ACTA SCIENTIAE

ISSN: 2178-7727
DOI: 10.64856/acta.scientiae.8308

Use of mathematical errors to address difficulties and reinforce learning about linear equations

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ABSTRACT

Background: We use mathematical errors as learning opportunities. **Objective:** This study describes the implementation of one of the activities with mathematical errors from a teaching proposal that addressed the difficulties and promoted learning of linear equations with one unknown. **Design:** Through a qualitative approach, specifically with a case study. **Setting and Participants:** We chose 18 Costa Rican students in their final year of secondary education, the closest level to higher education. **Data collection and analysis:** Information was collected from participants' written productions, and a content analysis was conducted using the previously constructed categorisation from the theory. **Results:** Although the students demonstrated the ability to identify errors and propose corrections, some solutions revealed weaknesses in reasoning regarding the use of algebraic symbolism and the interpretation of the equal sign in equations. We also identified difficulties in the manipulation of fractions and radicals. Nevertheless, several responses provided evidence of conceptual understanding when students identified and corrected errors in the equations presented. **Conclusions:** We consider it crucial to create learning situations that promote the use of errors as learning opportunities by including them in the development of activities in the mathematics classroom, as these situations could benefit students and provide information on measures to improve their learning.

Keywords: learning; difficulties; mathematical errors; error management, error pedagogy.

Uso de errores matemáticos para atender dificultades y reforzar el aprendizaje de ecuaciones lineales

RESUMEN

Antecedentes: Utilizamos los errores matemáticos como una oportunidad de aprendizaje. **Objetivo:** Describimos la aplicación de una de las actividades con errores matemáticos de una propuesta de enseñanza que atendió las dificultades y promovió el

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aprendizaje del tema de ecuaciones lineales con una incógnita. **Diseño:** Mediante un enfoque cualitativo, específicamente un estudio de casos. **Entorno y Participantes**: Elegimos 18 estudiantes costarricenses del último año de educación secundaria, nivel más próximo a la educación superior. **Recopilación y Análisis de datos:** Se recolectó información a través de las producciones escritas de los participantes y realizamos un análisis de contenido, utilizando la categorización creada desde la teoría. **Resultados**: Aunque los estudiantes mostraron capacidad para identificar errores y proponer correcciones, algunas soluciones presentaron fallas en el razonamiento relacionadas con el uso del simbolismo algebraico o con la interpretación del signo igual en una ecuación. Asimismo, se identificaron dificultades en la manipulación de fracciones y radicales. No obstante, en varias respuestas se observaron indicios de comprensión conceptual al identificar y corregir errores presentes en las ecuaciones planteadas. **Conclusiones**: Consideramos que es crucial generar situaciones de aprendizaje que incorporen el uso de los errores como oportunidades para el aprendizaje, al incluirlos en el desarrollo de actividades en el aula de matemática, pues esto puede beneficiar a los estudiantes y brindar información sobre algunas medidas para mejorar el aprendizaje.

Palabras clave: aprendizaje; dificultades; errores matemáticos; atención de errores; pedagogía del error.

Uso dos erros matemáticos para abordar dificuldades e reforçar a aprendizagem de equações lineares

RESUMO

Contexto: Utilizamos os erros matemáticos como uma oportunidade de aprendizagem. **Objetivo:** Descrevemos a aplicação de uma das atividades com erros matemáticos de uma proposta de ensino orientada a atender dificuldades e promover a aprendizagem do tema de equações lineares com uma incógnita. **Design:** Foi utilizada uma abordagem qualitativa, especificamente um estudo de caso. **Cenário e participantes:** Participaram 18 estudantes costarriquenhos do último ano do ensino médio, o nível mais próximo do ensino superior. **Coleta e Análise de Dados:** As informações foram coletadas por meio das produções escritas dos participantes e realizamos uma análise de conteúdo, utilizando a categorização criada a partir da teoria. **Resultados:** Embora os estudantes tenham demonstrado capacidade de identificar erros e aplicar correções, algumas das soluções apresentaram falhas de raciocínio relacionadas ao uso do simbolismo algébrico ou ao significado do sinal da igualdade em uma equação. Também foram identificadas dificuldades na manipulação de frações e radicais. No entanto, em várias respostas observaram-se indícios de compreensão conceitual ao identificar e corrigir erros presentes nas equações propostas. **Conclusões:** Acreditamos que é fundamental criar situações de aprendizagem que promovam o uso do erro como uma oportunidade de aprendizagem, incluindo-o no desenvolvimento de atividades na aula de matemática. Isso pode beneficiar os alunos e fornecer informações sobre as medidas que podem ser tomadas para melhorar a aprendizagem.

Palavras-chave: aprendizagem; dificuldades; erros matemáticos; atenção do erro, pedagogia do erro.

INTRODUCTION

Mathematics is very useful in the exact and natural sciences, social sciences, and technology, among others, and serves as a tool for solving various real-world problems (Mulero et al., 2013). In particular, algebra is considered a necessary and very useful area, as it is an effective means of expressing mathematical ideas, especially in equations, where relationships are established among variables, patterns, and algebraic structures, among others (Socas, 2011). On the other hand, Rico (1995) points out that errors are common and can occur at any time during learning, and that examining them as a subject of permanent interest, especially in mathematics education, constitutes a possibility for the acquisition and consolidation of knowledge (Rico, 1995). We understand the error from the perspective of Godino et al. (2004), who states that it constitutes a student action or production that is not valid from an educational point of view; thus, to achieve adequate learning, incorrect answers are considered difficulties and even failures.

Despite the above, mathematical errors are usually considered harmful elements and, in many cases, do not receive adequate didactic attention (Mancera & Basurto, 2015). In addition, some research reports that some difficulties are manifested through the errors that students make when performing algebraic tasks, in particular when they enter the higher level (Gamboa et al., 2019; García, 2015; Olivar et al., 2018), which is noteworthy given the number of years of education and mastery of relevant mathematical concepts that students should have at this level. Finally, Rodríguez et al. (2012) note that numerous investigations focus on students' difficulties and errors when learning algebra and, in addition, on the causes underlying them. However, the authors point out that references are rarely provided on how to make them useful, thereby hindering a better understanding of the contents.

Therefore, this research aimed to describe the application of one of the activities in a teaching proposal, in which mathematical errors were used to address difficulties and promote learning of linear equations with an unknown. It is relevant to research various aspects of algebraic knowledge, such as students' errors identified in the last year of high school, as they are of utmost importance for subsequent educational levels (García et al., 2011; Kayani & Ilyas, 2014; Pianda, 2018). In addition, this research aims to use errors as a learning opportunity by incorporating them into the mathematics classroom, which could benefit students and provide information on measures to overcome them (García, 2015).

THEORETICAL FRAMEWORK

The idea of an error is associated with the quest for truth, which is sought by examining errors through rational criticism and self-criticism (Rico, 1997). In this sense, error is a sample of partially developed knowledge, characterised by inadequate perceptions of certain mathematical objects and the use of incorrect procedures from a mathematical point of view (Olmedo et al., 2015; Rico, 1997). In addition, according to Parra (2021), difficulties in mathematical learning can be conceived as "a lack, deficient, incomplete or contrary knowledge, which is the cause of one or more mathematical errors" (p. 22), which are due to various situations that are intertwined and include from poor planning to the very nature of mathematics (Herrera, 2010).

From a constructivist perspective, errors are not seen as faults to be punished or failures to be regretted (Astolfi, 1999). On the contrary, they are interesting characteristics of the difficulties that students manifest. Therefore, the teacher should be interested in them, since errors are indicators of students' progress and conceptual understanding. Despite the importance of errors in learning, current research has focused on error correction from a more general perspective. For example, in relation to algebraic learning, the research has sought to "capture the difficulties of students in all aspects of algebraic language learning" (Pérez et al., 2019, p. 85), specifically in conceptualising, symbolising, generalising, and reasoning algebraically (Pérez et al., 2019). Some examples of contributions in this line include the work of Bolaños and Lupiáñez (2021), as well as Parra (2021). However, as it is a very broad field of research, it is difficult to observe in depth the errors and difficulties of a specific topic (Pérez et al., 2019), such as linear equations. Therefore, it is necessary to conduct studies focused on the errors and difficulties of the mathematical object in question (Pérez et al., 2019; Hall, 2002), as proposed in this work, to develop correction strategies that address the specific difficulties students present.

Categories of errors in solving linear equations

This section presents a theoretical synthesis of errors in solving linear equations, the product of an adaptation of the categorisations of Movshovitz-Hadar et al. (1987), Hall (2002), Pérez et al. (2019), Rodríguez (2015), and Rosas (2013). Specifically, errors are classified into the following categories:

algebra errors that are in arithmetic, algebraic errors inherent to equations, and procedural errors when solving equations.

Algebra errors that are in arithmetic

Here are the errors that are a consequence of the difficulties that were not solved in the learning of arithmetic and have an impact on algebraic knowledge (Rodríguez, 2015). Concerning the above, Pérez et al. (2019) indicate three types of errors: (a) errors when performing basic operations with integers, (b) errors when performing basic operations with rational numbers and (c) errors in the distributive property.

Regarding *errors when performing basic operations with integers*, there may be difficulties in performing additions, subtractions, multiplication, and division with positive and negative integers. For example, $3x = -5 + 3 \Rightarrow 3x = -8$. *Errors when performing basic operations with rational numbers* refer to incorrectly performing the algorithms when operating with fractions, for example $\frac{5x}{2} + \frac{x}{3} = 2 \Rightarrow \frac{6x}{5} = 2$. Finally, *errors in the distributive property* are due to performing this property incompletely or incorrectly, as in example $3(x + 5) = 2 \Rightarrow 3x + 5 = 2$.

Conceptual errors related to the nature of algebraic symbolism

During primary education and part of the student's secondary education, the solution methods of some exercises focus on performing arithmetic operations, i.e., working with numbers, so that the result at the end is also a number. However, when starting the topic of linear equations with an unknown, the operations become more structured, dependent on algebraic knowledge and procedures whose structure is not as evident to students, generating confusion (Hall, 2002; Movshovitz-Hadar et al., 1987). This type of error is strictly algebraic in nature and has no explicit reference in arithmetic (Rodríguez, 2015). These errors are classified as follows: (a) errors when confusing a term with an unknown and a constant term, (b) errors when incorrectly using the equal sign (=) and (c) errors when applying algebraic notation conventions.

Regarding the *errors that occur when a term is confused with an unknown and an independent term*, Rosas (2013) notes that letters or unknowns are treated as labels rather than an equivalence relationship between variables and numbers. In this case, the error consists of adding a term with an unknown to a constant; in other words, one operates on dissimilar terms. That is, the student adds a term with an unknown to one that does not have it, thereby simplifying dissimilar terms (Pérez et al., 2019). For example,

$$5(2x + 1) = 7 \Rightarrow 5(3x) = 7$$

Regarding errors when using the equal sign incorrectly, *these* correspond to not using the symbol of equality as a symmetrical relationship, and more precisely, an equivalence relationship (Rosas, 2013). That is, "=" in its passage from arithmetic to algebra and formal substitution (Rodríguez, 2015). For example, when solving equation $2x + 1 = 7 + x$, students start from the fact that they must perform operations somewhere in the equation and perform $2x + 1 = 7 + x \Rightarrow 2x + 1 = 7 + x - 1 \Rightarrow 2x + 1 = x + 6$.

Finally, in cases of errors in applying algebraic notation conventions, different situations are identified: for example, while in arithmetic concatenation represents addition, in algebra it denotes product. In addition, certain symbols are used for fundamental operations. For example, in algebra, the multiplication symbol changes from being represented with an "×" to being represented with a "·" and using x for the representation of variables (Rosas, 2013). More precisely, when solving $2x = 24 \Rightarrow x = 4$, as the student assumes that expression $2x$ represents a number whose digit in the units is x, they cancel digit 2 on both sides of the equation and therefore obtain an equality with the digits in the units.

Procedural errors when solving equations

When using equation-solving algorithms, they confuse and mix up the operations that must be performed. Here we present the classifications: (a) errors due to the misuse of the scale method, (b) errors due to incorrect transposition, (c) errors due to the application of the inverse operations incorrectly, (d) errors due to the lack of understanding of algorithms, (e) errors due to the lack of verification of the solution, and (f) errors due to misused data.

For Pérez et al. (2019), *errors arising from the misuse of the scale method* occur when applying the same operation to both sides of the equality, but doing so incorrectly on only one side. For example: $7x - 8 = 3 \Rightarrow 7x - 8 + 8 = 3$.

In *errors due to incorrect transposition*, there is difficulty in the exchange of addends, i.e., the inability to understand transposition of terms, such as changing sides and changing signs (Rosas, 2013). This type of error arises when the order of priority or hierarchy of the operations is not recognised, leading to inconsistencies when transposing terms from one side to the other (Hall, 2002; Pérez et al., 2019). For example: $\frac{3x}{2} + 6 = 4 \Rightarrow 3x + 6 = 8$, where an error is made when multiplying by 2, or also in the following case,

$\frac{x}{2} + 3 = 5 \Rightarrow x + 3 = 2 \cdot 5 \Rightarrow x + 3 = 10$, where we note that the denominator 2 has been transposed in the right part of the equation before the value of 3, the value 3 is multiplied by 5, resulting in 10; so an error is incurred in the priority order of the operations involved in the equation.

Errors due to the incorrect application of inverse operations stem from a lack of understanding of the concept and objective of each algebraic operation and its inverse (Hall, 2002; Pérez et al., 2019). An example is: $4x = 1 \Rightarrow x = 1 - 4$.

According to Hall (2002), *errors due to the lack of understanding of algorithms* occur when students do not understand that, to solve a linear equation, one method is to repeat two processes: deduction and reduction. The first process (deduction) consists of applying the same algebraic operation to both sides of the equation to maintain equality. The second (reduction) involves replacing an algebraic equation with an equivalent one by performing the necessary operations with similar terms at both ends of the equation. This is exemplified below: $3x + 7 = 2x \Rightarrow 3x + 7 - 2x = 2x - 2x\ (deducción) \Rightarrow x + 7 = 0\ (reducción)$. Students can get confused when they do not know at which step they are or what they are working on. In fact, they probably have not even noticed that the algorithm used to solve equations has only two steps, repeated a certain number of times.

Regarding errors due to the lack of solution verification, Movshovitz-Hadar et al. (1987) argue that these errors occur when each step of task performance is correct. However, the result at the end does not correspond to a plausible solution. In other words, the possible solution does not meet one or more required hypotheses, which are independent of the algorithmic process to obtain the possible solution. It can also be caused by omitting necessary conditions when solving an equation. For example, every denominator of a fraction must be different from zero.

Finally, *errors due to misused data* result from a mismatch between the data and the treatment given by the student. In other words, these errors arise during data collection or processing. For example, forgetting important data for the solution of a problem, as well as errors in the transcription or translation of the semantic language to algebraic expressions with numbers, operations, and variables (Movshovitz-Hadar et al., 1987; Rosas, 2013).

A summary of these error types in solving linear equations is presented in Table 1.

Table 1

Categories of errors in solving linear equations

Category	Type of error
CE1. Algebra errors that are in arithmetic.	E1.1 Errors when performing basic operations with integers. E1.2 Errors when performing basic operations with rational numbers. E1.3 Errors in the distributive property.
CE2. Errors due to algebraic symbolism.	E2.1 Errors due to confusing the term of the unknown and the independent term. E2.2 Errors due to incorrect use of the equal sign (=). E2.3 error due to algebraic notation conventions.
CE3.Procedural errors.	E3.1 error due to the misuse of the scale method. E3.2 Errors due to incorrect transposition. E3.3 Errors due to the application of the inverse operations incorrectly. E3.4. Errors due to the lack of understanding of the algorithms. E3.5 Errors due to the lack of verification of the solution. E3.6 Errors due to misused data.

Source: Movshovitz-Hadar et al. (1987), Hall (2002), Pérez et al. (2019), Rodríguez (2015), Rosas (2013) and authors.

Pedagogy of error: a teaching proposal for error treatment and correction

By understanding the teaching process as a sequence of pedagogical actions that influence student learning (Calzado, 2004), and error as an inseparable component of life and the learning process, teaching strategies can be generated, and errors made useful, in the sense that they allow learning to be improved (Torre, 2004). Under these guidelines emerges the pedagogy of error, a teaching proposal aimed at analysing the knowledge students build. Using their errors, we can diagnose which concepts or content present difficulties and need improvement, so we can provide them with adequate help and eradicate or reduce their occurrence (Torre, 2004). Then, instead of creating distance from the errors, one can deepen their logic and take advantage of it for improvement (Astolfi, 1999). Likewise, when we correct an error, we prevent its recurrence. Therefore, Torre (2004) proposes three phases of treatment of mathematical errors, which are presented below:

1. Error detection: This phase is of the utmost importance, because, until they are located and recognised, it is impossible to move forward.
2. Error identification: In this phase, an attempt is made to diagnose errors to provide sufficient information for their subsequent rectification.
3. Error rectification: Here, the process aims to change students' knowledge and, unlike a traditional approach to teaching, the pedagogy of error incorporates errors into their correction.

Once errors in the mathematics class have been identified, the next step is to determine how they can be corrected. For this, Torre (2004) proposes a series of strategies and activities as follows:

a) Error record sheet: This strategy is mainly based on the systematic observation and recording of students' most frequent errors.

b) Correct or improve an exercise: With the support of the previous strategy, we can introduce the most frequent errors in exercises or texts, or ask students, individually or in groups, to locate, identify and correct them.

c) Second chance: It is about giving students a second chance to present their work or exercises after the teacher has provided corresponding observations.

d) Cooperative correction: Understanding learning as a social-constructive process, this strategy is based on the rectification of errors through teacher and other students' support.

e) Review of incorrect exercises: This strategy helps identify processes, from their approach to their execution.

f) On the hunt for the teacher's error: The teacher proposes a game with different types of exercises that contain errors, and students must find them. If the student discovers the error, they score in their favour; otherwise, the teacher scores.

g) Self-reflection-metacognition: It consists of an analysis strategy of the same failure, in the case of unexpected results (low results). Here, it is useful to describe the errors made and to think about how and why they occurred.

With the theoretical support presented, the importance of integrating errors into the teaching of mathematics and the existence of various strategies to do so are evident through a methodological proposal: the pedagogy of error.

That is, we propose a teaching process that begins with the characterisation and diagnosis of students' difficulties, followed by the execution of active learning actions to contribute to the development of students' significant mathematical knowledge, specifically about first degree equations with an unknown.

METHODOLOGY

Type of research and participants

The approach was qualitative (Hernández-Sampieri et al., 2010), based on an interpretative paradigm. Specifically, the design of the research was a collective case study (Cohen et al., 2007), because the personal perceptions of the informants were analysed, and the underlying meanings were explored, by understanding in depth the errors that eleventh-year students made when solving linear equations with an unknown, and their subsequent integration into the teaching proposal (Gil et al., 2017). The study involved a class of 18 students from the province of Heredia, in Costa Rica, who were in the last grade of secondary education in 2022, with an average age of 17 years and a medium-low socioeconomic background. The group was selected through convenience sampling, given the institutional access for fieldwork and the willingness of the teacher responsible for the course to collaborate on the research. Secondary school was chosen because it is the closest level to higher education, at which, ideally, students are expected to have the fundamental algebraic knowledge required before progressing to the next level (García, 2010; Chávez, 2018).

Information collection instruments

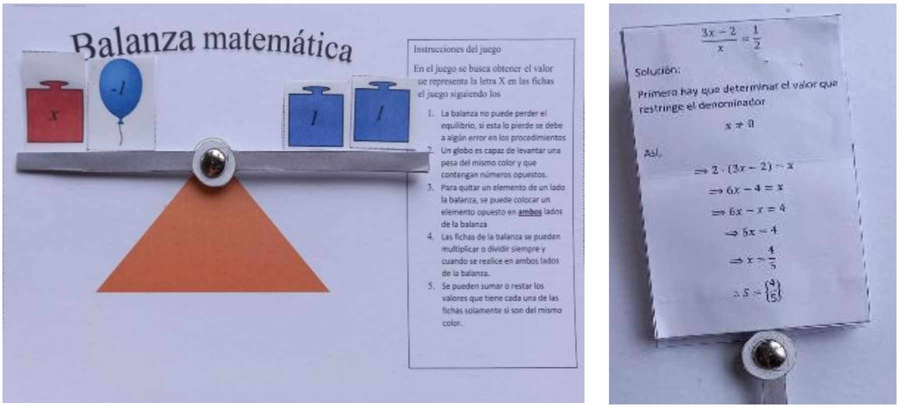
To collect information, different instruments were used. Firstly, a diagnostic questionnaire (pretest) was administered to identify the mathematical errors students made when solving linear equations with an unknown. This instrument was designed considering the contents and skills established in the Mathematics Study Programs of the Ministry of Public Education of Costa Rica (MEP, 2012). Subsequently, a teaching proposal composed of learning activities designed based on the errors identified in the pretest was implemented. Finally, the students' written productions were analysed during the activities, which served as the main source of information for the qualitative analysis of the results.

Math errors activity and fieldwork

In activity *A poner en equilibrio la balanza* [Balancing the Scales] (see Appendix 2), we present linear equations with errors, represented by unbalanced scales due to failures in their resolution. The equations were illustrated both algebraically and with manipulatives from the mathematical balance game (see Figure 1), which facilitated the step-by-step representation of the equations and their solution until the solution was completed. The activity was part of three learning activities from a teaching proposal established as part of a previous study (Sequeira, 2024). The activities were validated by two researchers in this study and four expert professors.

Figure 1

Manipulative materials provided in the activity



The fieldwork was carried out in several stages. The first stage consisted of designing the teaching proposal. First, we administered a pretest questionnaire and categorised the responses to understand the types of errors made within each analysis category and their frequency. Based on this information, a correspondence was established between the errors identified in the test and the approaches used in each activity of the teaching proposal, particularly in the activity *Balancing the Scales*. More precisely, the results helped develop exercises to address the errors raised in each activity. The exercises were proposed at three levels of complexity. Likewise, in the proposed activity, we tried to correct some errors or improve the solution to an exercise through cooperative correction. In addition, this activity was carried out according to the strategies proposed by Torre (2004), thereby implementing the phases of detection, identification, and error rectification in the pedagogy of error (Torre, 2004).

As a second stage, we apply the activity. The necessary material was provided to the students, including detailed instructions for the activity of this study and the required manipulatives. The activity was applied in groups of up to three people. The students identified the errors and tried to provide solutions to the equations raised, correcting or avoiding the errors. Likewise, they were encouraged to correct these equations using manipulative scales and to comment on the errors detected in each proposed solution. For the activity, 80 minutes of class time were available.

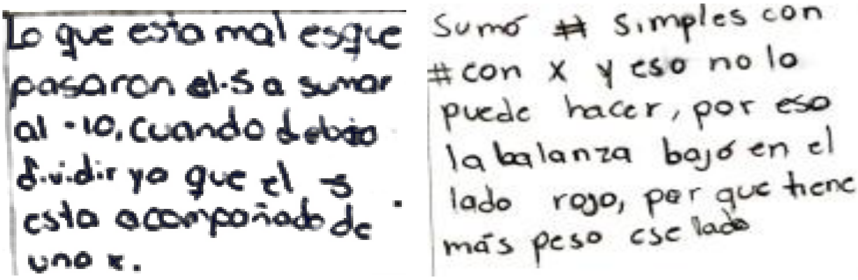
Analysis of information

The analysis of the information was carried out through a content analysis process, which allowed identifying and classifying the errors present in the students' written productions. For this purpose, the error classifications described in the theoretical framework were used as analytical categories, particularly those proposed by Movshovitz-Hadar et al. (1987), Hall (2002), Pérez et al. (2019), Rodríguez (2015), and Rosas (2013). These categories enabled the encoding of student responses and the analysis of different types of errors in solving linear equations with an unknown. The process consisted of identifying fragments of written productions that evidenced errors or indicated conceptual understanding, classifying them according to the corresponding category, and interpreting their meaning in relation to the objectives of the research. Specifically, fragments of text that provided explanations or interpretations indicating possible improvements in relation to the errors identified in the pretest questionnaire were used as thematic analysis units (Krippendorff, 1990). For example, in exercise: $-5x = -10 \Rightarrow x = -10 + 5$, an error corresponding to the category of incorrect transposition is observed. In this case, group G1 identified the error and subsequently proposed a correct solution, correcting the initially proposed procedure, as shown in Figure 2a.

Another example is the answer of the G8 group when proposing the solution to exercise $8x + 5 = 6x + 2 \Rightarrow 13x = 8$, which contains an *error due to confusing the term with an unknown with a constant term* (see Figure 2b). In the students' answer, the phrase refers to an incorrect reduction of similar terms, whose justification is related to the manipulative material proposed in the activity.

Figure 2

Examples of indicators of improvement of the errors proposed in the activity.



(a) Group 1 solution phrase. (b) Group 8 solution phrase.

To strengthen the consistency of the analysis, the categorisation was reviewed by all study investigators. Subsequently, the discrepancies in the classification were discussed until consensus was reached in the interpretation of the data.

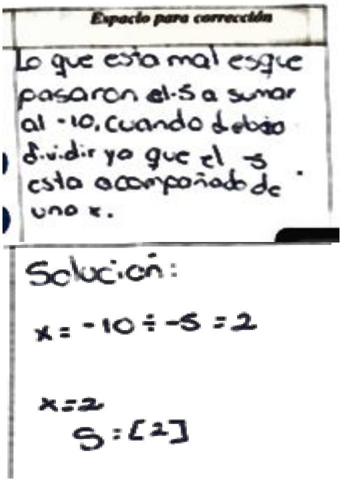
RESULTS

The analysis of the results obtained in solving the questions raised in the activity is shown below. Regarding the exercises of level 1 of the activity, we present to the students exercise 1: $-5x = -10 \Rightarrow x = -10 + 5$, with an error *due to an incorrect transposition.* The images in Figure 3 show the answers provided by groups G1 and G6, respectively.

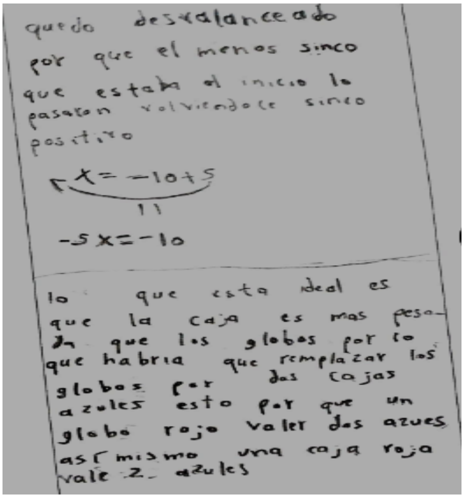
Figure 3(a) shows that the students identified the error raised, pointing out that -5 should not be *transposed* using addition but rather treated as a division. This can be interpreted as an indication of an understanding of the error *due to an incorrect transposition*, when identifying that the arithmetic operation between the constant -5 and the variable was a multiplication, and consequently, to solve the equation correctly, it was necessary to transfer the term to the other side, using the inverse operation, in this case, division.

Figure 3

Solution of exercise 1 of level 1 of the activity



(a) Group 1 solution.



(b) Group 6 solution.

Similarly, in Figure 3(b), the students in group G6 were able to identify the raised error, which was related to transposing the term -5 at the right side of the equation as a sum, although they did not clearly detail the solution process in their answer (Pérez et al., 2019). However, they provided the sentence: "*a red balloon is worth two blue ones, as well as a red box is worth two blue boxes*" and later detailed that variable x was equal to 2, which corresponded to the correct solution of the equation.

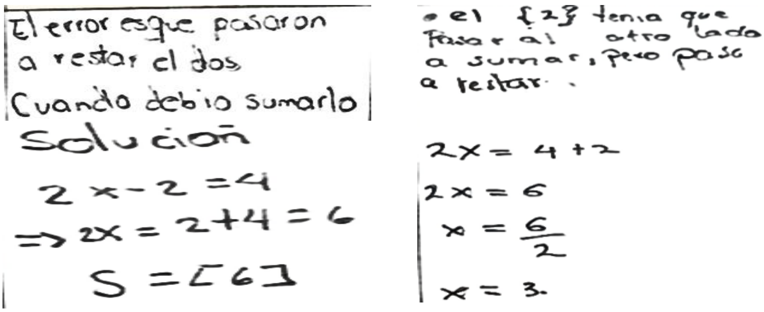
Figure 4(a) shows how the students identified the *error* in the solution of the original equation and provided a corrected solution. However, in the last procedure they performed, the coefficient of the variable x was not considered. In addition, they provided the solution set without fully clearing the variable, i.e., it was still necessary to perform additional procedures to obtain the solution set and complete the process. Despite not making any additional mistakes, the solution provided is incorrect. In contrast, the way the solution set is written, using brackets "[ ]" instead of braces "{ }", suggests that the students were not confident in their use or did not understand the meaning of the solution set.

For exercise 2, students were asked to analyse the *errors in applying inverse operations* (Pérez et al., 2019; Hall, 2002). The proposed exercise was

$2x - 2 = 4 \Rightarrow 2x = 4 - 2$. The answers of groups G1 and G7, respectively, are shown in Figure 4.

Figure 4

Solution of exercise 2 of level 1 of the activity



(a) Group 1 solution. (b) Group 7 solution.

Figure 4(b) shows how the students detected and corrected the error present in the solution of the proposed equation. In this case, they pointed out that *the 2 had to go to the other side to add, but it subtracted*. More precisely, this group of students correctly identified the error and solved the equation. However, in their response, they omitted the solution set.

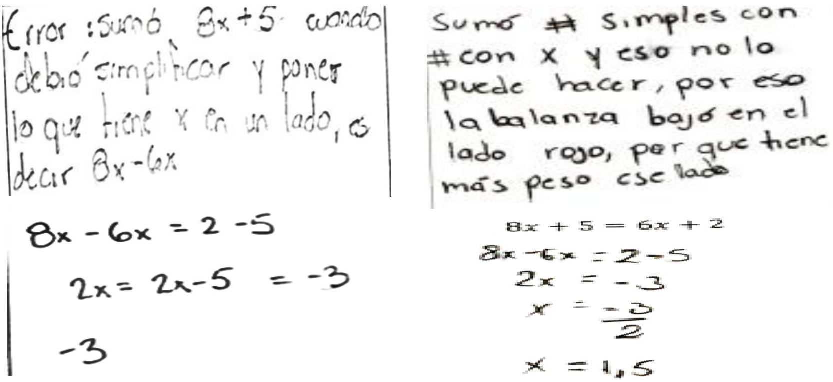
In exercise 3, we identified two errors: the first corresponded to the *confusion between the unknown and independent terms*, and the second was *an incorrect transposition* (Pérez et al., 2019). The proposed example is:

$$8x + 5 = 6x + 2 \Rightarrow 13x = 8 \Rightarrow x = 8 - 13$$

The answers of groups G4 and G8, respectively, are presented in Figure 5. In Figure 5(a), as for exercise 3, the students indicated that the sum $8x + 5$ was presented in the first step, when, before performing any operation, they had *to put what they had* x *on one side*, referring to the fact that the order used to solve the equation was not adequate. Continuing with the explanation, although in their solution they correctly transferred the terms with variables and constants to the opposite sides and simplified adequately, they did not complete the last necessary step: transferring number 2 to the other side of the equality to be divided, to obtain the correct final solution. In addition, by not verifying the solution set, a different response was given to the correct solution set of the exercise (Movshovitz-Hadar et al., 1987).

Figure 5

Solution of exercise 3 of level 1 of the activity



(a) Group 4 solution. (b) Group 8 solution.

Similarly, in Figure 5(b), the group pointed out the error of *adding simple numbers to expressions containing x,* noting that it is impossible to add or subtract variables with independent terms (Pérez et al., 2019). Although they did not formally present the solution set, they showed the correct solution in both its fractional and decimal representations. We highlight that both groups focused on the first type of error shown, leaving aside the second proposed error. To this end, we show the answers for groups G4 and G8 in Figure 5.

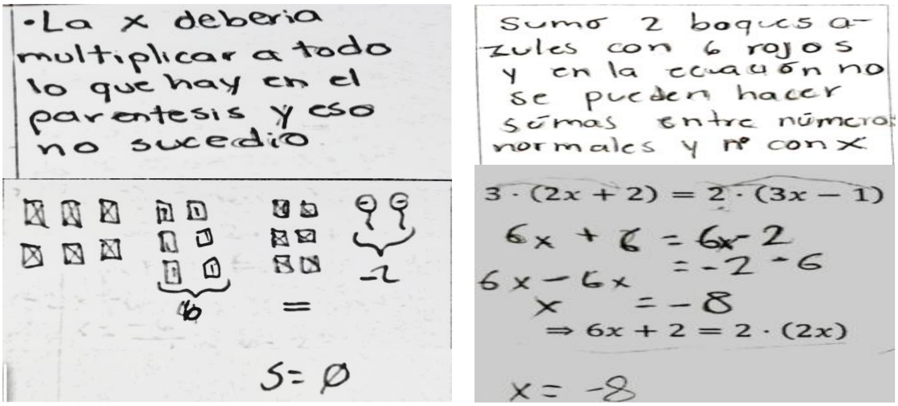
Regarding the exercises of level 2 of the activity, in exercise 1, students were shown two types of errors in *the distributive property*, in which they *confused between the term with the unknown and the independent term* (Pérez et al., 2019) as follows:

$$3 \cdot (2x + 2) = 2 \cdot (3x - 1) \Rightarrow 6x + 2 = 2 \cdot (2x) \Rightarrow 8x = 4$$

The answers of groups G5 and G8, respectively, are presented in Figure 6. In Figure 6(a), students attempted to describe the error shown. Although their explanation was not entirely clear, students emphasised the error in the distributive property, noting that x *must have multiplied everything in the parentheses* (Pérez et al., 2019). On their side, when solving the exercise, the students chose to present their reasoning through drawings rather than a formal mathematical procedure. They illustrated the elements of a scale; a tool used in the activity. Through this graphical representation and following the rules of the proposed game, their solution can be translated into the algebraic form: $6x + 6 = 6x - 2$, which they simplified to $6 = -2$. From this reasoning, they concluded that the solution of the exercise is $S = \emptyset$. This allows us to observe that students understand the concept of the solution set of an equation, even when it corresponds to the empty set. Likewise, the visual representation used in the activity helped them correctly identify the answer to the exercise. However, these results suggest that some students find it challenging to formally express procedures in algebraic language that can be represented visually or with manipulatives. This type of difficulty is not exclusive to this context, as it has also been reported at the university level, where a high percentage of students make errors in content they are supposed to master (Gamboa et al., 2019; Parra, 2021).

Figure 6

Solution of exercise 1 of level 2 of the activity



(a) Group 5 solution. (b) Group 8 solution.

In the solution shown in Figure 6(b), students identify an error in the distributive property on the second line of the solution. They indicated that *blue blocks* were added to *red blocks*, however, pointing out that they could not be added because they represented dissimilar terms. They focused on proposing an algebraic solution and correctly developing it until, in the last step, they made an error. For the expression $6x - 6x$ they incorrectly obtained the result x, when it should have been 0. Despite performing most procedures correctly, this error was due to the fact that they sometimes made logically invalid inferences, resulting in invalid deductions from a mathematical point of view (Movshovitz-Hadar et al., 1987). This may be because variables are sometimes treated as labels, and the relationships between them and the constants they accompany are not considered (Rosas, 2013).

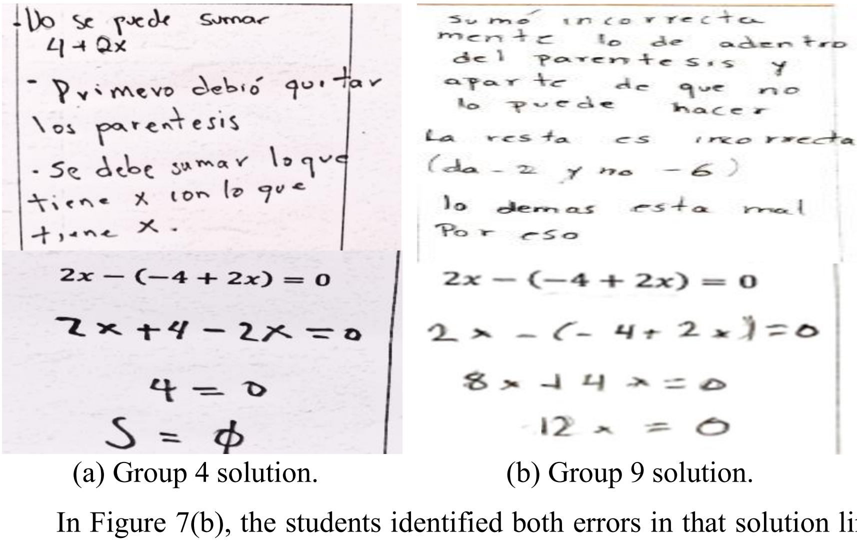
Next, the solutions provided by groups G4 and G9 for exercise 2 of the same level of complexity are shown. This addressed *errors in performing basic operations with integers* and *in confusing the unknown term with the independent term* (Pérez et al., 2019). The example presented is:

$$2x - (-4 + 2x) = 0 \Rightarrow 2x - (-6) = 0$$

In Figure 7(a), the students indicated that the error made was related to the *order in which the operations are performed* (Hall, 2002), as they noted that, when solving the exercise correctly, the first step should be to eliminate the parentheses from the equation and then simplify the similar terms. In addition, they noted that only *what x has should be added to what x has*, highlighting that, in the solution presented, the error stemmed from their confusion between the unknown and the independent term (Pérez et al., 2019).

Figure 7

Solution of exercise 2 of level 2 of the activity



(a) Group 4 solution. (b) Group 9 solution.

In Figure 7(b), the students identified both errors in that solution line and argued that, in this one, the terms within the parentheses were added incorrectly, since the addition could not be made because the terms were not similar. In addition to this, they mentioned that, when performing addition $-4 + 2x$, the result was –6, instead of -2, alluding to an *error when performing operations with whole numbers* (Pérez et al., 2019). On their side, when solving exercise 2, they made several mistakes. When trying to eliminate the parentheses from the equation, they obtained the result $8x + 4x$, as if the expression to be simplified had been $2x\ (- - 4 + 2x)$. These errors may be due to difficulties not resolved in the learning of arithmetic and may subsequently affect algebraic knowledge (Rodríguez, 2015).

At level 3 of the activity, we proposed first degree equations involving irrational numbers or fractions. It is important to mention that these contents do not correspond properly to the eleventh level but are part of the student's prior knowledge (MEP, 2012). We realised some students did not know how to approach this type of equation and could not answer. The answers collected from exercises 1 and 2 of this level, provided by groups G1 and G7, are shown in Figures 8 and 9.

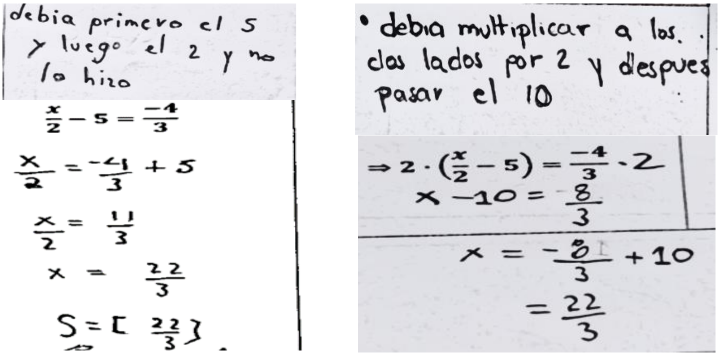
In the first exercise, the solution proposed to the students for review contains an *error in the scales method* (Pérez et al., 2019). The proposed example is as follows:

$$\frac{x}{2} - 5 = \frac{-4}{3} \Rightarrow 2 \cdot \left(\frac{x}{2} - 5\right) = \frac{-4}{3} \Rightarrow\ x - 10 = \frac{-4}{3}$$

The answers of groups G1 and G7, respectively, are presented in Figure 8.

Figure 8

Solution of exercise 1 of level 3 of the activity



(a) Group 1 solution. (b) Group 7 solution.

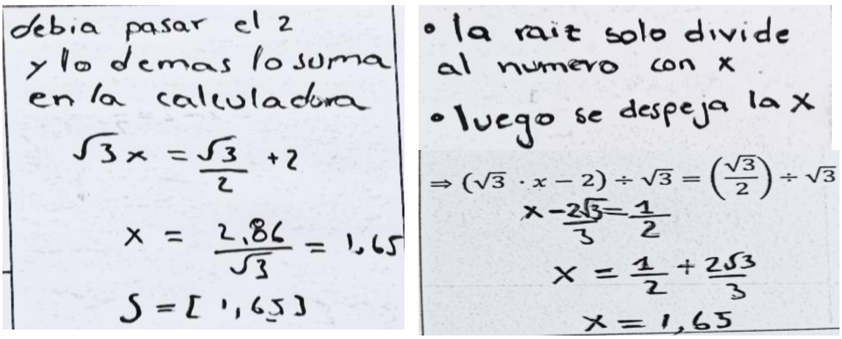
In Figure 8(a), students pointed out that the error in the solution was associated with the hierarchy of operations to be performed (Pérez et al., 2019). For this, they proposed a different sequence in the solution: first, move the constant terms to the right side to simplify them, and then treat the denominator of the variable as a multiplication. Following these steps, they correctly found the solution set of the equation. However, they provided it in brackets "[ ]", when braces "{}" are used instead, which may relate to difficulties students encounter with the characteristics of the symbolism that establish a specific use for each symbol (Rodríguez, 2015). In Figure 8(b), students recognised the error in *the scale method*, highlighting that the failure was to multiply by 2 on only one side of the equality and not both (Pérez et al., 2019). By correcting this error and applying a method different from that used in the G1 group, they also found the correct solution to the equation. Despite this, they did not provide the solution set or verify the solution found.

A similar situation occurred in exercise 2 and the last exercise of the activity, since the same groups followed different solution paths and performed other procedures. However, the result obtained at the end of the process matched the correct solution to the exercise. This exercise was designed to help students recognise an error *in the distributive property*. The specific error that involves dividing a parenthesis between the same element or —alternatively— multiplying it by the inverse of that element (Pérez et al., 2019). More precisely, the following example was proposed:

$$\sqrt{3}\cdot x-2=\frac{\sqrt{3}}{2}\Rightarrow\left(\sqrt{3}\cdot x-2\right)\div\sqrt{3}=\left(\frac{\sqrt{3}}{2}\right)\div\sqrt{3}\Rightarrow x-2=\frac{1}{2}$$

Figure 9

Solution of exercise 2 of level 3 of the activity



(a) Group 1 solution. (b) Group 7 solution.

Figure 9(a) shows that the students reported that the error corresponded to inconsistencies in the order of the operations to be performed to find the solution to the exercise (Hall, 2002). Specifically, they indicated that they *first had to add 2, and the rest can be added in the calculator*, alluding to the process to follow to find the solution. To perform the addition involving numbers with radicals, they used a scientific calculator and presented the results as a decimal approximation. In contrast to what was mentioned above, in Figure 9(b) they pointed out that the error was that the two numbers had to be divided by *the root,* alluding to an *incomplete distributivity error* (Pérez et al., 2019). They corrected this error and then transposed the constant term to the opposite side, clearing the variable without committing inconsistencies. Finally, they used the calculator to simplify the expression and provide a decimal approximation of the solution.

Finally, we highlight that across all the afore mentioned results, we found no indication that the students verified the solution obtained to determine the valid solution set of the equation.

CONCLUSIONS

In this work, we described the application of one of the activities of a teaching proposal that addressed the difficulties and promoted the learning of the topic of linear equations with an unknown. A contribution of this study lies in the need to understand the errors students make in the last year of secondary education, as these errors are of utmost importance for their subsequent educational levels (García et al., 2011; Kayani & Ilyas, 2014; Pianda, 2018). The results showed, although not in general, that incorporating the analysis and correction of errors into the activity of this study revealed the state of knowledge of the participants (Del Puerto et al., 2006).

In the activity, varied results were presented in the understanding and application of algebraic concepts and procedures. Although in some exercises, participants showed the ability to identify errors and apply corrections, some solutions failed to follow a coherent line of reasoning (Movshovitz-Hadar et al., 1987). The errors committed were related to the use of algebraic symbolism and the meaning of the equal sign (=) in equations (Rodríguez, 2015; Rosas, 2013), among others. Also, it is possible to highlight persistent difficulties with the manipulation of fractions and radicals, as participants provided fewer answers. In addition, we highlight the lack of verification of the proposed solution as a method for finding the valid set of the equation. Specifically, this could be due to the previous education received, possibly focused only on equations defined over the set of rational numbers, and to the lack of verification of the possible solution. This lack of knowledge poses a challenge for current learning and its future application in higher education courses, since students are supposed to master the prior knowledge necessary for this (Gamboa et al., 2019; Parra, 2021).

Regarding the phases of treatment of mathematical errors, according to Torre (2004), we follow the three phases: detection, identification, and rectification. This is evidenced by our previously detecting and identifying errors, the results of which enabled the design and application of an activity consistent with them. It is important to highlight that the activity was built with the aim of correcting or reducing these types of errors, carried out both through cooperative correction and in the review of exercises that were resolved incorrectly. In the activity, students enjoyed a playful experience identifying and correcting errors in linear equations. In addition, using a manipulative scale, they visualised the balance of the scale as a comparison with the application of operations on both sides of the equations, and how errors influenced their balance. In turn, it was evidenced that the adoption of didactic strategies that focus on errors as opportunities for learning develops critical and analytical skills in mathematics. Throughout the session, a high level of student interest and participation was observed, which created an environment conducive to learning. Although some students presented greater challenges than others, we observed notable improvements in their understanding and resolution of first degree equations, reflected in a more reflective and critical approach to learning.

Based on the findings, we believe it is crucial to create learning opportunities that incorporate the use of errors as part of the teaching process in the mathematics classroom, thereby benefiting students and providing relevant information to improve their learning (García, 2015). In this sense, the intentional use of errors in learning activities can be a valuable didactic strategy to foster mathematical reflection and deepen understanding of algebraic procedures.

AUTHORSHIP CONTRIBUTION STATEMENT

KPL, JRJ, and DSL conceived the idea presented. KPL and DSL developed the theory and methodology. KPL, JRJ, and DSL designed instruments that allowed the information to be collected. DSL collected and analysed the data and wrote the conclusions. All the authors actively discussed the results and reviewed and approved of the final version of the paper.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, KPL, upon reasonable request.

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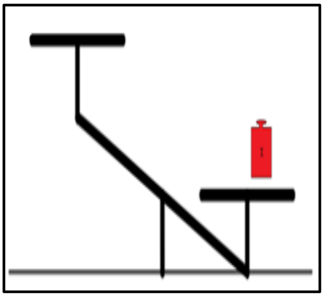
APPENDIX

1. Pretest and posttest questionnaire

In order to access the pretest and posttest questionnaire instrument that allowed us to collect part of the information necessary for the construction of activity 1, we invite you to scan and enter the following QR code adjacent to this paragraph.

2. Activity 1 of the teaching proposal: *Balancing the Scales*

This activity is intended to help students identify errors in a linear equation that has already been solved and illustrated by a scale, showing where they lose their balance when making an erroneous movement. For that, students will use the scale method to solve first degree linear equations with an unknown. On the other hand, it is divided into three levels of complexity, in which the materials the teacher provides are indispensable, in addition to the rules that allow relating the game Balancing the Scales to each of the processes for solving a linear equation with an unknown.



To access the total activity, please scan and enter the following QR code.