ACTA SCIENTIAE

ISSN: 2178-7727
DOI: 10.64856/acta.scientiae.8383

Teaching and Learning Variation and Problem Solving: Beliefs of Pre-Service Mathematics Teachers

Cristian Camilo Fúneme Mateus[a]

[a] Universidad Distrital Francisco José de Caldas, Facultad de Ciencias y Educación, Bogotá, Colombia.

ABSTRACT

Background: Problem solving is currently regarded in mathematics education as a cornerstone of learning processes, particularly for transversal topics such as variation. Nevertheless, how to integrate it effectively into teacher preparation remains an open issue. **Objectives**: To characterise the beliefs of preservice mathematics teachers at a Colombian university about the use of problem solving in teaching and learning variation. **Design**: Adopting a qualitative approach, this study is theoretically and methodologically grounded in the analytical tools and strategies of the onto-semiotic approach to mathematical knowledge and instruction, embracing its pragmatic perspective that views beliefs as dispositions for action. **Participants**: Thirty preservice mathematics teachers. **Data collection and analysis**: Epistemic configurations were constructed to describe the design and analysis practices employed by the participants when working on variation-related problems. The justifications they offered for these practices were then collected to identify their underlying beliefs. **Results**: Four stances regarding the role of problem solving in the teaching-learning process of variation emerged: (1) central axis, (2) goal, (3) part of the process, and (4) final stage. **Conclusions**: Preservice teachers display considerable diversity in their interpretations of problem solving, highlighting the need to further align current theoretical developments with actual educational practice.

Keywords: Beliefs; Preservice mathematics teachers; Problem solving; Variation.

La Resolución de Problemas en la Enseñanza y Aprendizaje de la Variación: Creencias de Profesores de Matemáticas en Formación Inicial

RESUMEN

Contexto: La resolución de problemas es considerada en la educación matemática actual como un eje fundamental para los procesos de aprendizaje, especialmente en objetos transversales como la variación. Sin embargo, el cómo integrarla eficazmente en la formación docente es un problema abierto. **Objetivos**: La caracterización de las creencias de profesores de matemáticas en formación inicial, de una universidad colombiana, sobre el uso de la resolución de problemas en la enseñanza

Corresponding author: Cristian Camilo Fúneme Mateus.
Email: ccfunemem@udistrital.edu.co

y aprendizaje de la variación. **Diseño**: Desde un enfoque cualitativo, este estudio se fundamenta teórica y metodológicamente en las herramientas y estrategias de análisis del Enfoque Ontosemiótico del Conocimiento y la Instrucción Matemática, adoptando su perspectiva pragmática que reconoce a las creencias como disposiciones para la acción. **Participantes:** 30 futuros profesores de matemáticas en formación inicial. **Análisis de datos**: A través de la elaboración de configuraciones epistémicas se describen las prácticas de diseño y análisis realizadas por los futuros profesores al trabajar problemas relacionados con la variación, luego se recolectan las justificaciones que dan los docentes a sus prácticas y se identifican sus creencias. **Resultados**: Se caracterizan cuatro posturas alusivas al papel de la resolución de problemas en el proceso de enseñanza y aprendizaje de la variación: (1) eje central, (2) finalidad, (3) parte del proceso y (4) etapa final. **Conclusiones**: Existe una diversidad considerable en la interpretación de la resolución de problemas por parte de los futuros docentes, lo que evidencia la necesidad de profundizar en cómo articular los desarrollos teóricos actuales con la práctica educativa real.

Palabras clave: Creencias; Profesores de matemáticas; Resolución de problemas; Variación.

Ensino e a Aprendizagem da Variação e a Resolução de Problemas: Crenças de Professores de Matemática em Formação Inicial

RESUMO

Contexto: A resolução de problemas é considerada, na educação matemática atual, como um eixo fundamental para os processos de aprendizagem, especialmente em objetos transversais como a variação. Contudo, a questão de como integrá-la eficazmente na formação docente permanece aberta. **Objetivos**: Caracterizar as crenças de professores de matemática em formação inicial, de uma universidade colombiana, sobre o uso da resolução de problemas no ensino e aprendizagem da variação. **Design**: A partir de uma abordagem qualitativa, este estudo fundamenta-se teórica e metodologicamente nas ferramentas e estratégias de análise do Enfoque Ontossemiótico do Conhecimento e instrução Matemática, adotando sua perspectiva pragmática, que reconhece as crenças como disposições para a ação. **Ambiente e participantes**: 30 futuros professores de matemática em formação inicial. **Coleta e análise de dados**: Por meio da elaboração de configurações epistêmicas, descrevem-se as práticas de desenho e análise realizadas pelos futuros professores ao trabalharem problemas relacionados à variação. Em seguida, coletam-se as justificativas que os professores dão às suas práticas e identificam-se suas crenças. **Resultados**: Foram caracterizadas quatro posturas relativas ao papel da resolução de problemas no processo de ensino e aprendizagem da variação: (1) eixo central, (2) finalidade, (3) parte do processo e (4) etapa final. **Conclusões**: Existe uma considerável diversidade na interpretação da resolução de problemas pelos futuros docentes, o que evidencia a necessidade de aprofundar como articular os atuais desenvolvimentos teóricos com a prática educativa real.

Palavras-chave: Crenças; Professores de matemática; Resolução de problemas; Variação.

INTRODUCTION

Problem solving has been a cornerstone of mathematics education for fostering critical, analytical, and reflective skills in students (Santos, 2024). In fact, various investigations have shown that teaching and learning mathematics through problems not only improves conceptual understanding but also stimulates reasoning and creativity in real, meaningful contexts (Miranda & Mamede, 2022). In this way, problem solving transcends the simple act of solving exercises and becomes a central process for encouraging mathematical thinking.

The importance that problem solving has acquired has led, for several decades, to numerous studies emphasising the need to integrate it into the initial and continuing education of mathematics teachers (Santos & Reyes, 2019; Schoenfeld, 2022). This incorporation seeks to develop in prospective teachers the competencies to create challenging and enriching didactic scenarios, in which problems are true articulating axes of mathematical learning. However, the effective transfer of these principles into the formative practices of prospective teachers requires more than their theoretical incorporation into curriculum plans (Parra & Breda, 2017).

In this sense, initial teacher education programs play a key role in the effective integration of problem solving into teaching practices, as they shape beliefs that largely guide prospective teachers' practices in real educational contexts (Wellberg, 2024). However, how problem solving has been linked to teacher education remains unclear.

That is, despite the broad consensus on the educational relevance of problem solving, there is still no consensus on how it should be developed in initial teacher education or on how its use demarcates the beliefs and teaching practices of prospective teachers (Cai & Hwang, 2023). Thus, it is necessary to explore the beliefs that prospective teachers build regarding the implementation of problem solving and how they affect their selection, design, and application of mathematical tasks in real contexts.

Moreover, it is important to understand how beliefs about problem solving constrain or favour the design and execution of situations for the teaching and learning of specific mathematical objects. For example, variation has been found to be a mathematical object that poses a particular challenge for work focused on problem solving, because its conceptual complexity leads teachers to focus on disjointed algorithmic and procedural aspects that divert students from a meaningful understanding (Cantoral et al., 2023).

In this framework, this research characterises the beliefs of a group of mathematics teachers in initial education at a Colombian university regarding the use of problem solving in the teaching and learning of variation. Starting with the presentation of the theoretical and methodological aspects that allow the study of beliefs from the onto-semiotic approach to mathematical knowledge and instruction (OSA), this is followed by the conceptual elements of problem solving and variation that are taken into account. Finally, we present the analysis of the beliefs and the research conclusions.

THEORETICAL ASPECTS

The study of beliefs in the field of mathematics education has been extensive and diverse, with a large number of positions that differ in their terminology, methods, and purposes (Goldin et al., 2009). Thus, when studying beliefs, there is no alternative but to adopt a theoretical position consistent with the epistemic and ontological assumptions about mathematics and its didactics (Törner, 2002). This is the case of this research, which adopts the onto-semiotic approach to mathematical knowledge and instruction (OSA), considering that its pragmatic and anthropological perspective has been consolidated in mathematical education as a reference that provides solid conceptual and methodological tools for the analysis of didactic-mathematical activity in its multiple dimensions and with problem solving as the central axis (Godino, 2024). The aspects of this position are detailed below, including conceptualising problem solving and the meaning of the variation taken as a reference.

Conceptualisation and identification of beliefs in the OSA

The OSA is a theoretical system that assumes an anthropological and pragmatic position of mathematics (Godino, 2024), implying that “the activity of people in solving problems is considered the central element in the construction of mathematical knowledge” (Godino et al., 2020, p. 6). Understanding *activity* as the putting into action of systems of practices through which a situation or problem is answered, and in which organised systems of culturally shared objects intervene (Godino, 2021). Whereas a *problem* is a “situation in which an individual is asked to perform a task for which they do not have an easily accessible algorithm that completely determines the method of solution” (Lester, 1980, p. 287).

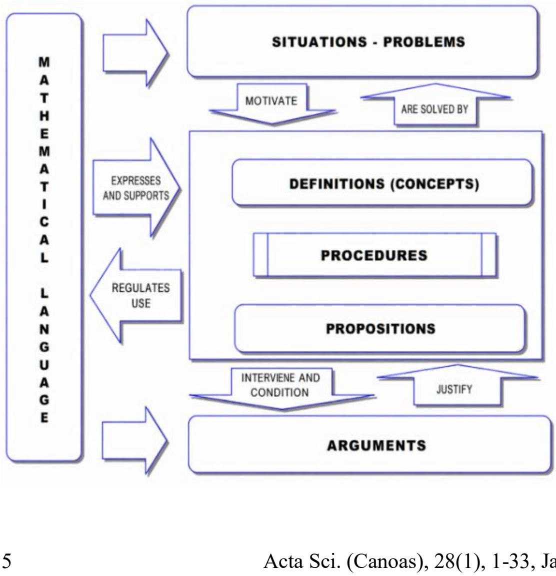
The above allows us to state that the activity around a problem leads to mathematical actions or expressions called *mathematical practices* (Godino, 2024). Only one person (personal practice) or a group of people (institutional practice) can be involved in these practices (Godino & Batanero, 1994). Expressions that support, regulate, or intervene in mathematical practices, whether material or immaterial, are referred to in the OSA as *mathematical objects* (Godino et al., 2020).

Mathematical objects are symbols of dynamic and constantly changing cultural units (Asenova, 2021). In this sense, the *meaning* attributed to a mathematical object is "the correspondence between an object and the system of practices where such an object intervenes" (Godino, 2021a, p. 8), established by the subjects involved in a practice. Of particular importance are the so-called *primary mathematical objects*: problems, languages, definitions, propositions, procedures, and arguments, since through them the meanings are articulated in mathematical practices (Godino et al., 2007).

In fact, given the variety of spatio-temporal and sociocultural scenarios, circumstances, or contexts in which mathematical practices are developed, which generate multiple processes of semiosis (Sáenz-Ludlow & Zellweger, 2016), the epistemic configuration of primary objects (Figure 1) has been developed in the OSA as a tool that makes visible, structures, analyses and discusses the emerging meanings of a mathematical practice.

Figure 1

Epistemic configuration of primary objects. (Font et al., 2010, p. 8)



In mathematical practices, some means of expression involve relationships of relative dependence between expressions and contents (Eco, 1979), giving mathematical objects a structural component that can be systematic (relationships between several objects) or unitary (an object as a whole) (D'Amore & Godino, 2007). Thus, in mathematical practices, objects emerge with different functions: representation, instrument, regulator, explanation, and justification (Godino et al., 2020).

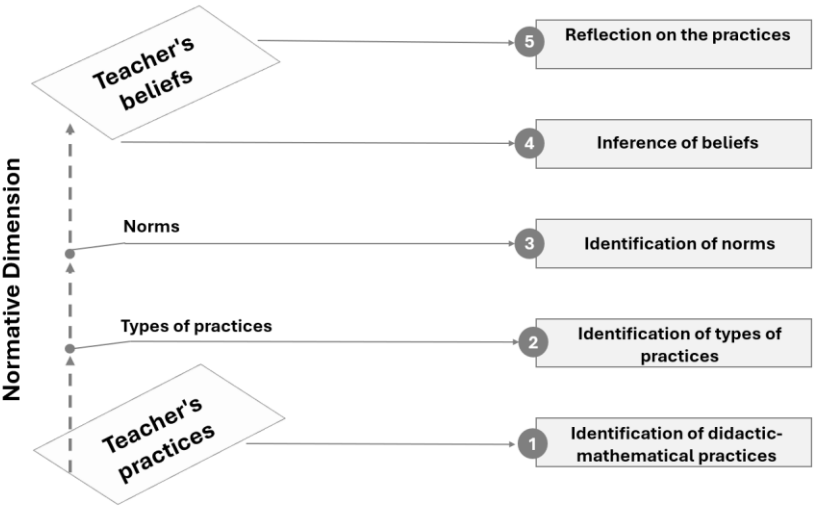
Precisely, in the scope of the objects of regulation and justification, in the OSA, beliefs appear. Although the discussion on the nature of beliefs has not reached a consensus and they have transited between clearly framing them in the cognitive field (Thompson, 1992; Goldin, 2002), or seeing them as objects of an exclusively affective nature (Nespor, 1987; DeBellis & Goldin, 2006), in the OSA they are recognised as objects that allow the two aspects to be connected.

In general, the OSA starts by considering the pragmatic position of mathematics, proposing that beliefs are dispositions for action, which become principles emerging from personal experience, with the intention of stimulating and inducing mathematical practices that give stability to a person's cognitive configuration (Godino et al., 2007). That is, they have an affective component, as they interact with emotions, attitudes, and values to stimulate mathematical activity (Fúneme, 2023), but they also have a cognitive scope, as they are mutually dependent on the meanings of the mathematical objects that emerge in mathematical practice (Beltrán & Godino, 2020).

This position allows an approach to a person's beliefs through the analysis of the design and implementation of practices, especially through reflection on them, because, through the justifications that teachers construct, the principles that stimulate or condition them emerge (Beltrán & Godino, 2017). A detailed description of how this process can occur is presented by Acevedo and Pino-Fan (2024), who propose that it should start from the identification and description of the teacher's practices through the configuration of primary objects, and then move towards reflection (Figure 2).

Figure 2

Approach to beliefs. (Acevedo & Pino-Fan, 2024)



In the first instance, the identification of teachers' didactic-mathematical practices is sought, understood as the actions or manifestations through which they approach the design, implementation, and evaluation of mathematics teaching and learning processes (Pino-Fan et al., 2022). To approach them, we use the epistemic configuration of primary objects as an instrument to visualise the mathematical objects to which teachers resort in their practices and the meanings they intend to achieve (Asenova et al., 2024).

Then, in the second moment, the practices are classified into: (1) *Planning* the teaching and learning situations or process; (2) *Motivation*, practices that seek the students' involvement in the mathematical activity; (3) *Assignment of tasks*, where the students' practices are directed and managed; (4) *Regulation* through rules and readjustments of the teaching and learning process; (5) *Evaluation* of the students' learning status; and (6) *Reflection* of the entire lived process (Acevedo & Pino-Fan, 2024).

However, to relate didactic-mathematical practices with beliefs, it is necessary to identify the norms that underlie teachers' actions and expressions, since these reveal the elements they take into account to regulate their practices (D'Amore et al., 2007). These norms correspond to the six facets of the teaching and learning process outlined in the OSA: epistemic, cognitive, affective, interactive, mediational, and ecological. This relationship is analysed in the third moment, in which the norms associated with the practices are identified according to the classification of Godino et al. (2009):

- Epistemic rules: regulation of mathematical knowledge through its decomposition into six primary objects.
- Cognitive norms: regulation of students' practices.
- Affective norms: regulation of the manifestations of students' emotions, attitudes, beliefs, or values about the objects of mathematical activity.
- Interactional norms: regulation of interactions among students, students and the teacher, students, the teacher, and learning situations.
- Mediation rules: regulation of the use of material, technological, spatial, and temporal resources.
- Ecological standards: regulation of the relationship with the contextual, social, political, economic, and curriculum scope, among others, that frame the teaching and learning process.

With the identification of practices and norms, the researcher has the necessary elements to establish regularities or patterns in teachers' actions and expressions; then, in the fourth moment, he/she proceeds to the initial inference of the beliefs. This inference aims to bring researchers and teachers together to dialogue about their practices and thus establish beliefs with greater certainty.

In this process of approaching beliefs, the first four moments fall to the researcher and are guided entirely by the principles and instruments of the OSA. As for the fifth moment, it is the final point in identifying beliefs; it plays a fundamental role in guiding reflection. For this reason Ledezma et al. (2022) propose as a process of emergence of beliefs to reflective analysis from: (1) *Confrontation* with the analysis of practices; (2) *Openness*, when the teacher receives comments and questions from other people; (3) *Argumentation*, where the teacher before the comments and questions alluding to his practices raises explanations of what he considered to develop them; and (4) *Conclusion*, in which what happened is jointly evaluated.

Problem solving in the OSA

Problem solving (PS) has been the subject of mathematical education since at least the beginning of the twentieth century, when two lines of research were clearly marked: (1) the development of heuristics, methods or schemes for PS and (2) PS as a trigger for mathematical activity.

Undoubtedly, the first line of research has highlighted the work of Polya (1945), Schoenfeld (1982), and Mason et al. (1982), who proposed models that are still frequently cited today. The success of their models was that they moved from approaching PS exclusively from the mathematician's work to studying how to link it to the teaching process. This allowed them to connect their ideas to the needs of teachers in educational institutions at different levels.

In the second line, the focus of analysis is different: researchers examine teaching and learning as a multidimensional process in which PS is the central axis of the mathematical activity of students, teachers, and researchers. Here are two great initial references, Gagné (1973) and Brousseau (1986), who rejected the search for models to follow in PS and proposed instead that freedom of thinking and action should be the protagonists in mathematics class, detaching it from the absolute control of the teacher as a transmitter of information and positioning it as a phenomenon of research and discussion for mathematical education. In this line, a large part of modern theories, such as mathematical modelling (Kaiser & Schwarz, 2010), ethnomathematics (D'Ambrosio, 2014), and objectivation theory (Radford, 2018), among many others, problematise the learning and teaching of mathematics and recognise the importance of PS.

The two lines of research in PS have contributed different elements to mathematics education (Fúneme et al., 2023). Therefore, in the OSA, their progress is recognised to consider PS as a macroprocess from which mathematical activity arises, which also allows us to conceive of it as a strategy that helps make sense of mathematical objects, their techniques, procedures, and, in general, their meanings (Font et al., 2013).

We also recognised that the mathematical practices that emerge in the classroom require teacher planning so that students' learning can be supported, managed, and encouraged. For this reason, the problem situations offered to the student must be carefully elaborated by the teacher so that they generate interactions between peers, various possibilities for exploratory approaches and, above all, communication, validation, and negotiation of meanings. That is, PS is understood as a macroprocess that promotes a dialogical-collaborative mathematical activity between students and teachers, using the instruments, techniques, and practices given by the sociocultural context (Godino et al., 2020).

The mathematical variation

Since prehistory, human beings have been concerned with the study of change (Collete, 2000), but it was only in the mid-fifth century that the mathematical problems associated with variation began to be identified, specifically with the emergence of the study of the immeasurable. In fact, rigorous and general mathematical approaches to it began only in medieval times, which triggered the study of motion from infinitesimal conceptions and the subsequent study of tangents, maxima, and minima, leading to what is recognised today as the derivative and the integral (Cantoral & Farfán, 2004).

In this sense, variation can be defined as the change or evolution of magnitudes, objects, or phenomena in relation to other magnitudes, variables, or parameters. This concept of variation is not limited to quantitative change in numerical magnitudes, but also encompasses the qualitative understanding of the dynamic behaviour of mathematical phenomena and the relationships that allow us to understand how one quantity can depend on another (Leung, 2023).

Moreover, mathematical variation is part of a broad structure that includes various forms of change, modelling an infinity of situations in which some type of transformation occurs, along with its forms of representation and quantification (Cantoral et al., 2023), which leads us to consider variation as an object that enables the mathematical study of situations involving functional relationships, rates of change, and optimisation processes, among others (Moreno, 2021).

The connection between variation and forms of algebraic representation and the objects of calculus, such as the derivative, often leads to a strictly formal, algebraic emphasis in teaching and learning. However, this distances the study from the variation in dynamic ways of thinking that allow us to perceive change and the constant in processes in which time, explicitly or implicitly, plays an essential role (Vasco, 2006).

That is, in the study of variation, it is important to identify the relationships between the variable elements and the invariants, thereby generating a model of the observed, building forms of mathematical representation that can be treated by metric, numerical, spatial, and random elements. This requires the recognition of how changes in one variable(s) generate simultaneous changes in others, in addition to the identification of the form of the relationship and its quantification (Cantoral et al., 2023).

Thus, this research recognises three reference meanings of the variation:

- Variation as change or evolution of magnitudes, objects, or phenomena in relation to other magnitudes, variables, or parameters (Leung, 2023).
- Variation as recognition and quantification of change in various classes of situations that involve continuous and discrete magnitudes (Cantoral et al., 2023).
- Mathematical variation as a model of change (in functional and covariation relationships) and its quantification in contexts that involve discrete and continuous magnitudes (Bonilla et al., 2015).

METHODOLOGY

This work is part of the qualitative paradigm and is developed as a case study that seeks to understand in depth the beliefs that underpin problem-solving practices in the teaching and learning of variation. The participating group was intentionally selected and consists of 30 prospective mathematics teachers in the final stretch of their initial formative career at a public university in Colombia. Specifically, they are students who take a formative space called Didactics of Variation, in which they must design, apply, and evaluate learning situations to address variation in the different educational levels of Colombian education.

To capture the complexity of the phenomenon, complementary data collection techniques were combined: documentary analysis of class plans produced by prospective teachers and semi-structured interviews based on those class designs to study variation. The data obtained were subjected to a deductive coding process based on the epistemic configuration of the primary objects of the OSA, the typology of didactic-mathematical practices, and the classification of norms, followed by a contrast with the emerging categories of beliefs.

Regarding the instruments, considering that the main source of information was class planning and that the epistemic configuration of primary objects was used for the identification and systematisation of the practices of prospective teachers, the instrument used to identify the beliefs was a semi-structured questionnaire in which each question is related to one of the types of norms defined in the OSA:

- Epistemic standards: What elements did you take into account to select the problem(s) that you presented to the students? How did you decide when to introduce the concept of variation and the procedure to quantify it?

- Cognitive norms: What difficulties do you think students have in solving the problem you pose? Could you describe to me how you would help students overcome those difficulties? Why did you choose to evaluate learning that way?
- Affective norms: Do you believe that when solving the problem, some positive or negative emotion may arise in the students that influences their learning? Should any negative emotion arise, would you do anything in particular to manage it?
- Cognitive-affective norms: In your opinion, are there any cognitive or affective factors that influence students to benefit from problem solving when studying variation?
- Interactional norms: Why did you organise the students to solve the problem in a(n) ___ way (individual or group)? What are your criteria for deciding when to intervene directly, when to ask, and when to let students continue without guidance in solving the problem?
- Mediational norms: Why did you include the resolution of a problem at this point in the class?
- Do you believe that the use or non-use of technological tools favours the understanding and resolution of the problem?
- Ecological standards: Do you believe that external factors, such as educational policies, in some way conditioned your way of working on problem solving in the class?
- If you had to redesign the problem, would it be relevant to include some aspect of the sociocultural context? If the answer is yes, could you give an example of how you would do it?

ANALYSIS AND RESULTS

The presentation of the data and results analysis begins with the description of the class plans made by the prospective teachers through their epistemic configurations. From this reconstruction, the types of emerging teaching practices are identified and categorised, detailing the context of problem solving implementation. On these practices, the norms that regulate them (epistemic, cognitive, affective, interactive, mediational, and ecological) are analysed. Finally, beliefs are interpreted through their connection to previously identified norms and practices.

Identification of didactic-mathematical practices

Each of the 30 prospective teachers designed a plan to address variation in teaching and learning in university education. In these plans, four types of design were identified, which differ in the problems or situations proposed but coincide in their structure. Identified structures are described below.

To start, we identified 12 plans in which problem solving appears as the final part of the teaching and learning process (Table 1). This type of planning was characterised by the teacher explaining the definition of instantaneous variation, then demonstrating, through algebraic examples, a procedure for quantifying variation in functions, and finally having students solve a problem.

Table 1

Example of primary objects in class structure 1

Type	Objects
Situations	After serving a cup of coffee, the temperature T (°C) varies with time t (minutes) according to the model: $T(t) = 80 + 15e^{-0.1t}$. Calculate the temperature variation between *t=2* and *t=6 min*. Then determine the temperature variation at *t=4 min*. Explain what that value physically means for the cup of coffee.
Language	Algebraic, numerical, graphic, and verbal.
Definitions	Variation is the measure of the change of a function.
Procedures	*1. Quantification of average variation:* Selection of data and unknowns; Substitution of data in the model; Calculation of average variation as a reason for change: $\frac{T(6)-T(4)}{6-4}$; Response to the problem. *2. Quantification of instantaneous variation:* Selection of known and unknown data; Derivation of model; Substitution of data; Response to the problem.
Propositions	- The average variation is calculated as a slope. - The derivative indicates how fast a variable changes at a point.
Arguments	- The average variation is obtained from the calculation of the slope of the secant line to a function. - The instantaneous variation is obtained through the calculation of the slope of the line tangent to a function.

The second structure (Table 2) corresponds to eight plans in which problem solving appears as part of the teaching and learning process. In these plans, the prospective teachers propose to start with an explanation of the definition of instantaneous variation through a problem situation that is represented in GeoGebra graphs, without solving it, then exemplifying the quantification through a problem in which there is an algebraic model; after that, another, in which the model must be determined, and finally the resolution of algebraic exercises in the class and the delivery of several problems for autonomous work.

Table 2

Example of primary objects in class structure 2

Type	Objects
Situations	During the irrigation season, a farm uses an inverted-conical reservoir to store rainwater. The cone is 5 m high and has a radius of 2 m. A pumping system brings water into the tank at a constant flow rate of 0.04 m^3/s. At what speed does the height of the water (in m/s) change the instant the water level reaches 3 m deep?
Language	Algebraic, numerical, graphic, and verbal.
Definitions	Variation is the recognition and quantification of change in functional relationships that model problems.
Procedures	*Quantification of the instantaneous variation:* - Graphic representation of the problem. - Selection of data and unknowns. - Construction of a model. - Derivation of the model. - Replacement of data. - Response to the problem.
Propositions	In any variation problem, you can find models that represent the change.
Arguments	- The graphical representation of a problem allows us to visualise what is variable and what is invariable. - The derivative is the mathematical object that measures the instantaneous variation.

The third structure (Table 3) was identified in three plans that used the resolution of variation problems as the purpose of the teaching and learning process. In this case, the prospective teachers do not give a definition of the variation, instead, they define an instantaneous variation problem as the one in which it is desired to know the change in a point, then they give a resolution scheme of this type of problems and proceed to give several examples how to apply it; finally, it closes with a set of problems that are handed to the students to solve them autonomously.

Table 3

Example of primary objects in class structure 3

Type	Objects
Situations	During an advertising campaign, a spherical balloon is inflated to serve as a floating advertisement. The pumping system introduces air at a constant speed of 0.05 m^3/s. At what speed does the radius of the balloon change when it reaches a radius of 1.2 m? Indicate the solution process step by step.
Language	Algebraic, numerical, graphic, and verbal.
Definitions	An instantaneous variation problem is one in which we want to calculate the rate of change of the variables involved at a specific point or time.
Procedures	*Fixing an instantaneous variation problem*: - Graphic representation of the problem. - Selection of data and unknowns. - Construction of a model. - Derivation of the model. - Replacement of data. - Response to the problem.
Propositions	- Without the graphic representation, one cannot understand the problem. - Variation, speed, rate of change, and velocity are the same in these problems. - You must always refer and replace.
Arguments	The derivative was created to solve problems involving variation.

In the fourth and final structure identified (Table 4), seven plans were found that visualise problem solving as the central axis of the teaching and learning of variation. This group of teachers poses a problem and expects students to solve it (in groups of three) without definitions, procedures, or initial explanations, allowing students to use any tools (books, computers, cell phones, etc.) to consult and find a solution. They also propose that their role would be to gather partial advances to institutionalise concepts and procedures progressively.

Table 4

Example of primary objects in class structure 4

Type	Objects
Situations	At a science fair, a water rocket is launched vertically from the ground. Measurements indicate that when: $t=0\ s$ the height of the rocket is $0\ m$; $t=3\ s$ the rocket is $36\ m$ above the ground; $t=6\ s$ touches the ground again. If the height of the rocket fits a quadratic function of time t. How quickly does the height of the rocket change when $t=4\ s$? When is the height of the rocket maximum? What is the variation of the height at that time?
Language	Algebraic, numerical, graphic, and verbal.
Definitions	Instantaneous variation is the measure of the instantaneous rate of change.
Procedures	*Solution to a variation problem:* Consultation of information; Negotiation of concepts and procedures; Execution of consulted procedures; and Response to the problem.
Propositions	At the maximum point, the rate of change is zero.
Arguments	- The instantaneous variation is the slope of the tangent line. - At the maximum point of a function, the tangent line is horizontal, and its slope is zero.

Identification of didactic-mathematical practices

From the objects identified in each structure, it is possible to classify the types of practices sought with each design. In the case of the first structure (Table 5), we find mostly regulation practices, since the teacher focuses his efforts on establishing conceptual and procedural rules to address instantaneous variation. In addition, problem solving is associated with evaluation practices.

Table 5

Types of practices in class structure 1

Code	Practice	Type
P1	Definition of instantaneous variation	Regulation
P2	Exemplification of the perception of the instantaneous variation	Motivation
P3	Exemplification of the measurement of variation in functions	Regulation
P4	Problem-solving assessment	Assessment

In the second structure, regulation is again the predominant practice type (Table 6). However, problem solving takes on a different role, as it is also associated with motivation and task-assignment practices. This, considering that prospective teachers seek to capture students' attention through everyday situations and aim for PS to be transversal in the planning process and, therefore, in the expected teaching and learning process.

Table 6

Types of practices in class structure 2

Code	Practice	Type
P5	Definition of instantaneous variation	Regulation
P6	Association of the definition with a problem	Motivation
P7	Graphic representation of the definition and the problem	Motivation
P8	Example of instantaneous variation PS	Regulation
P9	Example of modelling and variation PS	Regulation
P10	Assignment of algebraic practice exercises	Assign tasks
P11	Assignment of study problems	Assign tasks

The third planning structure focuses on regulating how to recognise, solve, and interpret problems of instantaneous variation. For this reason, the practices associated with PS are considered to regulate a set of rules of action that the teacher intends to establish in the group of students (Table 7). The only type of additional didactic-mathematical practice that appears in the planning is the assignment of tasks; however, the planning does not aim to develop this practice in the classroom, and students are expected to work independently.

Table 7

Types of practices in class structure 3

Code	Practice	Type
P12	Definition of the instantaneous variation problem	Regulation
P13	Solution outline presentation	Regulation
P14	Example of instantaneous variation PS	Regulation
P15	Assignment of study problems	Assign tasks

Finally, in the fourth type of planning, the teacher's practices begin with the assignment of tasks and then focus on regulating students' progress and difficulties, until the concepts and procedures put into play are institutionalised. That is, in this case, problem-solving appears throughout the planning to help students develop their mathematical activity, and the teacher can regulate the knowledge intended to be achieved.

Table 8

Types of practices in class structure 4

Code	Practice	Type
P16	Assignment of the problem	Assign tasks
P17	Validation of partial results for the problem	Regulation
P18	Collection of solutions to the problem	Regulation
P19	Institutionalisation of emerging knowledge of the problem	Regulation

Standard identification

Each of the identified practices aligns with the criteria and standards prospective teachers use to plan the teaching and learning of variation, as summarised in Table 9. In the case of the first type of planning, the predominant rules are epistemic in nature, since they focus on the meaning and procedures of the variation, as reflected in the teacher's actions. For this reason, we find that the standards refer to PS as an evaluative practice.

In the second planning structure, there is greater diversity of rule classes, since elements of the meaning of variation (epistemic), student learning considerations (cognitive), and instruments that can favour the mathematical activity of PS (mediational) are combined. In this context, prospective teachers seek to manage PS as an algorithmic exercise assigned by the teacher, in search of a "simple" understanding process.

In the third structure, PS is mainly part of cognitive practices and is related to the student's autonomous work. Its purpose is to ensure that all the actions the teacher carries out are captured and replicated by the students. Finally, in the fourth structure, PS is associated with prospective teachers' concerns about managing mediations and interactions that promote student work and teacher regulation of variation in teaching and learning.

Table 9

Types of standards in planning

Structure	Practice	Standards	Type
Type 1: PS as the final stage	P1	It must start by defining variation	Epistemic
	P2	Students' interest must be sought	Affective
	P3	It is necessary to give procedures	Epistemic
	P4	PS allows us to evaluate learning	Epistemic
Type 2: PS as part of the process	P5	It must start by defining variation	Epistemic
	P6 and P7	Students need multiple representations of the variation	Cognitive
	P7	The dynamic software makes the variation visible	Mediational
	P8 and P9	It is necessary to give procedures	Epistemic
	P10 and P11	Students must practice algorithms before solving problems	Cognitive
Type 3: PS as purpose	P12	No need to define variation	Epistemic
	P13	Student needs PS models	Cognitive
	P14	The teacher must give examples of PS	Regulation
	P15	Students must practice PS model autonomously	Cognitive
Type 4: PS as the central axis	P16	Teaching and learning variation is achieved with a problem	Cognitive
	P17	The role of the teacher is the validation of learning	Interactional
	P18 and P19	Institutionalisation is a final phase of learning in PS	Mediational

Belief inference

Because the main interest of this article is to address the beliefs alluding to the way in which PS is considered in the teaching and learning of variation, only those related to PS are selected from the inferred norms. Specifically, we infer that for the 12 prospective teachers who employed the first planning structure, the main belief is that PS is a form of learning assessment. For the 8 who designed the second type of planning, PS is an autonomous practice of the student with which they exercise algorithms given by the teacher, but it must be present in all planning, even if it is a practice of the teacher. The three prospective teachers who opted for the third structure believe that PS is a practice that can be learned from solution models, and the seven who designed the fourth structure believe that PS is the triggering activity for the teaching and learning of variation.

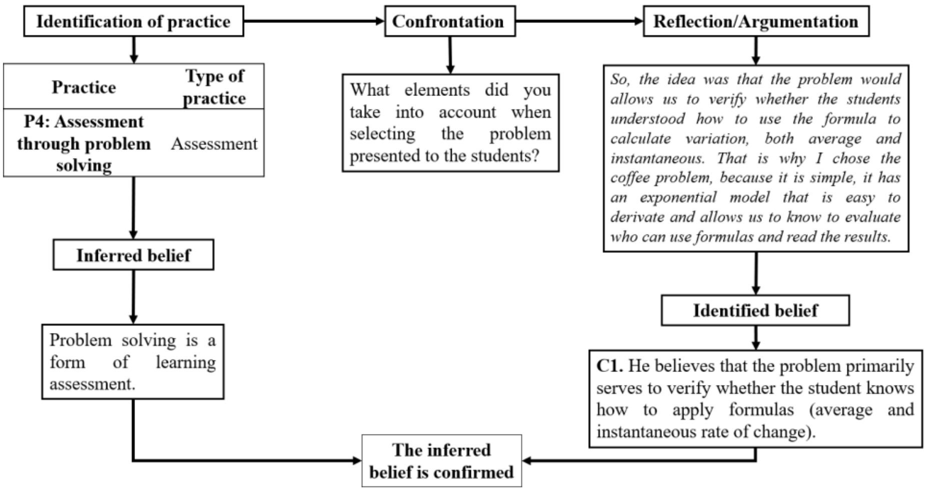
Reflection of prospective teachers

In the final stage, to deepen the inferred beliefs and verify which of them actually belong to the prospective teachers, the reflection process is developed. This stage begins from the confrontation of the ideas embodied in the plans with the practices identified by the researcher. Then, the questionnaire (previously presented) was applied so that the answers identify the arguments and reflections that account for the final list of beliefs.

To exemplify this process, Figure 3 shows how, from the answer to the first questionnaire question given by one of the prospective teachers who used the first planning structure, we determined that the belief that problem solving is a form of evaluation was correctly inferred.

Figure 3

Belief in reflection alluding to structure 1



As a result of this type of analysis, 14 beliefs were identified (Table 10) in the reflections of the prospective teachers who used the first type of planning. We can note that they resort to different types of standards to justify that PS is a personal activity developed to assess learning. In addition, positive and negative emotions are believed to emerge in PS, so it should occur at the end of a class, so that students already have the basic knowledge necessary to face problems and also so that the teacher can supervise the class group's understanding.

Table 10

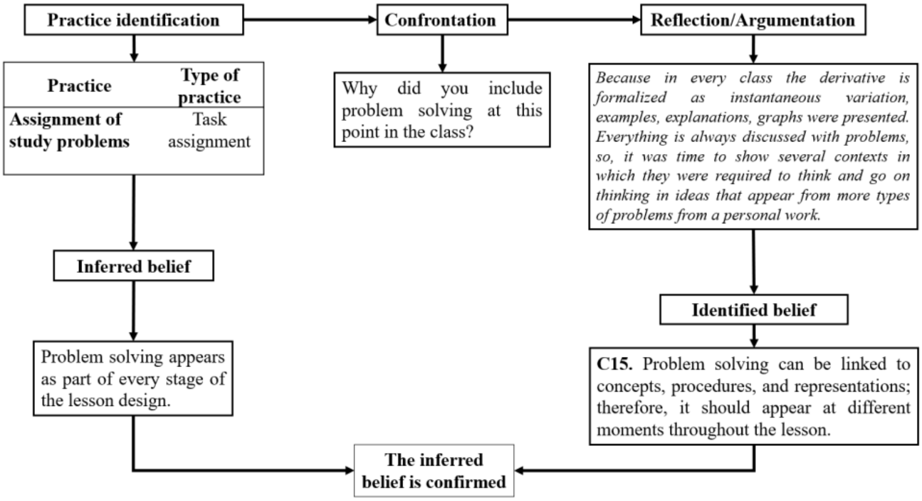
Beliefs in planning Type 1: PS as the final stage

Beliefs	Type
C1. The problem mainly serves to check whether students know how to apply formulas (average and instantaneous variation). **C2.** First comes the formal explanation, and then the problem as proof of what was taught.	Epistemic
C3. The difficulties in PS are mainly procedural (derivation, chain rule, interpretation of units). **C4.** The help in PS must be masterful and guided (review steps, give specific clues, model the solution). **C5.** PS is a valid evaluation strategy, meaning that students solved it because they "understood" it.	Cognitive
C6. Affective factors (anxiety/confidence) mediate the benefit of PS, but do not reconfigure its evaluative function.	
C7. Emotions play a role: success reinforces security and error frustrates. **C8.** The error in PS must be normalised, and calm, supportive feedback must be given to manage negative emotions.	Affective
C9. The work in PS must be individual to obtain personal evidence of learning (avoid "copying"). **C10.** The teacher must intervene when many have difficulties.	Interactional
C11. The problem lies at the end because its function is to check what has been seen. **C12.** Technology is not essential in variation PS; the key is the mastery of calculations.	Mediational
C13. Educational policies have little influence on how PS is used. **C14.** Employing a close context (climate or food) in PS gives additional value to the assessment.	Ecologic

The reflections of the prospective teachers who elaborated plans aligned with the second structure confirm that the central belief that guided their design was correctly inferred, as they place PS as a constitutive part of each moment of the teaching and learning process of variation (Figure 4). Well, they consistently justify their didactic decisions by claiming that they arise from the need to keep PS active at all times.

Figure 4

Belief in reflection alluding to structure 2



These preservice teachers proposed nine beliefs (Table 11), stating that PS is constantly related to concepts, procedures, and representations that cannot be presented simultaneously and therefore must appear at different times during the class. However, they assume that variation problems are addressed only after formalising the derivative and require constant dynamic visualisation through interactive graphics, in addition to a progressive sequence to mitigate the typical anxiety of working with rates. They even try to legitimise their ideas by interpreting the curriculum guidelines of the Ministry of Education of Colombia as requirements that mandate PS as the way in which mathematical competence should be developed.

Table 11

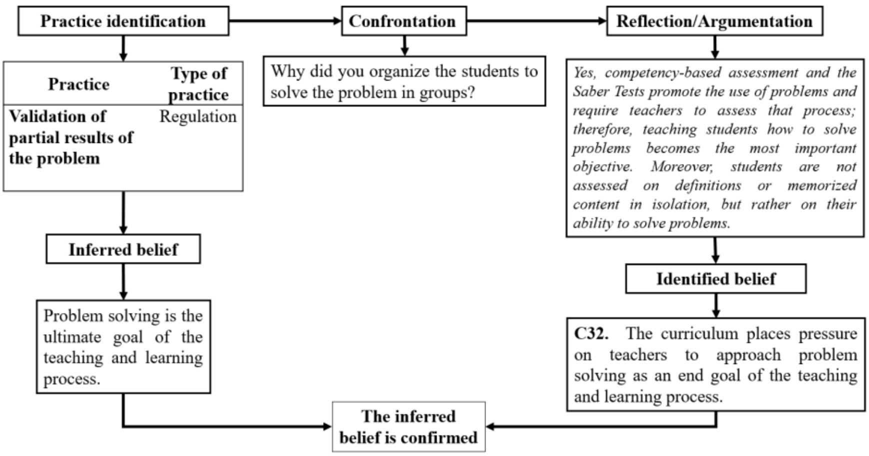
Beliefs in planning Type 2: PS as part of the process

Beliefs	Type
C15. The PS can be related to concepts, procedures, and representations, so it must appear at all times during the class.	Epistemic
C16. Students have difficulty identifying time-dependent variables. **C17.** Dynamic problems allow obstacles to be overcome.	Cognitive
C18. Variation PS generates anxiety; it must be worked on in phases.	Affective
C19. Although collaboration in pairs helps to understand the problems, only individual work gives evidence of learning. **C20.** The work with PS requires the teacher's follow-up.	Interactional
C21. Variation problems can only be worked on by students once the concept of the derivative is formalised.	Mediational
C22. Working with variation must be in accordance with the PS; it is a requirement of the Ministry of Education. **C23.** Context is important; that's why a problem is realistic.	Ecologic

In the third group, the reflections are consistent with the identified practices and norms, thereby confirming the central belief previously inferred: PS must constitute the purpose of teaching and learning (Figure 5).

Figure 5

Belief in reflection alluding to structure 3



Specifically, they stated ten beliefs (Table 12) in which variation PS is conceived as a rigorous, essentially individual exercise in which each task forces the student to read the situation, model it mathematically, and derive and justify each step, which can be taught as a resolution model. With this belief, they proposed that understanding emerges from practical training before formalising the definition of variation, and that any difficulties encountered are overcome through guided examples and explicit metacognition regarding the use of the model. In addition, for them, the affective-motor aspect of PS is the intrinsic pleasure of finding a solution and visualising the usefulness of variation; therefore, contextualising a problem becomes irrelevant, as do personal or social factors.

Table 12

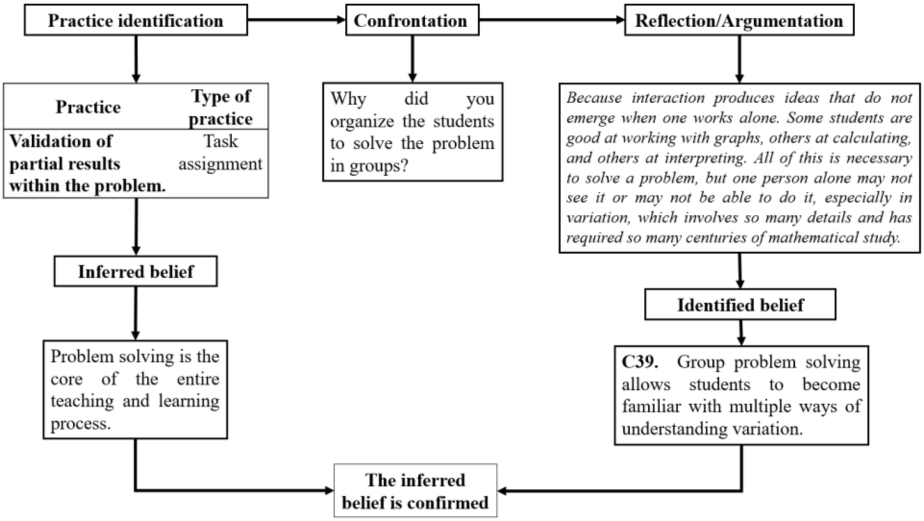
Beliefs in planning Type 3: PS as purpose

Beliefs	Type
C24. The variation problems must demand: reading, modelling, deriving, and justifying. **C25.** The definition of variation can wait; the PS method is first trained and then formalised.	Epistemic
C26. Difficulties are overcome through examples and metacognition (what they ask for, what changes, and how to isolate the rate of change). **C27.** With a good PS model, students can overcome difficulties alone.	Cognitive
C28. The pleasure of solving problems and seeing the usefulness of variation is the most important emotion.	Affective
C29. Acquiring the method of variation PS requires a personal effort. **C30.** The student only needs to listen to the explanation of the PS model and practice; group work is only distracting.	Interactional
C31. The use of technologies damages the reasoning that a student of variation PS must develop; they must abstract the variation.	Mediational
C32. The curriculum exerts pressure to work on PS as a purpose of the teaching and learning process. **C33.** Personal and social factors must be removed from the classroom, otherwise no one could solve a variation problem.	Ecologic

Finally, in the group reflections whose designs allowed inferring that PS is seen as the central axis of the teaching and learning of variation, arguments related to interactional norms predominate (Figure 6).

Figure 6

Belief in reflection alluding to structure 4



Through ten beliefs, they argue that a well-placed problem serves as a trigger that reveals all aspects of variation and exposes the real difficulties students must learn to face. For this reason, they believe that PS not only exercises calculation but also develops argumentation, fuels curiosity and creativity, and keeps students motivated.

Precisely, given the need to keep students involved in PS, this group of prospective teachers believes that student communication builds confidence and allows contrasting multiple views of variation to validate emerging ideas; therefore, PS must be conducted in a collaborative environment. For them, the teacher must focus their interventions and stimulate learning with the PS of family contexts, so that mathematical knowledge is related to everyday life and prepares the student to face contemporary social and political challenges.

Table 13

Beliefs in planning Type 4: PS as central axis

Beliefs	Type
C34. A good problem brings out all aspects of variation.	Epistemic
C35. The objective of PS is precisely that difficulties emerge and that the student learns to face them.	Cognitive
C36. With the variation PS, the student develops their ability to argue.	Affective
C37. PS fosters curiosity and creativity and keeps students motivated.	Affective
C38. Communication with colleagues generates trust, which is why PS must be worked on in groups. **C39**. Group PS allows students to learn multiple ways of seeing variation. **C40**. Group work allows the student to validate their ideas. **C41**. With PS, the teacher can focus their actions and enhance learning about variation.	Interactional
C42. In PS, students must have access to all tools.	Mediational

In this way, through prospective teachers' reflections on their designs, we find that the inferences drawn from the classification of practices and norms reflect the beliefs that guided the plans. Moreover, no inconsistencies or conflicts were identified between the designs, the practices identified, and the beliefs expressed in the reflection. With this, we consider that the process developed allows classifying the beliefs of prospective teachers about the role of PS in the teaching and learning of variation in four types: The first group understands PS as an assessment and places it at the end of the work sequence in the classroom; the second group conceives PS as part of the teaching and learning process, which is why it appears at the beginning, during and at the end of the planning; the third group, which makes PS the purpose, prioritises the resolution method (reading, modelling, deriving, and justifying) before the formalisation of concepts; and finally, in the fourth group, PS is considered as a structuring axis that allows concepts, arguments, procedures, and rules of interaction to emerge.

CONCLUSIONS

This article presents an example of the articulation between two emerging proposals within the theoretical-methodological framework of the OSA for identification, structuring, and analysis of beliefs (Ledezma et al., 2022; Acevedo and Pino-Fan, 2024). As a result, we could identify preservice mathematics teachers' central beliefs in an initial education program in Colombia, to design class sessions that use problem solving in the teaching and learning process of variation: (1) final stage, (2) part of the process, (3) purpose, and (4) central axis. In addition, we detailed 45 beliefs, specifically describing the epistemic, cognitive, affective, mediational, interactional, and ecological aspects of each of the four considerations.

The analysis developed enabled us to examine both discursive and operational practices to obtain a broad yet very close view of what mathematics teachers really think and do, as suggested by multiple investigations (Moreno & Azcarate, 2003; Speer, 2005; Calleja, 2022). The study evidenced that, although all participants were in the same formative environment, personal beliefs ultimately guide very different problem-solving practices. This result highlights the importance of reflecting on and discussing the didactic-mathematical practices of teachers in initial education, as they make explicit the beliefs that condition the interpretation and application of the emerging knowledge of mathematics education.

Regarding variation, the identification and classification of didactic-mathematical practices revealed that prospective teachers incorporate it differently into their class plans. Two groups considered it only a quantification of change; another emphasised the importance of its perception before measurement; and the remaining evaded its conceptualisation. This shows that the epistemic beliefs held about the mathematical objects involved in the practices strongly condition the design of teaching and learning situations based on problem solving, since the regulatory practices and the rules of interaction in the plans were derived from the meaning intended by the teachers.

Finally, the exposed belief analysis offers a replicable process that can strengthen the knowledge in the scientific community of mathematics education about the factors that favour or limit the transformation of traditional mathematics teaching and learning practices, recognising that, by expanding the explanation of the criteria and norms that motivate teachers' actions, it is also possible to investigate, diversify, and agree on formative practices that allow prospective teachers to transform the beliefs that hinder and limit their professional development.

AUTHORSHIP CONTRIBUTIONS STATEMENTS

CCFM conceived and developed the research, organising the theoretical part, the methodological design, and the data collection and analysis. Finally, CCFM wrote this article, published as a result of the research.

DATA AVAILABILITY DECLARATION

The data supporting the results of this study are available upon request to the author CCFM.

ACKNOWLEDGEMENTS

This research is funded by the Universidad Distrital Francisco José de Caldas, in the framework of the *Convoca UD* program *(Pr2-2025)* which supports research projects, research-creation, and innovation.

REFERENCES

Acevedo, G., & Pino-Fan, L. (2024). A proposal for the study of mathematics teachers' beliefs through the analysis of their practices. *Mathematics Education Research Journal*, 1–25. https://doi.org/10.1007/s13394-024-00496-y

Asenova, M. (2021). *Definizione categoriale di Oggetto matematico in Didattica della matematica*. Pitagora.

Asenova, M., D'Amore, B., Fandiño, M., Fúneme, C., Iori, M., & Santi, G. (2024). *Teorías relevantes en Educación Matemática*. Magisterio.

Beltrán, P., & Godino, J. (2017). Aplicación de indicadores de idoneidad afectiva en un proceso de enseñanza de probabilidad en educación secundaria. *Perspectiva Educacional, Formación de Profesores*, *56*(2), 92–116.

Beltrán, P., & Godino, J. (2020). An onto-semiotic approach to the analysis of the affective domain in mathematics education. *Cambridge Journal of Education*, *50*(1), 1–20.

Bonilla, M., Romero, J., Narváez, D. & Bohórquez, L. (2015). Características del proceso de construcción del significado del concepto de variación matemática en estudiantes para profesor de matemáticas. *Avances de investigación en educación matemática: AIEM*, (7), 73–93.

Brousseau, G. (1986). Fondements et méthodes de la didactique des Mathématiques. *Recherches en didactique des mathématiques*, *7*(2), 33–115.

Cai, J., & Hwang, S. (2023). Making mathematics challenging through problem posing in the classroom. En R. Leikin, (Ed.), *Mathematical challenges for all* (pp. 115-145). Springer.

Calleja, J. (2022). Changes in mathematics teachers' self-reported beliefs and practices over the course of a blended continuing professional development programme. *Mathematics education research journal*, *34*(4), 835–861.

Cantoral, R., Espinoza, L. y Gaete, C. (2023). Exponential behaviour and variational practices in Chilean newscasts: a socioepistemological view. *ZDM Mathematics Education*, *55*(1), 147–161.

Cantoral, R., & Farfán, R. (2004). Desarrollo conceptual del cálculo. Thomson.

Collette, J. (2000). *Historia de las matemáticas I y II*. Siglo Veintiuno editores.

D'Ambrosio, U. (2014). Las bases conceptuales del Programa Etnomatemática. *Revista Latinoamericana de Etnomatemática Perspectivas Socioculturales de la Educación Matemática*, *7*(2), 100–107.

D'Amore, B., Font, V., & Godino, J. (2007). La dimensión metadidáctica en los procesos de enseñanza y aprendizaje de la matemática. *Paradigma*, *28*(2), 49–77. https://doi.org/10.37618/PARADIGMA.1011-2251.2007.p49-77.id386

D'Amore, B., & Godino, J. (2007). El enfoque ontosemiótico como un desarrollo de la teoría antropológica en didáctica de la matemática. *Revista Latinoamericana de Investigación en Matemática Educativa*, *10*(2), 191–218.

DeBellis, V., & Goldin, G. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics*, *63*, 131–147.

Eco, U. (1979). *A theory of semiotics*. Indiana University Press.

Font, V., Godino, J., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*, *82*, 97–124.

Font, V., Planas, N., & Godino, J. (2010). Modelo para el análisis didáctico en educación matemática. *Infancia y Aprendizaje*, *33*(1), 89–105.

Fúneme, C. (2023). Una interpretación del papel de las creencias en algunos modelos de conocimiento del profesor. *Paradigma*, *44*(2), 190–211.

Fúneme, C., Linares, L., & Cáceres, L. (2023). Conocimiento didáctico-matemático de profesores colombianos sobre los procesos de generalización y particularización en la resolución de problemas. *Paradigma*, *44*(Edição Temática: EOS. Questões e Métodos), 194–220.

Gagne, R. M. (1965). *The conditions of learning*. Holt, Rinehart and Winston.

Godino, J. (2021). Emergencia, estado actual y perspectivas del enfoque ontosemiótico en educación matemática. *Revista Venezolana de Investigación en Educación Matemática (REVIEM)*, *1*(1), 1–21.

Godino, J. (2024). *Enfoque ontosemiótico en educación matemática. Fundamentos, herramientas y aplicaciones*. McGraw Hill-Aula Magna.

Godino, J., & Batanero, C. (1994). Significado institucional y personal de los objetos matemáticos. *Recherches en Didactique des Mathématiques*, *14*(3), 325–355.

Godino, J., Batanero, C., & Font, V. (2007). The ontosemiotic approach to research in mathematics education. *ZDM Mathematics Education*, *39*(1-2), 127–135.

Godino, J., Batanero, C., & Font, V. (2020). El enfoque ontosemiótico: Implicaciones sobre el carácter prescriptivo de la didáctica. *Revista Chilena de Educación Matemática*, *12*(2), 3–15.

Godino, J., Burgos, M., & Wilhelmi, M. (2020). Papel de las situaciones adidácticas en el aprendizaje matemático. Una mirada crítica desde el enfoque ontosemiótico. *Enseñanza de las Ciencias*, *38*(1), 147–164. 147 https://doi.org/10.5565/rev/ensciencias.2906

Godino, J., Font, V., Wilhelmi, M., & De Castro, C. (2009). Aproximación a la dimensión normativa en didáctica de las matemáticas desde un enfoque ontosemiótico. *Enseñanza de las ciencias*, *27*(1), 59–76. https://raco.cat/index.php/Ensenanza/article/view/132207

Goldin, G. (2002). Affect, meta-affect, and mathematical belief structures. En G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics Education?* (pp. 59–72). Kluwer Academic Publishers.

Goldin, G., Rösken, B., & Törner, G. (2009). Beliefs–no longer a hidden variable in mathematical teaching and learning processes. In J. Maaß, & W. Schlöglmann (eds.), *Beliefs and attitudes in mathematics education: New research results* (pp. 9–28). Sense Publishers.

Kaiser, G., & Schwarz, B. (2010). Authentic modelling problems in mathematics education—examples and experiences. *Journal für Mathematik-Didaktik*, *31*(1), 51–76.

Ledezma, C., Sol, T., Sala, G., & Font, V. (2022). Knowledge and beliefs on mathematical modelling inferred in the argumentation of a prospective teacher when reflecting on the incorporation of this process in his lessons. *Mathematics*, *10*(18), 3339.

Lester, F. (1980). Research on mathematical problem solving. En R. Shumway (Ed.), *Research in mathematics education* (pp. 286–323). National Council of Teachers of Mathematics.

Leung, A. (2023). A pedagogical reflection on the interplay between variation and invariant: Variational thinking. *Asian Journal for Mathematics Education*, *2*(3), 261–273.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Pearson Prentice Hall.

Miranda, P., & Mamede, E. (2022). Appealing to creativity through solving and posing problems in mathematics class. *Acta scientiae*, *24*(4), 109-146.

Moreno, L. (2021). The theory of calculus for calculus teachers. *ZDM Mathematics Education*, 53, 621–633. https://doi.org/10.1007/s11858-021-01222-9

Moreno, M., & Azcárate, C. (2003). Concepciones y creencias de los profesores universitarios de matemáticas acerca de la enseñanza de las ecuaciones diferenciales. *Enseñanza de las ciencias*, *21*(2), 265–280.

Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies*, *19*(4), 317–328.

Parra, Y., & Breda, A. (2017). La enseñanza de o desde la resolución de problemas matemáticos: concepciones de profesores de Matemática en formación. *Acta Scientiae*, *19*(2), 277–295.

Pino-Fan, L., Castro, W., & Font, V. (2022). A Macro tool characterize and Develop Key Competencies for the Mathematics Teacher' Practice. *Internacional Journal of Science and Mathematics Education*, *21*(5), 1407–1432. https://doi.org/10.1007/s10763-022-10301-6

Polya, G. (1945). *How to solve it*. Princeton University Press.

Radford, L. (2018). On theories in mathematics education and their conceptual differences. In B. Sirakov, P. De Souza, & M. Viana (Eds.), *Proceedings of the International Congress of Mathematicians Vol. 4* (pp. 4055–4074). World Scientific Publishing Co.

Sáenz-Ludlow, A. & Zellweger, S. (2016). Classroom mathematical activity when it is seen as an inter-intra double semiotic process of interpretation. En A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a Tool for Learning Mathematics* (pp. 43–68). Sense Publishers.

Santos, M. (2024). Problem solving in mathematics education: tracing its foundations and current research-practice trends. *ZDM Mathematics Education*, *56*, 211–222. https://doi.org/10.1007/s11858-024-01578-8

Santos, M., & Reyes, I. (2019). High school prospective teachers' problem-solving reasoning that involves the coordinated use of digital technologies. *International Journal of Mathematical Education in Science and Technology*, *50*(2), 182–201.

Schoenfeld, A. (1982). Some thoughts on problem-solving research and mathematics education. In F. Lester, & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research* (pp. 27–37). Franklin Institute Press.

Schoenfeld, A. (2022). Why are learning and teaching mathematics so difficult? In M. Danesi (Ed.), *Handbook of cognitive mathematics* (pp. 1–35). Springer.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In A. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 127–146). Macmillan.

Törner, G. (2002). Mathematical beliefs - a search for a common ground. In G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 73–94). Kluwer Academic Publishers.

Vasco, C. (2006). *Didáctica de las Matemáticas. Artículos Selectos. 1a edición*. Editorial Universidad Pedagógica Nacional.

Wellberg, S. (2024). An examination of pre-service mathematics teachers' course-taking, beliefs, and preferred assessment practices. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-024-09640-8