical contents addressed by the generator problems of circumference and parabola, as highlighted by Allevato and Onuchic (2019). Figure 13 shows the representation of the understanding of that group.

Figure 13

Representation built by G3-3rd B group (Data from Engineering problem)

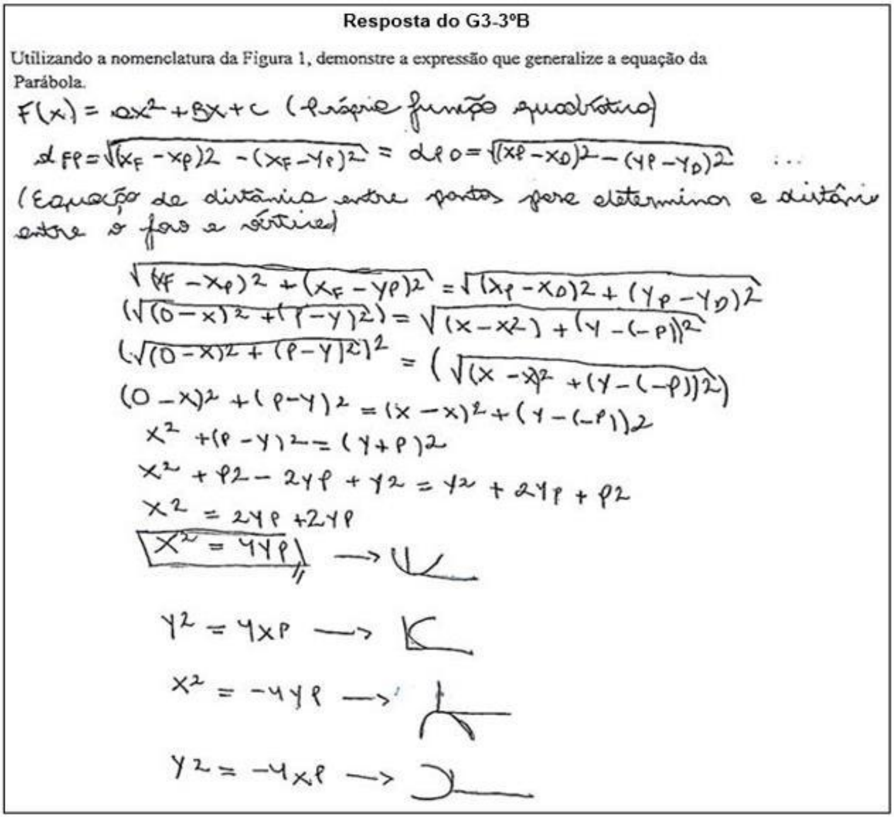


Considering those facts, it became clear that the dynamic geometry software contributed for the understanding of concepts and the necessary relations for the mathematical deduction of the parabola equation by the students. Those aspects were confirmed in the written responses, when it was mentioned that the distance from (F) focus to a curve point (P) is the same as the distance from (P) point to the directrix ($\overline{FP} = \overline{PD}$).

In the end, item 'c' confirmed what we had foreseen: the generator problem favoured both knowledge construction and understanding of the mathematical content, as well as the development of deductive reasoning. The students developed a mathematical proof by using relations, concepts, properties and mathematical language. These aspects are seen in G3-3rd B resolution, as it is shown in the following figure.

Figure 14

G3-3rd B deduction of the parabola equation (Data from Engineering problem)



The group started the equation deduction by specifying that the distances between ($\overline{FP}$) and ($\overline{DP}$) segments are equally long, and it can be calculated by using the expression to determine the distance between two points, i.e., $d_{FP} = d_{PD}$. The students, by using the coordinates of F, P and D points, wrote:

- $d_{FP} = \sqrt{(x_F - x_P)^2 - (y_F - y_P)^2} =$

$d_{PD} \sqrt{(x_P - x_D)^2 - (y_P - y_D)^2}$

Although the students had used the wrong sign in the beginning of the equation deduction mentioned above, in the sequence, they used the right plus sign and replaced the corresponding values in each coordinate; they made the algebraic manipulations involving potentiation, notable product, simplification, and grouped similar terms in a consistent way, presenting the parabola equation family and its graphic representations when its vertex point coincides with the origin of the coordinated axis system.

The resolution built by G3-3rd B and the ones from the other groups presented a structured reasoning, an explanation of mathematical ideas, concepts and contents to validate their answer, as highlighted by Hanna (1990). It is important to point out that those aspects are in accordance with what Balacheff (2019) e Pedemonte (2007) emphasized, because, when the argumentation presents organization in a logical chain that validates the answer, such argument earns the status of mathematical proof.

Although some groups were not able to develop the deduction of the parabola equation, specially due to their difficulties concerning the algebraic manipulations, the students could understand the logic and the deduction process presented by their classmates. Therefore, the activity as a whole contributed to the construction of concepts and contents, as well as to the deduction of the parabola equation by using the formal aspects that are so advocated by official documents and by Mathematics Education researches (Allevato; Onuchic, 2021; Balacheff, 2019; Pedemonte, 2007; Stylianides, 2007).

In agreement, the students realized that the mistakes they had made during problem solving were related to former knowledge, contents they had forgotten or never learned, or aspects they were learning in that moment. However, they were essential to clear doubts, learn new concepts and contents, and remember others that had supposedly been learned before. Despite the difficulties, the students demonstrated effort and curiosity to find an answer, which increased their self confidence to face challenges and overcome obstacles they found in that problem and in mathematical learning through that new approach.

When questioned by the researcher about mathematical learning provided by that problem solving, the participants answered in an assertive and clear manner, highlighting the acquisition of several mathematical skills, such as: development of a mathematical proof; contents related to parabola; algebraic manipulations; notable product; rules of potentiation and root extraction; problem solving strategies, among others. Besides that, the evaluation process occurred throughout the activity. Although it was not the main purpose of this study, we emphasize the evaluation contributed to detect the students' doubts and mistakes during problem solving, which allowed the researcher to mediate and help students correct and understand their mistakes and advance in their knowledge construction, according to recommendations of Allevato and Onuchic (2021).

Finally, we point out that the participants showed a meaningful evolution in the construction of mathematical knowledge, since it was not necessary to make so many interventions as in the problem of the radio station involving circumference, which had been solved before. In other words, the Methodology of Mathematics Teaching-Learning-Assessment through Problem Solving contributed to the construction and understanding of concepts and contents regarding parabola and the formal deduction of its equation when its vertex coincides with the origin of orthogonal axis system.

FINAL CONSIDERATIONS

The present work aimed to analyse the contributions of the Methodology of Mathematics Teaching-Learning-Assessment through Problem Solving to the construction of mathematical knowledge regarding the concept of parabola and the deduction of its equation, developed by 3rd-year students of a Technical High School located in the metropolitan region of São Paulo. During its development our goal was to answer the following question: How does the construction of knowledge regarding parabola and the deduction of its formula occur through problem solving?

The discussions held during the activity, as well as the analyses of the protocols produced by the participants allowed us to identify the contributions of the moments of elaboration of resolutions and the plenary session, in the implementation of the Methodology, which aims to lead students to a conscious, responsible and creative construction of mathematical knowledge, aligned with the teacher's established objectives for that class.

We emphasize the valorization of students' protagonism – expressed in different elaborated approaches, in the reasoning processes and in the presented justifications – as well as in the interaction and the leaning promoted in the plenary and in the search for consensus, and also in the acceptance of their mistakes as a path for knowledge construction, which were essential for the development of the content understanding in that learning process. In that context, the teacher could also evaluate and identify the progress, the understandings and the difficulties presented by the students, and help them in their learning process. Besides, it is important to point out that the teacher's mediation, the use of the dynamic geometry software GeoGebra and the

Methodology of Mathematics Teaching-Learning-Assessment through Problem Solving, well planned and implemented, were crucial for the construction of concepts and contents and for the deduction, in this research, of the parabola equation.

Although both classes presented difficulties, the students clearly managed to build knowledge and to understand either the parabola content or the importance of the elaboration of the mathematical proof, in a context of development of high cognitive abilities and promotion of the understanding and the mathematical proof in school context, as advocated by official documents (Brazil, 2018; France, 2015; England, 2014; NCTM, 2000; Portugal, 2013; Singapore, 2020) and by researches (Allevato; Onuchic, 2021; Balacheff, 2019; Pedemonte, 2007; Ponte et al., 2012; Stylianides, 2007).

Therefore, the presented reflections show it is possible to give the Basic Education student the opportunity to build and understand mathematical contents, as well as to develop creativity, generalization, mathematical proof and demonstration, and mathematical reasoning, helped by GeoGebra, in a context that is grounded on the work through problem solving. The Methodology of Mathematics Teaching-Learning-Assessment through Problem Solving has the potential to promote such aspects.

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